LINEAR CONTROL SYSTEMS OVER ORE ALGEBRAS: EFFECTIVE ALGORITHMS FOR THE COMPUTATION OF PARAMETRIZATIONS

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Abstract: In this paper, we study linear control systems over Ore algebras. Within this mathematical framework, we can simultaneously deal with different classes of linear control systems such as time-varying ordinary differential systems, differential time-delay systems, partial differential equations, multidimensional discrete systems, etc. We give effective algorithms which check whether or not a linear system over some Ore algebra is controllable, parametrizable or flat.

Keywords: Linear systems over Ore algebras, differential time-delay systems, controllability, parametrization, flatness, Gröbner bases.

1. INTRODUCTION

Over the last thirty years, for practical and theoretical reasons, different new classes of linear control systems have been introduced such as differential time-delay systems, multidimensional systems, partial differential equations, hybrid systems... All these classes of systems are characterized by the fact that they are governed by new types of mathematical equations and need new techniques in order to analyze their structural properties and to synthesize new control laws. With this growth of new types of control systems, we are led to generalize some previously known results and techniques so that they can be used for more general classes of systems. Hence, we get similar concepts, techniques and algorithms for studying different classes of systems.

In this paper, we study linear control systems over *Ore algebras*. An Ore algebra is an algebra of non-commutative polynomials in functional operators which satisfy certain commutation rules. For instance, differential/time-delay/discrete shift operators are examples of elements of some Ore algebras. Within this mathematical framework, we can simultaneously deal with different classes of linear control systems such as time-varying ordinary differential systems (ODEs), differential time-delay systems (TDSs), partial differential equations (PDEs), multidimensional discrete systems... Moreover, the recent extension of *Gröbner bases* to some non-commutative polynomial rings allows us to work effectively in some Ore algebras (Chyzak and Salvy, 1998).

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The purpose of this paper is to give effective algorithms which check whether or not a linear control system over some Ore algebras is controllable, parametrizable or flat. These problems have been largely studied in (Fliess and Mounier, 1998) for linear differential time-delay systems and, in (Pommaret and Quadrat, 1999a; Pommaret and Quadrat, 1999b; Wood, 2000), for linear multidimensional systems. The main novelty of this paper is to present some algorithms which work for both classes of systems as well as for new ones. In particular, this approach allows us to effectively obtain some parametrizations of a controllable plant and the flat outputs of a flat system. Let us notice that such algorithms were missing for linear differential time-delay systems and they could play important roles for the study of motion planning. See (Fliess and Mounier, 1998) and the references therein for more details.

All the presented algorithms have been implemented in the package *Oremodules* of Maple based on the library *Mgfun* (Chyzak, 1998). For a lack of space, we were not able to present the Maple-worksheets in the final version of this paper. We refer the reader to (Chyzak *et al.*, 2003) for more results, algorithms and illustrating examples obtained using Oremodules.

2. ORE ALGEBRAS

2.1 Definitions and examples

Matrices over Ore algebras provide a unified framework for different classes of linear systems (e.g. ODEs, PDEs, TDSs, multidimensional systems).

Definition 1. (1) (McConnell and Robson, 2000) Let A be an integral domain (i.e. $a\,b = 0,\ a \neq 0 \Rightarrow b = 0$). The *skew polynomial ring* $A[\partial;\sigma,\delta]$ is the non-commutative ring consisting of all polynomials in ∂ with coefficients in A obeying the commutation rule

$$\partial\, a = \sigma(a)\,\partial + \delta(a), \quad a \in A, \qquad (1)$$

where $\sigma : A \to A$ is a *k-algebra endomorphism of* A, namely

$$\begin{cases} \sigma(1) = 1, \\ \sigma(a+b) = \sigma(a) + \sigma(b), \quad a, b \in A, \\ \sigma(a\,b) = \sigma(a)\,\sigma(b), \quad a, b \in A, \end{cases}$$

and $\delta : A \to A$ *is a σ-derivation of* A, namely

$$\begin{cases} \delta(a+b) = \delta(a) + \delta(b), \quad a, b \in A, \\ \delta(a\,b) = \sigma(a)\,\delta(b) + \delta(a)\,b, \quad a, b \in A. \end{cases}$$

