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On a generalization of the Youla–Kučera parametrization. Part I: the fractional ideal approach to SISO systems

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Abstract

In this paper, we show how to use the theory of fractional ideals in order to study the fractional representation approach to analysis and synthesis problems for SISO systems. Within this mathematical framework, we give necessary and sufficient conditions so that a plant is internally/strongly/bistably stabilizable or admits a (weak) coprime factorization. Moreover, we show how to generalize the Youla–Kučera parametrization of the stabilizing controllers to any stabilizable plant which does not necessarily admit a coprime factorization. This parametrization is generally affine in two free parameters.

Keywords: Fractional representation approach to analysis and synthesis problems; Internal stabilizability; Youla–Kučera parametrization of the stabilizing controllers; Infinite-dimensional linear systems; Fractional ideals; isomorphy class

1. Introduction

In this paper, we show why the *theory of fractional ideals* [3,6,16] is a natural mathematical framework for the *fractional representation approach to analysis and synthesis problems* [4,5,7,18,20] in the case of single input single output (SISO) systems. Within this algebraic framework, we prove that some analysis problems (stability, existence of (weak) coprime factorizations) as well as synthesis problems (internal/strong/bistable stabilization, parametrization of all the stabilizing controllers, etc.) have simple and tractable formulations.

For finite-dimensional systems (rational transfer functions), internal stabilizability is equivalent to the existence of a coprime factorization [20] but this is not true anymore in the general setting (non-rational transfer functions coming from differential time-delay systems or partial differential equations) [4,5,20]. The Youla–Kučera parametrization was developed in [8,21] in order to parametrize all the stabilizing controllers of a plant admitting a *coprime factorization*. Hence, it is natural to ask whether or not it is possible to parametrize all the stabilizing controllers of an *internally stabilizable plant* which does not admit a coprime factorization. In this paper, we exhibit such a parametrization without any assumption on the existence of a coprime factorization for the plant. We prove that this parametrization is affine like the Youla–Kučera parametrization and generally has two free parameters. Moreover, we show that if p admits a coprime factorization, then this parametrization is nothing else than the Youla–Kučera one and

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if p admits no coprime factorization but p^2 does, then this parametrization has a unique free parameter. We illustrate all these results using explicit examples coming from [1,4,9,11].

The fractional ideal approach also allows us to show that all the plants belonging to the same *isomorphy class* [3,6,16] share the same structural properties (internal stabilizability, existence of a (weak) coprime factorization, etc.). Using the concept of *class semigroup* [3,6,16], we give an explicit relation between two transfer functions so that the corresponding plants belong to the same isomorphy class, and thus, have the same structural properties. Moreover, using the concept of *Picard group* [3,6,16], we show how to check whether or not every stabilizing plant has a Youla–Kučera parametrization or a generalized parametrization with one or two parameters.

Finally, let us notice that K. Mori has also attempted in [10] to generalize the Youla–Kučera parametrization to any stabilizable plant using a different approach. However, contrary to the fractional ideal approach developed in this paper, the number of free parameters in his parametrization is not minimal and no system interpretation to the number of free parameters is given.

2. Fractional representation approach to stabilization problems

In the course of the text, A will denote a commutative integral domain with a unity ($ab=0, a \neq 0 \Rightarrow b=0$), $U(A)=\{a \in A \mid \exists b \in A\colon ab=1\}$ the group of the invertible elements of A and

$$K = Q(A) = \{n/d \mid 0 \neq d,\ n \in A\}$$

the field of fractions of A [17]. We shall denote the set of $q \times p$ (resp. $p \times p$) matrices with entries in A by $M_{q\times p}(A)$ (resp. $M_p(A)$) and I_p the identity matrix. If $a_1,\ldots,a_n \in K=Q(A)$, then $(a_1,\ldots,a_n)$ will denote the A-module defined by $Aa_1+\cdots+Aa_n$, and, if M and N are two A-modules, $M \cong N$ will mean that M and N are isomorphic. Finally, the notation $\triangleq$ will mean 'by definition'.

The fractional representation approach to analysis and synthesis problems was developed in the 1980s in order to unify in a common mathematical framework certain questions arising from different synthesis problems (internal, robust, strong or simultaneous stabilization problems, parametrization of the stabilizing controllers, graph metric, H_2 or H_∞-optimal controllers, etc.) [4,5,20]. In this input–output framework, the class of linear time-invariant SISO systems is defined by means of the quotient field $K=Q(A)$ of an *integral domain A of SISO stable plants*. Examples of such rings A of SISO stable plants are RH_∞, $H_\infty(\mathbb{C}_+)$ or the Wiener algebras $\hat{\mathscr{A}}$ and $l_1(\mathbb{Z}_+)$ [4,5,20]. Hence, the fact that a transfer function does not belong to the ring A of SISO stable plants means that it is not stable, i.e. unstable. For example, the unstable transfer function $p=1/(s-1)$ does not belong to RH_∞ but to $K=Q(RH_\infty)=\mathbb{R}(s)$ because it can be written as $p=n/d$ with $n=1/(s+1) \in RH_\infty$ and $0 \neq d=(s-1)/(s+1) \in RH_\infty$. Therefore, one of the main ideas of the fractional representation approach to systems is to replace the verification of the stability of a plant p by the membership problem $p \in A$ [5,20].

We shall need the following definitions.

Definition 1.

- A transfer function $p \in K = Q(A)$ admits a *weakly coprime factorization* if there exist $0 \neq d,\ n \in A$ such that $p=n/d$ and [13,14,19]:

  $\forall k \in K = Q(A)\colon kn,\ kd \in A \Rightarrow k \in A.$

- A transfer function $p \in K=Q(A)$ admits a *coprime factorization* if there exist $0 \neq d,\ n,x,y \in A$ such that $p=n/d$ and $dx-ny=1$ [4,19,20].
- A plant, defined by a transfer function $p \in K$, is *internally stabilizable* iff there exists a controller $c \in K$ such that [19,20]

$$H(p,c) = \begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{1-pc} & \frac{p}{1-pc} \\ \frac{c}{1-pc} & \frac{1}{1-pc} \end{pmatrix} \in M_2(A),$$

  where $H(p,c)$ represents the transfer matrix from $(u_1 : u_2)^T$ to $(e_1 : e_2)^T$ (see Fig. 1 for more details).
- A plant, defined by a transfer function $p \in K = Q(A)$, is *strongly stabilizable* (resp. *bistably stabilizable*) if there exists a stable (resp. stable with a stable inverse) controller $c \in A$ (resp. $c \in \mathrm{U}(A)$) which internally stabilizes p [20].

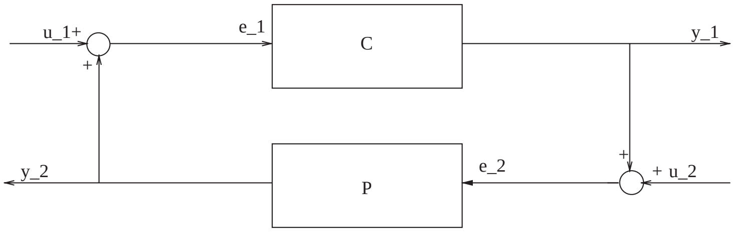

Fig. 1. Closed-loop system.

3. SISO systems & fractional ideals

Let us give few definitions on fractional ideals. See [3,6,16,17] for more details.

Definition 2.

- A *fractional ideal* I of A is an A-submodule of $K = Q(A)$ such that there exists $0 \neq a \in A$ satisfying $aI \subseteq A$.
- A fractional ideal I of A is *principal* if $I = (k) \triangleq Ak$ for a certain $k \in K$.
- An ideal $I \subseteq A$ is a fractional ideal of A called *integral ideal* of A.

If I and J are two fractional ideals of A, then

$$\begin{cases} I \cap J = \{a \in I,\ a \in J\}, \\ I + J = \{a + b \mid a \in I,\ b \in J\}, \\ IJ = \left\{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I,\ b_i \in J,\ n \in \mathbb{Z}_+ \right\}, \\ I : J = \{k \in K = Q(A) \mid (k)J \subseteq I\}, \end{cases}$$

are also fractional ideals. Hence, the fractional ideals are stable under intersections, finite sums, products and residuals. Let us denote by $\mathscr{F}(A)$ the set of non-zero fractional ideals of A.