(2) (Chyzak and Salvy, 1998) Let $A = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k. The skew polynomial ring

$$D = A[\partial_1;\sigma_1,\delta_1]\ldots[\partial_m;\sigma_m,\delta_m]$$

is called *Ore algebra* if the σ_i's and δ_j's commute for $1 \leq i, j \leq m$ and satisfy:

$$\sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad j < i.$$

If $D = A[\partial;\sigma,\delta]$ is a skew polynomial ring, then every element P of D has a unique normal form $P = \sum_{i=1}^n a_i\,\partial^i$ for suitable $a_i \in A$ and $n \in \mathbb{N}$. For every Ore algebra, we get a similar normal form of its elements by moving $\partial_1, \ldots, \partial_m$ on the right in each summand.

Example 2. The Weyl algebra $A_1 = k[t][\partial;\sigma,\delta]$, where $\sigma = \mathrm{id}_{k[t]}$, $\delta = \frac{d}{dt}$, is a skew polynomial ring. We interpret (1) as a rule of differentiation:

$$\partial\, a = a\,\partial + \frac{da}{dt}, \quad a \in k[t].$$

Similar to polynomial rings in $2\,n$ indeterminates, we can define the Weyl algebra

$$A_n = k[x_1, \ldots, x_n][\partial_1;\sigma_1,\delta_1]\ldots[\partial_n;\sigma_n,\delta_n],$$

where σ_i and δ_i on $k[x_1, \ldots, x_n]$ are the maps

$$\sigma_i = \mathrm{id}_{k[x_1,\ldots,x_n]}, \quad \delta_i = \frac{\partial}{\partial\, x_i}, \quad i = 1, \ldots, n,$$

and every other commutation rule is prescribed by Def. 1. In particular, we have:

$$\partial_i\, x_j = x_j\,\partial_i + \delta_{ij}, \quad 1 \leq i, j \leq n,$$

where $\delta_{ij} = 1$ if $i = j$ and 0 else.

Example 3. The algebra of shift operators with polynomial coefficients $S_h = k[t][\delta_h;\sigma_h,\delta]$, defined by $\sigma_h(a)(t) = a(t-h)$, $\delta(a) = 0$, $a \in k[t]$ and $h \in \mathbb{R}$, is a skew polynomial ring. Hence, the commutation rule $\delta_h\, t = (t-h)\,\delta_h$ actually represents the action of the shift operator on polynomials. ∂_h is a *time-delay operator* if $h > 0$ and an *advance operator* if $h < 0$.

Example 4. In order to treat differential time-delay systems, we mix the constructions of the two preceding examples. We define the Ore algebra

$$D_h = k[t][\partial;\sigma_1,\delta_1][\delta_h;\sigma_2,\delta_2],$$

$$\sigma_1 = \mathrm{id}_{k[t]},\ \delta_1 = \frac{d}{dt},\ \sigma_2(a)(t) = a(t-h),\ a \in k[t],$$

with $\delta_2 = 0$, $h \in \mathbb{R}_+$. If the system also involves the advance operator, then we may work with

$$H_h = k[t][\partial;\sigma_1,\delta_1][\delta_h;\sigma_2,\delta_2][\tau_h;\sigma_3,\delta_3],$$

where σ_i, δ_i, $i = 1, 2$, are as above and:

$$\sigma_3(a)(t) = a(t+h), \quad \delta_3 = 0, \quad a \in k[t].$$

Ore algebras with other functional operators can also be defined (e.g. divided differences, q-shift, Eulerian operators). We refer to (Chyzak and Salvy, 1998; McConnell and Robson, 2000).

2.2 Properties & Gröbner bases

We summarize the most important properties of Ore algebras that will enable us to computationally deal with modules over Ore algebras.