Lemma 1 (Fuchs and Salce [6]). *If I, J, L are three fractional ideals of A, then we have*

(1) $I(J + L) = IJ + IL$,
(2) $I : (J + L) = (I : J) \cap (I : L)$,
(3) $(I : J) : L = I : (JL) = (I : L) : J$.

Definition 3. A non-zero fractional ideal I of A is *invertible* if there exists $J \in \mathscr{F}(A)$ such that $IJ = A$.

Let us note that an invertible fractional ideal I of A is *cancellative*, namely

$$\forall J, L \in \mathscr{F}(A):\ IJ = IL \Rightarrow J = L.$$

In particular, this means that every invertible fractional ideal has a unique inverse. We denote this inverse by I^{-1}.

Lemma 2 (Bourbaki [3], Fuchs and Salce [6], Rosenberg [16], Rotman [17]). *If I is an invertible fractional ideal of A, then we have*

(1) *I is a finitely generated projective A-module, namely an A-module I such that there exist $r \in \mathbb{Z}_+$ and an A-module P which satisfy $I \oplus P \cong A^r$,*
(2) $I^{-1} = A : I = \{k \in K = Q(A) \mid (k)I \subseteq A\}$,
(3) *I^{-1} is invertible and $(I^{-1})^{-1} = I$.*

We have the following theorem.

Theorem 1. *Let A be an integral domain of stable SISO systems, $K = Q(A)$, $p \in K$ a transfer function and $J = (1, p) \triangleq A + A\,p$ the fractional ideal defined by* 1 *and p. Then, we have*

(1) *p is stable, i.e. $p \in A$, iff $J = A$, or equivalently, iff $A : J = A$.*
(2) *p admits a weakly coprime factorization iff the fractional ideal $A : J$ is a principal integral ideal, i.e. there exists $0 \neq d \in A$ such that $A : J = (d)$. Then, $p = n/d$, $n \triangleq d\,p \in A$, is a weakly coprime factorization of p.*

(3) *p is internally stabilizable iff the fractional ideal J is invertible, namely we have $J(A:J)=A$, i.e. iff there exist $a,b\in A$ such that*

$$a - pb = 1,$$

$$pa \in A. \quad (1)$$

If $a\neq 0$, then $c=b/a$ is a stabilizing controller of p and $J^{-1}=(a,b)$. Moreover, c internally stabilizes p iff we have the following equality:

$$(1,p)(1,c)=(1-pc). \quad (2)$$

(4) *p admits a coprime factorization iff the fractional ideal J is a principal fractional ideal of A, i.e. there exists $0\neq k\in K=Q(A)$ such that $J=(k)$. Moreover, there exists $0\neq d\in A$ such that $k=1/d$ and $p=n/d$ is a coprime factorization where $n\triangleq d\,p\in A$.*

(5) *p is strongly stabilizable iff there exists $c\in A$ such that $J=(1-pc)$.*

(6) *p is bistably stabilizable iff there exists $c\in \mathrm{U}(A)$ such that $J=(1-pc)$.*

Proof. (1) If p is stable, i.e. $p\in A$, then $J=(1,p)=(1)=A$, and thus, $A:J=A:A=A$. Now, if $A:J=A$, then we have

$$1\in A:J=\{k\in K\mid k,\ kp\in A\}$$
$$=\{d\in A\mid d\,p\in A\}\Rightarrow\ p\in A.$$

(2) If p has a weakly coprime factorization $p=n/d$ $(0\neq d,n\in A)$, then we have $A:(d,n)=\{k\in K\mid kd,\ kn\in A\}=A$. By (3) of Lemma 1, we obtain

$$A:J=(A:(d^{-1})(d,n))=(A:(d,n)):(d^{-1})$$
$$=A:(d^{-1})=(d).$$

Thus, $A:J$ is a principal integral ideal. Conversely, let us suppose that $A:J$ is a principal integral ideal, i.e. $A:J=(d)$, $0\neq d\in A$. Then, we have

$$A:J=\{k\in K\mid k,\ kp\in A\}$$
$$=\{d'\in A\mid d'\,p\in A\}=(d),$$

and thus, $d\,p\in A$. If we note $n=d\,p\in A$, then $p=n/d$. Moreover, by (3) of Lemma 1, we have

$$A:(d,n)=A:((d)J)=(A:J):(d)$$
$$=(d):(d)=A,$$

i.e. $A:(d,n)=\{k\in K\mid kd,\ kn\in A\}=A$, and thus, $p=n/d$ is a weakly coprime factorization of p.

(3) $\Rightarrow$ If p is internally stabilizable, then there exists a controller $c\in K$ such that $1-pc\neq 0$ and

$$H(p,c)=\begin{pmatrix}1 & -p\\ -c & 1\end{pmatrix}^{-1}$$
$$=\begin{pmatrix}\frac{1}{1-pc} & \frac{p}{1-pc}\\ \frac{c}{1-pc} & \frac{1}{1-pc}\end{pmatrix}\in M_2(A). \quad (3)$$

Let us note $a=1/(1-pc)\in A$, $b=c/(1-pc)\in A$. The integral ideal $L=(a,b)$ of A, defined by a and b, satisfies

$$1=a-b\,p\in JL=(a,\ b,\ a\,p,\ b\,p)\subseteq A$$
$$\Rightarrow JL=A\ \ \Rightarrow\ \ L=J^{-1},$$

i.e. J is an invertible ideal of A and $J^{-1}=(a,b)$.

$\Leftarrow$ If $J=(1,p)$ is an invertible ideal of A, then we have $(A:J)J=A$, with $A:J=\{d\in A\mid d\,p\in A\}$. Thus, there exist $a,b\in A:J$, i.e. $a,b\in A$ and $a\,p,\ b\,p\in A$, which satisfy $a-b\,p=1$, and thus, we have (1) because $b\,p=1-a\in A$. If $a\neq 0$, then $c=b/a$ is a stabilizing controller of p because

$$\begin{cases}1/(1-pc)=a\in A,\\ p/(1-pc)=a\,p\in A,\\ c/(1-pc)=b\in A,\end{cases}$$

$$\Rightarrow\begin{pmatrix}1 & -p\\ -c & 1\end{pmatrix}^{-1}=\begin{pmatrix}a & pa\\ b & a\end{pmatrix}\in M_2(A).$$

Finally, if c internally stabilizes p, then $J^{-1}=(a,b)$, and thus

$$A=JJ^{-1}=(1,p)(a,b)=(1,p)(a)(1,b/a)$$
$$\Rightarrow(1,p)(1,c)=(a^{-1})=(1-pc).$$

Conversely, if we have (2), then we have

$$(1,p)(1/(1-pc),c/(1-pc))=A,$$

which shows that $J=(1,p)$ is invertible and

$$J^{-1}=\{d\in A\mid d\,p\in A\}=(1/(1-pc),c/(1-pc)).$$

If we define $a\triangleq 1/(1-pc)\in A$, $b\triangleq c/(1-pc)\in A$, then $a\,p\in A$ and $a-b\,p=1$, which shows that $c=b/a$ is a stabilizing controller of p.

(4) If p has a coprime factorization $p=n/d$ ($0\neq d,n\in A$), then there exist $x,y\in A$ such that $dx-ny=1$. Therefore, $I=(d,n)=A$, and thus, $J=(1,p)=(d^{-1})I=(d^{-1})$ is a principal fractional ideal of A. Conversely, if J is a principal fractional ideal of A, then there exists $k\in K$ such that $J=(k)$. Then, there exist $x,y,n,d\in A$ such that we have

$$\begin{cases} k=x-y/p,\\ 1=dk,\\ p=nk.\end{cases}$$

From the second and third equations, we deduce that $k=1/d$ and $p=n/d$. Therefore, in substituting $k=1/d$ and $p=n/d$ into the first equation, we obtain that $1/d=x-y(n/d)$, i.e. $dx-ny=1$, and thus, $p=n/d$ is a coprime factorization of p.

(5) (resp. 6). If p is strongly (resp. bistably) stabilizable, then there exists a stable controller $c\in A$ (resp. $c\in \mathrm{U}(A)$) such that we have (2). But, we have $(1,c)=A$ because $c\in A$ (resp. $c\in U(A)$), and thus, (2) implies $(1,p)=(1-pc)$.

Conversely, if there exists $c\in A$ (resp. $c\in\mathrm{U}(A)$) such that $J=(1-pc)$, then there exist $0\neq d,\ n\in A$ such that we have

$$\begin{cases} 1=d(1-pc),\\ p=n(1-pc),\end{cases} \Rightarrow \begin{cases} 1-pc\neq 0,\\ p=n/d,\\ p=(d-1)/(dc),\\ p=n/(1+nc).\end{cases}$$

Also, from the first previous system, we obtain

$$\begin{aligned} 1+p(-c)&=d(1-pc)+n(1-pc)(-c)\\ &=(1-pc)(d-nc). \end{aligned} \tag{4}$$

Then, from the fact that $1-pc\neq 0$ and (4), we obtain that $d-nc=1$, and thus, $p=n/d$ is internally stabilized by the stable (resp. bistable) controller c because we have

$$\begin{pmatrix} 1 & -n/d\\ -c & 1\end{pmatrix}^{-1}=\frac{1}{(d-nc)}\begin{pmatrix} d & n\\ dc & d\end{pmatrix} =\begin{pmatrix} d & n\\ dc & d\end{pmatrix}\in M_2(A). \qquad \square$$

Example 1. Let us consider the wave equation [4]:

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(x,t)-\frac{\partial^2 z}{\partial x^2}(x,t)=0,\\ \frac{\partial z}{\partial x}(0,t)=0,\ \frac{\partial z}{\partial x}(1,t)=u(t),\\ y(t)=\frac{\partial z}{\partial t}(1,t).\end{cases}$$

$$\Rightarrow \hat{y}(s)=\frac{(\mathrm{e}^{s}+\mathrm{e}^{-s})}{(\mathrm{e}^{s}-\mathrm{e}^{-s})}\hat{u}(s).$$

The transfer function

$$p=(\mathrm{e}^{s}+\mathrm{e}^{-s})/(\mathrm{e}^{s}-\mathrm{e}^{-s})=(1+\mathrm{e}^{-2s})/(1-\mathrm{e}^{-2s})$$

belongs to the field of fractions of $A=H_\infty(\mathbb{C}_+)$ because $1+\mathrm{e}^{-s},1-\mathrm{e}^{-s}\in A$. Let us consider the fractional ideal $J=(1,p)$ of A. We can check that 1 is the greatest common divisor of $1-\mathrm{e}^{-2s}$ and $1+\mathrm{e}^{-2s}$ (A is a *greatest common divisor domain* [13,19]). Thus, we have $A:J=\{d\in A\mid dp\in A\}=(1-\mathrm{e}^{-2s})$ and, by (2) of Theorem 1, $p=(1+\mathrm{e}^{-2s})/(1-\mathrm{e}^{-2s})$ is a weakly coprime factorization.

Moreover, using the fact that $((1+\mathrm{e}^{-2s})+(1-\mathrm{e}^{-2s}))/2=1$, we obtain $J(A:J)=(1+\mathrm{e}^{-2s},1-\mathrm{e}^{-2s})=A$, and thus, by (3) of Theorem 1, p is internally stabilizable. Let us compute a stabilizing controller c of p. Using the fact that we have $a-bp=1$, with $a=-b=(1-\mathrm{e}^{-2s})/2\in(A:J)$, we obtain that $c=b/a=-1$ is a stabilizing controller of p.

The fact that p is internally stabilizable implies that $J^{-1}=A:J=(1-\mathrm{e}^{-2s})$, and thus, $J=(J^{-1})^{-1}=(1/(1-\mathrm{e}^{-2s}))$ is a principal fractional ideal of A. By (4) of Theorem 1, p admits the coprime fatorization $p=(1+\mathrm{e}^{-2s})/(1-\mathrm{e}^{-2s})$, with $\frac{1}{2}(1+\mathrm{e}^{-2s})-\left(-\frac{1}{2}\right)(1-\mathrm{e}^{-2s})=1$. Finally, we have $1-pc=1+p=2/(1-\mathrm{e}^{-2s})$, and thus, $J=(1/(1-\mathrm{e}^{-2s}))=(1-pc)$. Therefore, by (5) of Theorem 1, we find that p is bistably stabilizable ($-1\in\mathrm{U}(A)$ is a bistable stabilizing controller).

Example 2. Let us consider $A=H_\infty(\mathbb{C}_+)$,

$$p=\mathrm{e}^{-s}/(s-1)\in K=Q(A)$$

and the fractional ideal $J=(1,p)$ of A. Then, we have $A:J=((s-1)/(s+1))$, because $p=n/d$, where $d=(s-1)/(s+1)\in A$ and $n=\mathrm{e}^{-s}/(s+1)\in A$, is a weakly coprime factorization of p (n and d have no common factor [13,14]). By (3) of Theorem 1, p is internally stabilizable iff there exist $a,b\in A:J=(d)$ such that $a-bp=1$, i.e. iff there exist $x,y\in A$

such that

$$a = ((s-1)/(s+1))x, \quad b = ((s-1)/(s+1))y,$$
$$a - bp = 1. \tag{5}$$

Then, we need to understand what are the constraints on x and y so that (5) is satisfied [2]. From (5), we obtain

$$b = \frac{(a-1)}{p} = \left(\frac{s-1}{s+1}\right)\left(\frac{(s-1)x-(s+1)}{e^{-s}}\right)$$
$$\Rightarrow y = \frac{(s-1)x-(s+1)}{e^{-s}} \Leftrightarrow x = \frac{(s+1)+e^{-s}y}{s-1}.$$

Thus, the numerator $(s+1)+e^{-s}y(s)$ of x must have a zero at $s=1$ if we want to have $x \in A$, i.e. we must have $y(1) = -2e$. Let us take $y = y(1) = -2e \in A$, then we obtain

$$x = ((s+1) - 2e^{-(s-1)})/(s-1)$$
$$= 1 + 2((1-e^{-(s-1)})/(s-1)) \in A.$$

Therefore, we have

$$\begin{cases} a = \left(\frac{s-1}{s+1}\right)\left(1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) \in A : J, \\ b = -2e\left(\frac{s-1}{s+1}\right) \in A : J, \\ a - bp = 1, \end{cases}$$
$$\Rightarrow c = \frac{b}{a} = \frac{y}{x} = \frac{-2e}{1+2\left(\frac{1-e^{-(s-1)}}{s-1}\right)}$$

is a stabilizing controller of p. Finally, we have $J = (1/d)$ because

$$1 = d/d, \quad p = n/d,$$
$$1 + 2\left(\frac{1-e^{-(s-1)}}{s-1}\right) + 2ep = 1/d.$$

By (4) of Theorem 1, p admits the coprime factorization $p = n/d$.

Definition 4. A fractional ideal I of A is *divisorial* if $I = A : (A : I)$.

By (2) and (3) of Lemma 2, every invertible ideal of $\mathscr{F}(A)$ is divisorial.

Corollary 1. *Let $p \in K = Q(A)$ be a transfer function. Then, we have the following equivalences:*

(1) *p admits a coprime factorization,*
(2) *p admits a weakly coprime factorization and the fractional ideal $J = (1, p)$ of A is divisorial.*

Proof. 1 ⇒ 2. By hypothesis, p admits a coprime factorization. Thus, by (4) of Theorem 1, there exists $0 \neq d \in A$ such that we have $J = (1/d)$. In particular, J is invertible and $J^{-1} = A : J = (d)$. Thus, by (2) of Theorem 1, p admits a weakly coprime factorization. Moreover, we have $J = A : (d) = A : (A : J)$, i.e. J is a divisorial ideal of A.