Proposition 5. (Chyzak and Salvy, 1998) If A has the *left Ore property*, namely, for each pair $(a_1, a_2) \in A^2$, there is a pair $(0, 0) \neq (b_1, b_2) \in A^2$ such that $b_1\, a_1 = b_2\, a_2$, then so is $A[\partial; \sigma, \delta]$.

Proposition 6. (McConnell and Robson, 2000) If A is an integral domain and σ is injective, then the skew polynomial ring $A[\partial, \sigma, \delta]$ is an integral domain.

Proposition 7. (McConnell and Robson, 2000) If A is a left Noetherian ring and σ is an automorphism (e.g. A_n, S_h, D_h, H_h), then the skew polynomial ring $A[\partial; \sigma, \delta]$ is a left Noetherian ring.

In order to study effectively systems over (non-commutative) polynomial rings, we need to introduce some algorithmic methods based on Gröbner bases. We first need term orders in order to compare (non-commutative) polynomials.

Definition 8. Let D be an Ore algebra. A *term order* $<$ on D is an order on the set of monomials of $\mathrm{Mon}(D)$ which is compatible with the multiplication in D, i.e. $\forall\, m_1, m_2, n \in \mathrm{Mon}(D)$, we have:

$$m_1 < m_2 \quad \Rightarrow \quad n\, m_1 < n\, m_2.$$

The *leading monomial* $\mathrm{lm}(P)$ of $0 \neq P \in D$ is the largest (w.r.t. $<$) monomial in P with non-zero coefficient.

Definition 9. (Adams and Loustaunau, 1994) Let A be a polynomial ring and I be an ideal of A. A set of non-zero polynomials $G = \{g_1, \ldots, g_t\}$ is called a *Gröbner basis* for I if for all $0 \neq f \in I$, there exists $1 \leq i \leq t$ such that $\mathrm{lm}(g_i)$ divides $\mathrm{lm}(f)$.

A consequence of this definition is that every polynomial in I is *reduced* to 0 modulo G, i.e., by iterative division of the leading monomial of f by suitable $g_i \in G$, one obtains the zero polynomial.

For the case of commutative polynomial rings, Buchberger's algorithm ((Adams and Loustaunau, 1994), (Becker and Weispfenning, 1993)) computes Gröbner bases of polynomial ideals and modules. The next theorem states that this algorithm can be applied for certain Ore algebras. Every Ore algebra within our scope is of this kind.

Theorem 10. (Chyzak and Salvy, 1998; Kredel, 1993) Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring with coefficients in the field k and $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ an Ore algebra satisfying, $\forall\, i = 1 \ldots m, \quad \forall\, j = 1 \ldots n,$

$$\sigma_i(x_j) = a_{ij} x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij}, \qquad (2)$$

for certain $a_{ij} \in k \setminus \{0\}$ $b_{ij} \in k$ and $c_{ij} \in A$ is of total degree at most 1 in the x_i's. Then, a non-commutative version of Buchberger's algorithm terminates for every term order on $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_m$, and the result of this algorithm is a Gröbner basis w.r.t. the given term order.

An important technique that uses Gröbner bases is *elimination of variables*. By means of an *elimination order* $<$ (Adams and Loustaunau, 1994) one can force Buchberger's algorithm to give a Gröbner basis whose elements are preferably polynomials in the "small" (w.r.t. $<$) variables. Thus, the largest variables w.r.t. $<$ are eliminated (as far as possible). [2]

3. MODULE THEORY

Let us consider a system of equations

$$\sum_{j=1}^{p} R_{ij}\, y_j = 0, \quad 1 \leq i \leq q, \qquad (3)$$

where $R_{ij} \in D$, $p, q \in \mathbb{N}$. By collecting the coefficients R_{ij}, we obtain a matrix $R \in D^{q \times p}$ which, multiplied by $y = (y_1 : \ldots : y_p)^T$, yields system (3) again.

We set up the convention that D^r is always considered as the D-module of *row vectors* of length r ($r \in \mathbb{N}$). Let us consider the following left D-morphism (D-linear map):

$$\begin{aligned} D^q &\xrightarrow{.R} D^p, \\ (P_1 : \ldots : P_q) &\longmapsto (P_1 : \ldots : P_q)\, R. \end{aligned}$$

Then, $\mathrm{im}.R = D^q\, R$ is the left D-module generated by the left D-linear combinations of the rows of R.

Let us show that system (3) corresponds to the left D-module $M = D^p / D^q\, R$. Let $\{e_i\}_{1 \leq i \leq p}$ (resp. $\{f_j\}_{1 \leq j \leq q}$) be the canonical basis of D^p (resp. D^q). We denote by $\pi : D^p \rightarrow M = D^p / D^q\, R$ the left D-morphism which maps every element of D^p to its residue class in M. For $i = 1, \ldots, q$, we have

$$f_j\, R = (R_{j1} : \ldots : R_{jp}) = \sum_{i=1}^{p} R_{ji}\, e_i \in D^q\, R$$

$$\Rightarrow \pi(f_j\, R) = \pi\left(\sum_{i=1}^{p} R_{ji}\, e_i\right) = \sum_{i=1}^{p} R_{ji}\, \pi(e_i) = 0,$$

[2] In our implementation, we use the common order *lexdeg* of the Maple package *Groebner*.

and thus, if we denote by $y_i = \pi(e_i)$ the residue class of e_i in M, then M is defined by

$$\sum_{i=1}^{p} R_{ji}\, y_i = 0, \quad 1 \leq j \leq q, \quad \Leftrightarrow \quad R\, y = 0,$$

as well as by the left D-linear combinations of its equations. The left D-module M is *finitely generated* because every element $m \in M$ can be written as $m = \sum_{i=1}^{p} P_i\, y_i$, where $P_i \in D$.

Definition 11. The finitely generated left D-module $M = D^p / D^q\, R$ is *associated with (3).*

Example 12. Let us reconsider the Ore algebra $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ defined in Ex. 4 and the following wind tunnel model defined in (Manitius, 1984)

$$\left\{ \begin{array}{l} \dot{x}_1(t) = -a\, x_1(t) + k\, a\, x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\, x_2(t) - 2\, \zeta\, \omega\, x_3(t) + \omega^2\, u(t), \end{array} \right. \tag{4}$$

where a, k, ζ and ω are real constants. System (4) gives rise to the following matrix

$$R = \begin{pmatrix} \partial + a & -k\, a\, \delta_h & 0 & 0 \\ 0 & \partial & -1 & 0 \\ 0 & \omega^2 & \partial + 2\, \zeta\, \omega & -\omega^2 \end{pmatrix} \in D_h^{3 \times 4} \tag{5}$$

and thus, system (4) corresponds to the left D_h-module $M = D_h^4 / D_h^3\, R$.

Definition 13. (Rotman, 1979) A family $(M_i)_{i \in \mathbb{Z}}$ of D-modules together with a family $(d_i)_{i \in \mathbb{Z}}$ of D-module morphisms $d_i : M_i \longrightarrow M_{i-1}$ is a *complex*, if $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$. We write:

$$\ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots \tag{6}$$

Complex (6) is called *exact at position* i if the *defect of exactness of (6) at position* i,

$$H(M_i) = \ker d_i / \operatorname{im} d_{i+1},$$

is equal to 0 or, equivalently, if $\ker d_i = \operatorname{im} d_{i+1}$. Complex (6) is called *exact* if it is exact at every position. Finally, the exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0,$$

i.e. f is injective, g is surjective and $\ker g = \operatorname{im} f$, is called a *short exact sequence.*

We recall some properties of D-modules that will be important in the course of the paper.

Definition 14. (Rotman, 1979) Let D be a (left) Ore algebra and M a finitely generated (left) D-module.

(1) The D-module M is *free* if it is isomorphic to D^r for a certain $r \in \mathbb{Z}_{\geq 0}$.
(2) M is a *projective* D-module if there exist a free D-module F and a D-module N such that $F \cong M \oplus N$.
(3) The submodule of M

$$t(M) = \{ m \in M \mid \exists\, 0 \neq P \in D : \; P\, m = 0 \}$$

is called the *torsion submodule* of M. An element of $t(M)$ is a *torsion element* of M.
(4) M is called *torsion-free* if $t(M) = 0$.