2 ⇒ 1. By hypotheses, p admits a weakly coprime factorization. Thus, by (2) of Theorem 1, there exists $0 \neq d \in A$ such that $A : J = (d)$. Now, using the fact that J is divisorial, we obtain that

$$J = A : (A : J) = A : (d) = (1/d),$$

and thus, p admits a coprime factorization by (4) of Theorem 1. □

Example 3. Let $A = W_+$ be the algebra of analytic functions on the unit disc $\mathbb{D}$ whose Taylor series converge absolutely, i.e.

$$W_+ = \left\{ f(z) = \sum_{n=0}^{+\infty} c_n z^n \mid \sum_{n=0}^{+\infty} |c_n| < +\infty \right\}.$$

$A = W_+$ is the integral domain of unit-pulse responses of BIBO-stable causal digital filters [20]. In [11], it is shown that

$$\begin{cases} n = (1-z)^3 e^{-((1+z)/(1-z))} \in A, \\ d = (1-z)^3 \in A, \end{cases}$$

and thus, $p = n/d = e^{-((1+z)/(1-z))} \in K = Q(A)$. Let us define the fractional ideal $J = (1, p)$ of A and $A : J = \{d \in A \mid dp \in A\}$. In [11], it is proved that $A : J$ is not a finitely generated ideal of A, and thus, by (2) of Theorem 1, p does not admit a weakly coprime factorization. Moreover, by Corollary 1, p does not admit a coprime factorization. Finally, p is not internally stabilizable because, otherwise, $A : J$ would have been generated by at most two elements of A.

4. A generalization of the Youla–Kučera parametrization

Points (3) and (4) of Theorem 1 show that the stabilizability of a plant is generally not equivalent to the existence of a coprime factorization: if a plant has a coprime factorization, then it is internally stabilizable but the converse fails to be generally true. The Youla–Kučera parametrization of all the stabilizing controllers of a plant is based on the assumption that this plant admits a coprime factorization. Hence, the question of the possibility to parametrize all the stabilizing controllers of a stabilizable plant which does not admit coprime factorization naturally arises. In this section, we shall give a general answer to this problem. First of all, we shall need the following lemma.

Lemma 3. *Let $p \in K = Q(A)$, $J = (1, p)$ be an invertible ideal of A and $J^{-1} = (a, b)$, where a and b satisfy (1). Then, we have*

(1) *J^2 is an invertible ideal of A, $J^2 = (1, p^2)$, there exist r_1 and $r_2 \in A$ such that $r_1 - r_2 p^2 = 1$, $r_1 p^2 \in A$ and $(J^2)^{-1} = (r_1, r_2)$.*
(2) *$(J^{-1})^2 = (a, b)^2 = (a^2, b^2)$.*
(3) *$(J^2)^{-1} = (J^{-1})^2$, and thus, if we note $J^{-2} = (J^2)^{-1} = (J^{-1})^2$, then we have $J^{-2} = (a^2, b^2) = (r_1, r_2)$, where $r_1 - r_2 p^2 = 1$, $r_1 p^2 \in A$.*

Proof. (1) If J is invertible, then we have $JJ^{-1} = A$, and thus,

$$(JJ^{-1})(JJ^{-1}) = J^2 (J^{-1})^2 = A,$$

i.e. J^2 is invertible with inverse $(J^2)^{-1} = (J^{-1})^2$. Now, let us prove that we have $J^2 = (1, p^2)$. We have $(1, p^2) \subseteq J^2 = (1, p, p^2)$. Now, using the fact that J is invertible, then there exist $a, b \in A$ such that $a - b\,p = 1$ and $a\,p \in A$. Hence, by multiplying the last equality by p, we obtain $p = (a\,p) - (b)\,p^2$. Using the fact that $a\,p \in A$ and $b \in A$, then we have $p \in (1, p^2)$, and thus, $J^2 = (1, p^2)$. Finally, using that $J^2 = (1, p^2)$ is an invertible fractional ideal of A, we obtain that there exist $r_1, r_2 \in A$ such that $(J^2)^{-1} = (r_1, r_2)$ (see the proof of (3) of Theorem 1).

(2) We have $J^{-1} = (a, b)$, and thus, $(a^2, b^2) \subseteq (J^{-1})^2 = (a^2, ab, b^2)$. However, we have $a - b\,p = 1$ and $a\,p \in A$, and thus, $ab = (b)a^2 - (a\,p)b^2 \in (a^2, b^2)$, because $b, a\,p \in A$. Therefore, we have $(J^{-1})^2 = (a^2, b^2)$.

(3) In (1), we have proved that $(J^2)^{-1} = (J^{-1})^2$, and thus, with the notation $J^{-2} = (J^2)^{-1} = (J^{-1})^2$, from (1) and (2), we obtain $J^{-2} = (r_1, r_2) = (a^2, b^2)$.

Theorem 2. *Let A be an integral domain of SISO stable plants, $K = Q(A)$ its quotient field, $p \in K$ a plant and $J = (1, p) \triangleq A + A\,p$ the fractional ideal of A defined by 1 and p. If p is internally stabilizable, then all the stabilizing controllers of p are parametrized by*

$$c(q_1, q_2) = \frac{b + q_1 a^2 + q_2 b^2}{a + q_1\, p a^2 + q_2\, p b^2}, \tag{6}$$

where q_1 and q_2 are any element of A such that $a + q_1\, p a^2 + q_2\, p b^2 \neq 0$ and $c = b/a$ $(0 \neq a,\ b \in A)$ is a stabilizing controller of p and

$$\begin{cases} a - b\,p = 1, \\ a\,p \in A. \end{cases}$$

The parametrization of all the stabilizing controllers of p has also the form

$$c(q_1, q_2) = \frac{b + q_1 r_1 + q_2 r_2}{a + q_1\, p r_1 + q_2\, p r_2}, \tag{7}$$

where $J^{-2} = (r_1, r_2)$ (see Lemma 3), and q_1 and q_2 are any element of A such that $a + p q_1 r_1 + p q_2 r_2 \neq 0$.

Proof. Let $c_1, c_2 \in K = Q(A)$ be two stabilizing controllers of p. Then, by (3) of Theorem 1, we have $c_i = b_i / a_i$ $(a_i = 1/(1 - p c_i), b_i = c_i/(1 - p c_i) \in A)$, for $i = 1, 2$, where

$$\begin{cases} 0 \neq a_i, b_i \in A, \\ a_i - b_i\, p = 1, \\ a_i\, p \in A, \end{cases} \Rightarrow \begin{cases} (b_2 - b_1) \in A, \\ (b_2 - b_1)\, p = (a_2 - a_1) \in A, \\ (b_2 - b_1)\, p^2 = (a_2 - a_1)\, p \in A, \end{cases}$$

and thus, we have

$$(b_2 - b_1) \in (A : (1, p, p^2)) = A : J^2 = J^{-2}.$$

By Lemma 3, we know that there exist $r_1, r_2 \in A$ (we can choose $r_1 = a^2$ and $r_2 = b^2$) such that $J^{-2} = (r_1, r_2)$. Thus, there exist $q_1, q_2 \in A$ such that $b_2 = b_1 + q_1 r_1 + q_2 r_2$ and $a_2 = a_1 + p(b_2 - b_1) = a_1 + p q_1 r_1 + p q_2 r_2$. If $a_1 + p q_1 r_1 + p q_2 r_2 \neq 0$, then we have

$$c_2 = \frac{b_2}{a_2} = \frac{b_1 + q_1 r_1 + q_2 r_2}{a_1 + q_1\, p r_1 + q_2\, p r_2}.$$

Now, let us show that any controller c_2 of the previous form is a stabilizing controller of p. We have

$$(a_1 + q_1 p r_1 + q_2 p r_2) - (b_1 + q_1 r_1 + q_2 r_2) p$$
$$= a_1 - b_1 p = 1,$$

$$(a_1 + q_1 p r_1 + q_2 p r_2) p$$
$$= a_1 p + (q_1 r_1 + q_2 r_2) p^2 \in A,$$

because $a_1 p \in A$ and $q_1 r_1 + q_2 r_2 \in J^{-2}$. Thus, by (3) of Theorem 1, c_2 is a stabilizing controller of p. Therefore, if $c_1 = b_1/a_1$ is a stabilizing controller of p, then all the stabilizing controllers of p have the form (7), with $q_1, q_2 \in A$ such that $a_1 + q_1 p r_1 + q_2 p r_2 \neq 0$, and, in particular, of form (6). □