Proposition 15. (Rotman, 1979) We have the following implications:

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{torsion-free}.$$

Theorem 16.
- (McConnell and Robson, 2000; Rotman, 1979) If D is a *Dedekind domain* (e.g. $D = A_1$), then a finitely generated torsion-free D-module is projective. If D is a *principal ideal domain* (e.g. the commutative polynomial ring $D = k[x]$ with coefficients in a field k), then a finitely generated torsion-free D-module is free.
- (Rotman, 1979) Every projective module over a commutative polynomial ring with coefficients in a field is free.

In the following sections, we shall develop effective algorithms which check whether or not a left D-module M is torsion-free, projective or free.

4. SYSTEM INTERPRETATIONS

Let us give some system interpretations of the properties of modules (Fliess and Mounier, 1998; Pommaret and Quadrat, 1999*a*; Pommaret and Quadrat, 1999*b*; Wood, 2000).

Definition 17.
- An *observable* of a linear system $R\, y = 0$ is a *scalar* D-linear combination of the components of y (i.e. inputs, states, outputs...). An observable $\phi(y)$ is called *autonomous* if it satisfies some equations of the form $P_1\, \phi(y) = 0, \ldots, P_r\, \phi(y) = 0$, where $P_i \in D$. An observable is said to be *free* if it is not autonomous.
- A linear system is said to be *controllable* if every observable is free.
- A linear system $R\, y = 0$ is *parametrizable* if there exist a matrix R_{-1} with entries in D and arbitrary functions z such that the compatibility conditions of the inhomogeneous system $y = R_{-1}\, z$ is exactly generated by $R\, y = 0$, i.e. if there exists $R_{-1} \in D^{p \times m}$ such that $M = D^p / D^q\, R \cong D^p\, R_{-1}$. Then, R_{-1} is called a *parametrization* of the system $R\, y = 0$ and z is the *potential* of the system.
- A linear system is *flat* (or *free*) if it is parametrizable and every component z_i of

the potential z is an observable of the system, i.e. if there exists a parametrization $R_{-1} \in D^{p\times m}$ which admits a left-inverse $S_{-1} \in D^{m\times p}$, namely $S_{-1}\,R_{-1} = I_m$. Then, z is called a *flat output*.

Proposition 18. Let D be an Ore algebra, $R \in D^{q\times p}$ and $M = D^p/D^q\,R$ be the left D-module associated with the system $R\,y = 0$ (see Def. 11).

(1) An observable of the system $R\,y = 0$ is an element of the left D-module M.
(2) The autonomous elements of the system are in one-to-one correspondence with the torsion elements of M.
(3) The system is controllable iff M is a torsion-free left D-module.
(4) The system is parametrizable iff M is a torsion-free left D-module.
(5) The system is flat iff M is a free left D-module. Then, a basis of M is a flat output.

Definition 19. Let $R \in D^{q\times p}$ be a *full row rank matrix* with entries in a commutative polynomial $D = \mathbb{R}[x_1, \ldots, x_n]$ (i.e. the q rows of R are D-linearly independent). Then,

- R is *minor left-prime* if the greatest common factor of all the q by q minors of R is 1.
- R is *zero left-prime* if all the q by q minors of R does not vanish simultaneously in $\mathbb{C}^n$.

Theorem 20. Let $R \in D^{q\times p}$ be a full row rank matrix with entries in ring $D = \mathbb{R}[x_1, \ldots, x_n]$.

(1) R is minor left-prime iff $M = D^p/D^q\,R$ is a torsion-free D-module.
(2) R is zero left-prime iff $M = D^p/D^q\,R$ is a free D-module.

Hence, the concepts of torsion-freeness and projectiveness generalize to non-commutative polynomial rings the well-known concepts of primenesses (Pommaret and Quadrat, 1999*a*).