Example 4. In [9], the ring of discrete finite-time delay system $A = \mathbb{R}[x^2, x^3]$ has been used in order to modelize some high-speed electronic circuits. Let us consider the plant $p = (1 - x^3)/(1 - x^2) \in K = Q(A)$ [9] and the fractional ideal $J = (1, p)$ of A. Using the identity

$$(1 - x^3)(1 + x^3) = (1 - x^2)(1 + x^2 + x^4),$$

we obtain that

$$A : J = \{d \in A \mid d\, p \in A\} = (1 - x^2, 1 + x^3).$$

$A : J$ is not a principal ideal of A ($A : J$ is the principal ideal $(1 + x)$ over $\mathbb{R}[x]$ but not over A). Then, by (2) of Theorem 1, p does not admit a weakly coprime factorization, and thus, by Corollary 1, p has no coprime factorization. Therefore, it is not possible to parametrize all the stabilizing controllers of p by means of the Youla–Kučera parametrization. But, we have

$$J(A : J) = (1 - x^2, 1 + x^3, 1 - x^3, 1 + x^2 + x^4),$$

which shows that

$$(1 - x^3)/2 + (1 + x^3)/2 = 1 \in J(A : J),$$

and thus, by (3) of Theorem 1, p is internally stabilizable, $J^{-1} = A : J$ and

$$\begin{cases} a = (1 + x^3)/2 \in J^{-1}, \\ b = (x^2 - 1)/2 \in J^{-1}, \end{cases}$$
$$\Rightarrow c = b/a = -(1 - x^2)/(1 + x^3)$$

is a stabilizing controller of p. Let us compute the parametrization of all the stabilizing controllers of p. From (2) of Lemma 3, the ideal

$$J^{-2} = (J^{-1})^2 = ((1 - x^2)^2, (1 + x^3)^2)$$

is not a principal ideal of A, and thus, all the stabilizing controllers of p are of form (6), namely

$$c(q_1, q_2) = \frac{-(1 - x^2) + (1 - x^2)^2 q_1 + (1 + x^3)^2 q_2}{(1 + x^3) + (1 - x^2)(1 - x^3) q_1 + (1 + x^3)(1 + x^2 + x^4) q_2},$$

where q_1 and q_2 are two free parameters of A such that the denominator of $c(q_1, q_2)$ does not vanish.

Lemma 4. *Let $p \in K = Q(A)$ and $J = (1, p)$. We have the following results*:

(1) *If p is internally stabilizable, then p^2 admits a coprime factorization iff J^2 is a principal fractional ideal, or equivalently, iff J^{-2} is a principal fractional ideal.*
(2) *If p admits a coprime factorization, so does p^2.*

Proof. (1) By (4) of Theorem 1, p^2 has a coprime factorization iff $(1, p^2)$ is a principal fractional ideal. Using the fact that p is internally stabilizable, i.e. by (3) of Theorem 1, J is invertible, and, from (1) of Lemma 3, we have $J^2 = (1, p^2)$. Hence, p^2 admits a coprime factorization iff J^2 is a principal fractional ideal of A. Finally, if $0 \neq k \in K$, then we have

$$J^2 = (J^{-2})^{-1} = (k) \quad \Leftrightarrow \quad J^{-2} = (k^{-1}),$$

i.e. J^2 is a principal fractional ideal iff so is J^{-2}.

(2) If p admits a coprime factorization, then, by (4) of Theorem 1, there exists $0 \neq k \in K$ such that $J = (1, p) = (k)$. Thus, $J^2 = (k^2)$ is also a principal ideal and, using the fact that $J^2 = (1, p^2)$ (see (1) of Lemma 3), then, by (4) of Theorem 1, we obtain that p^2 admits a coprime factorization. □

Corollary 2. *Let p be an internally stabilizable plant. Then, all the stabilizing controllers of p can be parametrized by means of a parametrization with only one free parameter iff p^2 admits a coprime factorization. Moreover, we have*

(1) *If p has no coprime factorization but p^2 does admit one ($p^2 = s/r$ is a coprime factorization*

of p^2), *then, all the stabilizing controllers of* p *are of the form*

$$c(q) = \frac{b+qr}{a+qpr}, \tag{8}$$

where q *is any element of* A *such that* $a+pqr \neq 0$ *and* $c = b/a$ *is a stabilizing controller of* p, *i.e.* $0 \neq a, b \in A$ *satisfy* (1).

(2) *If* p *admits a coprime factorization* $p = n/d$,

$$dx - ny = 1 \ (0 \neq d, n \in A, x, y \in A),$$

then, all the stabilizing controllers of p *are of the form*

$$c(q) = \frac{y+qd}{x+qn}, \tag{9}$$

where q *is any element of* A *such that* $x+qn \neq 0$. *We recover the Youla–Kučera parametrization of the stabilizing controllers* [8,20,21].

Proof. By Theorem 2, a stabilizable plant p has a parametrization of all its stabilizing controllers with only one free parameter iff J^{-2} is a principal ideal of A. By, (1) of Lemma 4, J^{-2} is principal iff p^2 admits a coprime factorization. Then, point (1) comes directly from Theorem 2 with $J^{-2} = (r)$, $r \in A$.

Let us prove (2). From (4) of Theorem 1, we have $J = (1, p) = (d^{-1})$, and thus, $J^2 = (d^{-2})$. Moreover, we have

$$(dx) - (dy)p = 1, \quad a = dx, \quad b = dy \in A,$$

$$(dx)p = nx \in A.$$

Using (1), we obtain that all the stabilizing controllers of p have the form

$$c(q) = \frac{dy+qd^2}{dx+qd^2p} = \frac{dy+qd^2}{dx+qdn} = \frac{y+qd}{x+qn},$$

$$\forall q \in A : x+qn \neq 0. \qquad \square$$

Example 5. Let us consider $A = \mathbb{Z}[i\sqrt{5}]$ and $p = (1+i\sqrt{5})/2 \in K = Q(A)$ defined in [1]. The ideal $J = (1, p)$ is such that $A : J = (2, 1-i\sqrt{5})$ is not a principal ideal [16] which implies, by (2) of Theorem 1, that p does not admit any weakly coprime factorization, and thus, by Corollary 1, p does not admit any coprime factorization. However, we have $(-2) - (-1+i\sqrt{5})p = 1$, $-2p \in A$, which shows, by (3) of Theorem 1, that c internally stabilizes p. Moreover, we check that $(2, 1+i\sqrt{5})^2 = (2)$, and thus, $J^2 = (4^{-1})(2, 1+i\sqrt{5})^2 = (2^{-1})$. By (1) of Corollary 2, all the stabilizing controllers of p are of the form

$$c(q) = \frac{-1+i\sqrt{5}+2q}{-2+2((1+i\sqrt{5})/2)q} = \frac{1-i\sqrt{5}-2q}{2-(1+i\sqrt{5})q}, \quad q \in A.$$

Let us notice that for some classes of linear infinite-dimensional linear systems (e.g. transfer functions which belong to the quotient field of $\mathscr{A}$ or $\hat{\mathscr{A}}$ [4,20]) or multidimensional systems ($A = \{n/d \mid 0 \neq d, n \in \mathbb{R}[z_1, \ldots, z_m],\ d(z) = 0 \Rightarrow z \in \mathbb{C}^m \setminus \overline{\mathbb{D}}^m\}$, where $\overline{\mathbb{D}}^m$ is the closed polydisc) [10], it is still not known whether or not any internally stabilizable plant admits a coprime factorization, and thus, whether or not we can parametrize the stabilizing controllers of a general plant by means of the Youla–Kučera parametrization, by (6) or by (8).

5. Isomorphism classes of $\mathscr{F}(A)$

Let us denote the multiplicative group of the non-zero principal fractional ideals of A by $\mathscr{P}(A)$. We can define an equivalence relation on $\mathscr{F}(A)$ as follows: I and J are *equivalent*, denoted by $I \sim J$, if there exists $0 \neq (k) \in \mathscr{P}(A)$ such that $I = (k)J$.