5. SYZYGY MODULES

Let M be a finitely generated left module over a left noetherian ring D, i.e. there exists a surjective D-morphism $\varphi : D^p \to M$ which maps the ith canonical basis vector e_i of D^p to some y_i. We have the exact sequence $D^p \xrightarrow{\varphi} M \longrightarrow 0$. Then, φ may fail to be injective since there may be relations among the $\{y_i\}_{1\le i\le p}$:

$$\ker\varphi = \{\, P = (P_1 : \ldots : P_p) \in D^p \mid \qquad (7)$$
$$\phi(P) = \sum_{i=1}^{p} P_i\,\phi(e_i) = \sum_{i=1}^{p} P_i\,y_i = 0\}.$$

The D-linear relations among the $y_1, \ldots, y_p$ form the left D-module $S(M)$ defined by (7) and is called a *syzygy module* of M.

Since D is a left noetherian ring, $S(M)$ is a finitely generated left D-module. Thus, we can again find a suitable free D-module D^q and a map ψ sending the canonical basis vectors of D^q to the generators of $S(M)$. We have the exact sequence:

$$D^q \xrightarrow{\psi} D^p \xrightarrow{\varphi} M \longrightarrow 0.$$

This exact sequence is a *finite presentation* of the left D-module M and M is *finitely presented.* Finally, iterating the preceding construction, we get a *free resolution* of M (Rotman, 1979) .

Definition 21. (1) The exact sequence

$$\ldots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0 \quad (8)$$

is called a *free resolution* of M if the D-modules F_i are left free D-modules.
(2) If the D-modules F_i in (8) are projective, then (8) is a *projective resolution* of M.
(3) Let us consider a projective resolution of M:

$$0 \longrightarrow F_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0.$$

The *length* of this resolution is n.
(4) The minimal length of the left projective resolutions of the left D-module M is called the *projective dimension* $\mathrm{pd}_D(M)$ of M. The projective dimension may be infinite.
(5) The *left global dimension* of D is defined by:

$$\mathrm{lgld}\,D = \sup\{\mathrm{pd}_D(M) \mid M \text{ a left } D\text{-module}\}.$$

We describe the computational tools for the construction of free resolutions. The techniques to compute syzygy modules use Gröbner bases and elimination technique (see section 6.1 of (Becker and Weispfenning, 1993)). Let D be an Ore algebra which satisfies (2) and L a finitely generated left D-module which is a submodule of a free D-module D^p, $p \in \mathbb{N}$. Thus, a set of generators of L consists of row vectors in D^p.

Algorithm 1. INPUT: Generating set $\{R_1, \ldots, R_q\}$ of the D-module L, $R_i = (R_{i1} : \ldots : R_{ip}) \in D^p$.
OUTPUT: $S \in D^{r\times q}$ such that $D^r\,S$ is a generating set of the syzygy module $S(L)$, i.e. $S(L) = D^r\,S$.
SYZYGIES $(R_1, \ldots, R_q)$
  $P \leftarrow \{\sum_{j=1}^{p} R_{ij}\,\lambda_j - \mu_i \mid i = 1, \ldots, q\}$.
  $G \leftarrow$ Gröbner basis of P in $\oplus_{i=1}^{p} D\lambda_i \oplus_{i=1}^{q} D\mu_i$
    w.r.t. a term order that eliminates the λ_i's
  $S = (S_{ij}) \in D^{r\times q} \leftarrow G \cap \oplus_{i=1}^{q} D\,\mu_i =$
    $\{\sum_{j=1}^{q} S_{ij}\,\mu_j \mid i = 1, \ldots, r\}$.

Remark 22. Let us consider $R \in D^{q\times p}$ and the left D-module $M = D^p/D^q\,R$. Then, we can apply the preceding algorithm to the set formed by

$$R_i = (R_{i1} : \ldots : R_{ip}) \in L = D^q\, R \subseteq D^p,$$

$i = 1, \ldots, q$, in order to obtain $S = (S_{ij}) \in D^{r\times q}$ such that $S_2(M) = \ker .R = D^r\, S$ and we obtain the exact sequence:

$$D^r \xrightarrow{.S} D^q \xrightarrow{.R} D^p \xrightarrow{\pi} M \longrightarrow 0.$$

Iterating the process, we obtain a free resolution of the left D-module M.