Lemma 5 (Fuchs and Salce [6]). *Let* I *and* J *be two fractional ideals of* A. *Then, we have*

$$I \sim J \quad \Leftrightarrow \quad I \cong J.$$

Proof. $\Rightarrow$ If $I \sim J$, then there exists $0 \neq k \in K = Q(A)$ such that $I = (k)J$. Let us define the following two maps:

$$\phi: \quad I \to J, \qquad \psi: \quad J \to I,$$
$$\qquad a \to a/k, \qquad\qquad b \to bk.$$

We easily check that these two maps are A-morphisms and $\psi \circ \phi = \mathrm{id}_I$ and $\phi \circ \psi = \mathrm{id}_J$, which proves that $I \cong J$.

$\Leftarrow$ If $I \cong J$, then there exists an isomorphism $\phi : I \to J$. Let us fix $0 \neq a \in I$. The element a has the form $a = n/d$ with $0 \neq d$, $n \in A$. Let $i = x/y$ be any element of I ($0 \neq y, x \in A$). Then, we have $a = (ny)/(dy)$

and $i=(dx)/(dy)$ with $0 \neq dy \in A$. Hence, we can always suppose that we have $a=r/s$ and $i=r'/s$. Using the fact that ϕ is an A-morphism, then, for all $i \in I$, we obtain

$$a\phi(i)=(r/s)\phi(r'/s)=(1/s)\phi(rr'/s)$$
$$=(r'/s)\phi(r/s)=i\phi(a) \tag{10}$$

and thus, we have $aJ=\phi(a)I \Rightarrow J=(\phi(a)/a)I$, i.e. $I \sim J$ $(0 \neq \phi(a)/a \in K)$. □

Definition 5. Let us define $\mathscr{S}(A)=\mathscr{F}(A)/\sim$. If we denote by $[I]$ (resp. $[J]$) the class of $I \in \mathscr{F}(A)$ (resp. $J \in \mathscr{F}(A)$) in $\mathscr{S}(A)$, then $\mathscr{S}(A)$ has a natural product defined by $[I][J]=[IJ]$ and $[A]$ is a unit. $\mathscr{S}(A)$ is called the *class semigroup* of A.

Using Lemma 3, we can also see $\mathscr{S}(A)$ as the *semigroup of the isomorphism classes* of the non-zero fractional ideals of A. Let us define by $\mathscr{I}(A)$ the subgroup of $\mathscr{F}(A)$ formed by the invertible ideals of A. The group $\mathscr{I}(A)$ is called the *group of Cartier divisors* (a *Cartier divisor* is an invertible ideal of A).

Proposition 1. *The structural properties of a system, defined by a transfer function* $p \in K=Q(A)$, *only depend on the class* $[(1,p)]$ *in* $\mathscr{S}(A)$ *of the fractional ideal* $(1,p) \triangleq A+Ap$ *of* A.

Proof. The structural properties of a plant, defined by a transfer function $p \in K$, must not depend on the choice of fractional representations of p. If

$$p=n_1/d_1=n_2/d_2, \quad 0 \neq d_i,\ n_i \in A,\ i=1,2, \tag{11}$$

then the structural properties of p must not depend on the choice of the integral ideals $I_1=(d_1,n_1)$ and $I_2=(d_2,n_2)$. Using the fact that A is an integral domain, from (11), we obtain $d_2n_1=d_1n_2$. Thus, we have

$$(d_1)I_2=(d_1d_2,d_1n_2)=(d_1d_2,d_2n_1)$$
$$=(d_2)I_1 \Rightarrow I_2=(d_1/d_2)I_1 \Rightarrow [I_1]=[I_2].$$

Hence, the structural properties of p must only depend on $[I_1]=[I_2]$. Finally, using the fact that $(1,p)=(1/d_1)I_1$, we have $[(1,p)]=[I_1]=[I_2]$, which shows that the structural properties of p only depend on the class $[(1,p)]$. □

Proposition 2. (1) *p has a weakly coprime factorization iff* $[A:(1,p)]=[A]$.

(2) *p is internally stabilizable iff* $[(1,p)]$ *is invertible in* $\mathscr{S}(A)$. *Moreover, if p is internally stabilized by c, then we have* $[(1,p)]^{-1}=[(1,c)]$.

(3) *p admits a coprime factorization iff* $[(1,p)]=[A]$.

Proof. (1) By (2) of Theorem 1, p admits a weakly coprime factorization iff the fractional ideal $J=(1,p)$ of A satisfies that $A:J$ is a principal ideal of A, i.e. iff there exists $0 \neq d \in A$ such that $A:J=(d)$, and thus, iff $[A:J]=[A]$.

(2) If p is internally stabilizable, then $J=(1,p)$ is invertible by (3) of Theorem 1, and thus, $JJ^{-1}=A \Rightarrow [JJ^{-1}]=[A] \Rightarrow [J][J^{-1}]=[A]$. Conversely, if $[J]$ is invertible in $\mathscr{S}(A)$, then there exists $[I]$ of $\mathscr{I}(A)$ such that $[J][I]=A$, and thus, $[JI]=[A]$, i.e. there exists $0 \neq k \in K$ such that $JI=(k)$. Hence, $J((k^{-1})I)=A$, i.e. J is invertible, i.e. p is internally stabilizable by (3) of Theorem 1. Finally, by (3) of Theorem 1, we know that c is a stabilizing controller of p, iff we have

$$(1,p)(1,c)=(1-pc)$$
$$\Rightarrow [(1,p)][(1,c)]=[(1-pc)]=[A]$$
$$\Rightarrow [(1,p)]^{-1}=[(1,c)].$$

(3) By (4) of Theorem 1, p admits a coprime factorization iff $J=(1,p)$ is a principal fractional ideal of A, i.e. iff there exists $0 \neq k \in K$ such that $J=(k)$, and thus, iff $[J]=[A]$. □

Proposition 3. *Let* $p_1,p_2 \in K=Q(A)$ *and* $J_1=(1,p_1)$ *and* $J_2=(1,p_2)$ *be the fractional ideals of A defined by* 1 *and* p_i. *Then, we have the equivalences*:

(1) $J_1 \cong J_2$, *i.e.* $[J_1]=[J_2]$,
(2) *there exist* $\alpha=(\alpha_{ij}) \in M_2(A)$ *and* $\beta=(\beta_{ij}) \in M_2(A)$ *such that*

$$\begin{cases} (\alpha_{11}+\alpha_{12}p_2)(\beta_{11}+\beta_{12}p_1)=1, \\ p_1=\dfrac{\alpha_{21}+\alpha_{22}p_2}{\alpha_{11}+\alpha_{12}p_2}, \\ p_2=\dfrac{\beta_{21}+\beta_{22}p_1}{\beta_{11}+\beta_{12}p_1}. \end{cases} \tag{12}$$

If p_1 *and* p_2 *satisfy* (12), *then* p_1 *and* p_2 *have the same structural properties* (*e.g. existence of a* (*weakly*) *coprime factorization, internal stabilizability*).

Proof. We have the following equivalences:

$$J_1 \cong J_2$$
$$\Leftrightarrow \exists 0 \neq k \in K,\ J_1 = (k)J_2,\ \text{i.e.}\ (1, p_1) = (k, k p_2),$$
$$\Leftrightarrow \exists 0 \neq k \in K,\ \exists \alpha = (\alpha_{ij}),\ \beta = (\beta_{ij}) \in M_2(A): \tag{13}$$

$$\begin{cases} 1 = (\alpha_{11} + \alpha_{12} p_2)k, \\ k = \beta_{11} + \beta_{12} p_1, \\ p_1 = (\alpha_{21} + \alpha_{22} p_2)k, \\ k p_2 = \beta_{21} + \beta_{22} p_1. \end{cases} \tag{14}$$

From (14), we obtain (12). Conversely, if we note $k = \beta_{11} + \beta_{12} p_1 \in K$, then, from the first equation of (12), we obtain $0 \neq k = 1/(\alpha_{11} + \alpha_{12} p_2)$ and, by substitution in the second and third equations of (12), we obtain (14).

Finally, if p_1 and p_2 satisfy (12), then, by the previous equivalence, we obtain that $[(1, p_1)] = [(1, p_2)]$, and thus, by Proposition 2, p_1 and p_2 have the same structural properties. □

Example 6. For instance, the transformations, defined by

$$T_1 : p_1 \to p_2 = p_1 + a \quad (a \in A),$$

$$T_2 : p_1 \to p_2 = 1/p_1,$$

$$T_3 : p_1 \to p_2 = p_1/(1 + a p_1) \quad (a \in A),$$

$$T_4 : p_1 \to p_2 = u p_1 \quad (u \in \mathrm{U}(A)),$$

keep the structural properties of the systems (e.g. existence of a (weak) coprime factorization, internal stabilizability) because p_1 and p_2 satisfy (12).

6. The fractional representation approach to systems

Lemma 6. *Let $p = n/d$ $(0 \neq d,\ n \in A)$ be any fractional representation of $p \in K = Q(A)$. If we denote $J = (1, p)$, then we have*

(1) $[(1, p)] = [(d, n)]$.
(2) $(d) \cap (n) = (n)(A : J)$.

Proof. (1) We have $J = (1, p) = (1/d)(d, n)$, and thus, $[(1, p)] = [(d, n)]$.

(2) This point is a generalization of a result used in [11]. We have $A : J = \{d \in A \mid d p \in A\}$, and thus, an element $a \in (n)(A : J)$ is of the form $a = nb$, with $b \in A$ and $b p \in A$. Thus, $a \in (n)$ and $a = (b p)d \in (d)$, i.e. $a \in (d) \cap (n)$. Conversely, if $a \in (d) \cap (n)$, then there exist $u, v \in A$ such that $a = ud = vn$, and thus, $ud = (v p)d \Rightarrow u = v p$ because $d \neq 0$ and A is an integral domain. Thus, we have $v \in A : J$ which shows that $a = vn \in (n)(A : J)$. □

Theorem 3. *Let $p \in K = Q(A)$ and $p = n/d$ $(0 \neq d, n \in A)$ be any fractional representation of p. Let $I = (d, n)$ be the integral ideal of A defined by d and n. Then, we have*

(1) *p is stable iff $I = (d)$, or equivalently, iff $A : I = (d^{-1})$.*
(2) *p admits a weakly coprime factorization iff the ideal $(d) \cap (n)$ is principal.*
(3) *p is internally stabilizable iff I is an invertible integral ideal of A, namely we have $I(A : I) = A$, or equivalently, iff there exists $x, y \in K$ such that*

$$\begin{cases} dx - ny = 1, \\ dx, dy, nx \in A. \end{cases} \tag{15}$$

If $x \neq 0$, then $c = y/x$ internally stabilizes p and $I^{-1} = A : I = (x, y)$. Moreover, $c = s/r$ internally stabilizes $p = n/d$ iff we have

$$(d, n)(r, s) = (dr - ns). \tag{16}$$

(4) *p admits a coprime factorization iff $I = (d, n)$ is a principal integral ideal of A, i.e. there exists $0 \neq a \in A$ such that $I = (a)$.*
(5) *p is strongly stabilizable iff there exists $c \in A$ such that $I = (d - nc)$.*
(6) *p is bistably stabilizable iff there exists $c \in \mathrm{U}(A)$ such that $I = (d - nc)$.*

Proof. Let us denote by $J = (1, p)$ the fractional ideal defined by 1 and p.

(1) If p is stable, i.e. $p \in A$, then $I = (d, n) = (d, d p) = (d)$. Thus, we have $A : I = (d^{-1})$. Finally, if $A : I = \{k \in K \mid kd,\ kn \in A\} = (d^{-1})$, then we have $d^{-1} n = p \in A$, i.e. p is stable.

(2) Using (2) of Theorem 1, we know that p admits a weakly coprime factorization iff $A : J$ is a principal ideal of A. But, from the second point of Lemma 6, $A : J$ is principal iff $(d) \cap (n)$ is also a principal integral ideal of A.

(3) Using (3) of Theorem 1, p is internally stabilizable iff we have $J(A : J) = A$. But, we have $J = (1/d)I$, and thus, J is invertible iff so is I. Hence, p is internally stabilizable iff we have $I(A : I) = A$, that is to say, there exist $x, y \in A$ such that $dx, dy, nx, ny \in A$ and $dx - ny = 1$. Thus, we have $(d, n)(x, y) = A$, and thus, $I^{-1} = (x, y)$. Finally, if $x \neq 0$, then $c = y/x$ is a stabilizing controller of p because we have

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \frac{1}{(dx - ny)} \begin{pmatrix} dx & nx \\ dy & dx \end{pmatrix} = \begin{pmatrix} dx & nx \\ dy & dx \end{pmatrix} \in M_2(A).$$

Finally, from (3) of Theorem 1, $c = s/r$ internally stabilizes $p = n/d$ iff (2) is satisfied, and thus, iff we have $(1/(dr))(d, n)(r, s) = (1/(dr))(dr - ns)$, which is equivalent to (16).

(4) From (4) of Theorem 1, p has a coprime factorization iff the fractional ideal J of A is principal, i.e. there exists $0 \neq k \in K$ such that $J = (k)$. Thus, we have $J = (1/d)(d, n) = (k) \Rightarrow I = (d, n) = (kd)$, i.e. I is a principal integral ideal of A. Conversely, if there exists $0 \neq a \in A$ such that $I = (a)$, then there exists $d', n', x, y \in A$ such that

$$\begin{cases} d = d'a, \\ n = n'a, \\ a = dx + ny, \end{cases} \Rightarrow \begin{cases} p = n/d = n'/d', \\ d'x + n'y = 1, \end{cases}$$

i.e. p admits the coprime factorization $p = n'/d'$ with $d'x + n'y = 1$.

(5) (resp. 6) If p is strongly (resp. bistably) stabilizable then, by (5) (resp. 6) of Theorem 1, there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that $J = (1 - pc)$. Thus, we have $J = (1/d)(d, n) = (1/d)(d - nc) \Rightarrow (d)J = (d, n) = (d - nc)$. Conversely, if there exists $c \in A$ (resp. $c \in \mathrm{U}(A)$) such that we have $I = (d, n) = (d - nc)$, then $d \neq 0 \Rightarrow I = (d, n) \neq 0 \Rightarrow d - nc \neq 0$. Moreover, there exist $u, v \in A$ such that

$$\begin{cases} d = (d - nc)u, \\ n = (d - nc)v, \end{cases}$$

$$\Rightarrow p = n/d = v/u = (u - 1)/(uc) = v/(1 + vc).$$

Thus, we have $d - nc = (d - nc)(u - vc) \Rightarrow u - vc = 1$. Hence, we have

$$\begin{pmatrix} 1 & -p \\ -c & 1 \end{pmatrix}^{-1} = \begin{pmatrix} u & v \\ uc & u \end{pmatrix} \in M_2(A),$$

i.e. p is strongly (resp. bistably) stabilizable. □

In [13], we show how to use point (3) of Theorem 2 in order to recover the characterization of internal stabilizability given in [18]. Moreover, let us notice that if p is strongly stabilizable, then, in particular, p admits a coprime factorization $p = n'/d'$ and $I' = (d', n') = A$. Using the fact that (5) of Theorem 3 does not depend on the choice of the fractional representation of p, then there exists $c' \in A$ such that $I' = (d', n') = (d' - n'c') = A$, i.e. $d' - n'c' \in \mathrm{U}(A)$ [20].

Proposition 4. *If $p = n/d \in K = Q(A)$, $0 \neq d$, $n \in A$, is internally stabilizable, then all the stabilizing controllers of p have the form*

$$c(q_1, q_2) = \frac{y + q_1 d z_1 + q_2 d z_2}{x + q_1 n z_1 + q_2 n z_2}, \tag{17}$$

where q_1 and q_2 are two free parameters of A, $x, y \in A$ satisfy (15) and $I^{-2} = \{k \in K \mid kd^2,\ kn^2 \in A\} = (z_1, z_2)$. We can take $z_1 = x^2$ and $z_2 = y^2$.

Proof. Let $c_1, c_2 \in K$ be two stabilizing controllers of p and $c_i = s_i/r_i$, $0 \neq r_i, s_i \in A$, be any fractional representation of c_i. Then, by (3) of Theorem 3, c_i satisfies (16), and thus, $(d, n)(r_i/(dr_i - ns_i), s_i/(dr_i - ns_i)) = A$. If we note $I = (d, n)$ and

$$x_i = r_i/(dr_i - ns_i), \quad y_i = s_i/(dr_i - ns_i) \in K = Q(A),$$

then we have $I^{-1} = (x_i, y_i)$, $c_i = y_i/x_i$ and

$$\begin{cases} dx_i - ny_i = 1, \\ dx_i,\ dy_i,\ nx_i \in A. \end{cases} \quad \forall i = 1, 2,$$

$$\Rightarrow d(x_2 - x_1) = n(y_2 - y_1). \tag{18}$$

From (2) of Lemma 3, we obtain that $I^2 = (d^2, n^2, dn) = (d^2, n^2)$, and thus, there exist $z_1, z_2 \in K$ such that

$I^{-2} = \{k \in K \mid kd^2,\ kn^2 \in A\} = (z_1, z_2)$ (let us notice that we can choose $z_1 = x_1^2$ and $z_2 = y_1^2$ because we have $I^{-2} = (x_1^2, y_1^2)$). Using (18), we have

$$d^2((y_2 - y_1)/d) = d(y_2 - y_1) \in A,$$

$$n^2((y_2 - y_1)/d) = n(n(y_2 - y_1)/d)$$
$$= n(d(x_2 - x_1)/d) = n(x_2 - x_1) \in A,$$

which proves that $(y_2 - y_1)/d \in I^{-2}$, i.e. $y_2 - y_1 \in (d)I^{-2}$, and thus, there exist $q_1, q_2 \in A$ such that $y_2 = y_1 + q_1 d z_1 + q_2 d z_2$. Using (18), we obtain

$$x_2 = x_1 + p(y_2 - y_1) = x_1 + q_1 n z_1 + q_2 n z_2$$
$$\Rightarrow c_2 = \frac{y_1 + q_1 d z_1 + q_2 d z_2}{x_1 + q_1 n z_1 + q_2 n z_2}.$$

Finally, we have

$$d(x_1 + q_1 n z_1 + q_2 n z_2)$$
$$-n(y_1 + q_1 d z_1 + q_2 d z_2) = d x_1 - n y_1 = 1,$$

$$d(x_1 + q_1 n z_1 + q_2 n z_2)$$
$$= d x_1 + dn(q_1 z_1 + q_2 z_2) \in A,$$

$$d(y_1 + q_1 d z_1 + q_2 d z_2)$$
$$= d y_1 + d^2(q_1 z_1 + q_2 z_2) \in A,$$

$$n(x_1 + q_1 n z_1 + q_2 n z_2)$$
$$= n x_1 + n^2(q_1 z_1 + q_2 z_2) \in A,$$

because $dn, d^2 n^2 \in I^2$ and $(q_1 z_1 + q_2 z_2) \in I^{-2}$. Therefore, all the stabilizing controllers of p have the form of (17) with $x + q_1 n z_1 + q_2 n z_2 \neq 0$. □

7. Integral domains, Picard group and Class group

Corollary 3. *If A is a* greatest common divisor domain (*gcdd*) [3,17], *namely a domain such that two elements $a, b \in A$ have a greatest common divisor $[a, b]$, then*

(1) *Every $p \in K = Q(A)$ admits a weakly coprime factorization.*
(2) *p admits a coprime factorization iff $J = (1, p)$ is a divisorial ideal of A.*

Proof. (1) Let $p = n/d$ $(0 \neq d, n \in A)$ be a fractional representation of p. Then, we have $p = n/d = (n/[d,n])/(d/[d,n])$, and thus

$$(d/[d,n])\,p = n/[d,n] \in A \;\Rightarrow\; (d/[d,n]) \subseteq A : J.$$

If $d' \in A : J = \{d \in A \mid d\,p \in A\}$, then we have

$$p = n'/d' = (n/[d,n])/(d/[d,n]),$$

and thus, $d'(n/[d,n]) = n'(d/[d,n])$. Using the fact that A is a gcdd and $n/[d,n]$ does not divide $d/[d,n]$ and conversely, then there exists $a \in A$ such that $d' = (d/[d,n])a$, and thus, $d' \in A : J \subseteq (d/[d,n])$, which proves that $A : J = (d/[d,n])$ is an integral principal ideal of A, and thus, by (2) of Theorem 1, p admits the weakly coprime factorization $p = (n/[d,n])/(d/[d,n])$.

(2) This result directly follows from the first point and Corollary 1. □

The following corollary easily follows from (3) of Theorem 1 and (4) of Theorem 2.

Corollary 4.

- *Every plant $p \in K = Q(A)$ is internally stabilizable iff A is a* Prüfer domain [3,17], *namely an integral domain such that, for every $p \in K$, the fractional ideal $J = (1, p)$ is invertible* [14].
- *Every transfer function $p \in K = Q(A)$ admits a coprime factorization iff A is a* Bézout domain [17], *namely an integral domain such that every finitely generated integral ideal of A is generated by one element of A* [20].

Using the fact that $\mathscr{P}(A)$ is a subgroup of $\mathscr{I}(A)$, then $\mathscr{C}(A) = \mathscr{I}(A)/\mathscr{P}(A)$ is the *group of the isomorphy classes of (finitely generated) invertible ideals of A*. $\mathscr{C}(A)$ is sometimes called the *Picard group* of A. If A is a Prüfer domain (see Corollary 4) [3,6,16], then $C(A)$ is called the *class group* of A [6]. We have the following consequences of Corollary 2.

Corollary 5. (1) *If $C(A) \cong \mathbb{Z}/2\mathbb{Z}$, then all stabilizing controllers of a stabilizing plant $p \in K = Q(A)$ are parametrized by* (8), *i.e. by means of a parametrization with only one free parameter.*

(2) *If $C(A) \cong 1$, then every internally stabilizable plant admits a coprime factorization, and thus, all its stabilizing controllers are parametrized by the Youla–Kučera parametrization* (9). *This condition is satisfied if A is a* projective-free ring [14], *namely*

a ring such that every finitely generated projective A-module is free (*e.g. Bézout domains*, $H_\infty(\mathbb{C}_+)$ [13,14]).

Proof. (1) $C(A) \cong \mathbb{Z}/2\mathbb{Z}$ means that for every (finitely generated) invertible fractional ideal J of A, J^2 is a principal fractional ideal of A. In particular, if we take $J=(1,p)$, then J^2 is a principal fractional ideal of A, and thus, by Lemma 3, we obtain that $J^2=(1,p^2)$ is principal, i.e. p^2 admits a coprime factorization. The result follows directly from (1) of Corollary 2.

(2) $C(A) \cong 1$ means that every (finitely generated) invertible fractional ideal of A is principal. Hence, using (2) of Corollary 2, we obtain that every stabilizable plant has a Youla–Kučera parametrization (9). □

Example 7. It is known that $A=\mathbb{Z}[\mathrm{i}\sqrt{5}]$ is a Prüfer domain with a class group $C(A) \cong \mathbb{Z}/2\mathbb{Z}$ [16]. Therefore, every plant—defined by $p \in K = Q(A)$—is internally stabilizable but some plants fail to admit coprime factorizations (for instance, p defined in Example 5). However, using Corollary 5, we know that we can parametrize all their stabilizing controllers by means of (8).

8. Conclusion

We hope that we have convinced the reader that the theory of fractional ideals is a natural mathematical framework for the fractional representation approach to analysis and synthesis problems. In this approach, the characterizations of some structural properties become simple and tractable. Moreover, we were able to generalize the Youla–Kučera parametrization of the stabilizing controllers to any stabilizing plant which does not admit any coprime factorization. In [15], we show how these results can be extended to multi-input multi-output (MIMO) systems using the concept of *lattice* [3]. Finally, a duality between the fractional ideal approach and the operator-theoretic one [7] is developed in [12]. This duality allows us to give behavioral interpretations to stabilization problems.

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