Example 23. Let us reconsider Ex. 12 and define the D_h-module $L = D_h^4\, R^T$ generated by the rows of the matrix:

$$R^T = \begin{pmatrix} \partial + a & 0 & 0 \\ -k\,a\,\delta_h & \partial & \omega^2 \\ 0 & -1 & \partial + 2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{pmatrix} \in D_h^{4\times 3}.$$

The Gröbner basis of

$$\{(\partial + a)\,\lambda_1 - \mu_1, -k\,a\,\delta_h\,\lambda_1 + \partial\,\lambda_2 + \omega^2\,\lambda_3 - \mu_2, \\ -\lambda_2 + (\partial + 2\,\zeta\,\omega)\,\lambda_3 - \mu_3, -\omega^2\,\lambda_3 - \mu_4\}$$

w.r.t. the elimination ordering induced by the degree reverse lexicographical orderings on $\lambda_1 > \lambda_2$ and $\mu_1 > \mu_2 > \delta_h > \partial$ resp. is:

$$G = \{(\partial + a)\,\lambda_1 - \mu_1, \omega^2\,\lambda_2 + \partial\,\mu_4 + \omega^2\,\mu_3 + 2\,\zeta\,\omega\,\mu_4, \\ \omega^2\,k\,a\,\delta_h\,\lambda_1 + \omega^2\,\mu_2 + \omega^2\,\partial\,\mu_3 + (\partial^2 + 2\,\zeta\,\omega\,\partial + \omega^2)\,\mu_4, \\ \omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\partial + \omega^2\,a)\,\mu_2 + (\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\mu_3 \\ +(\partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 + \omega^2\,\partial + 2\,a\,\zeta\,\omega\,\partial + a\,\omega^2)\,\mu_4\}.$$

Intersecting G with $\oplus_{i=1}^3 D_h\,\mu_i$ we get

$$S = \{\omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\partial - \omega^2\,a)\,\mu_2 \\ +(\omega^2\,\partial^2 + \omega^2\,a\,\partial)\,\mu_3 + (\partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 \\ +\omega^2\,\partial + 2\,a\,\zeta\,\omega\,\partial + a\,\omega^2)\,\mu_4\}.$$

If we denote by R_{-1}^T the row vector

$$R_{-1}^T = (\omega^2\,k\,a\,\delta_h : \ \omega^2\,\partial + \omega^2\,a : \ \omega^2\,\partial^2 + \omega^2\,a\,\partial : \\ \partial^3 + 2\,\zeta\,\omega\,\partial^2 + a\,\partial^2 + \omega^2\,\partial + 2\,a\,\zeta\,\omega\,\partial + a\,\omega^2),$$

then we obtain the following free resolution of the D_h-module $N = D_h^3 / D_h^4\, R^T$:

$$0 \longrightarrow D_h \xrightarrow{.R_{-1}^T} D_h^4 \xrightarrow{.R^T} D_h^3 \xrightarrow{\pi} N \longrightarrow 0. \quad (9)$$

Proposition 24. (McConnell and Robson, 2000) Let A be an integral domain with $\operatorname{lgld} A < +\infty$ and σ an automorphism. The left global dimension of $A[\partial; \sigma, \delta]$ satisfies:

$$\operatorname{lgld} A \leq \operatorname{lgld} A[\partial; \sigma, \delta] \leq \operatorname{lgld} A + 1.$$

Moreover, if $\mathbb{Q} \subseteq k$ is a field, then we have $\operatorname{lgld} k[x_1, \ldots, x_n] = n$ and $\operatorname{lgld} A_n = n$.

6. INVOLUTIONS

Definition 25. Let k be a field and D a (noncommutative) k-algebra. An *involution* θ of D is a k-linear map $\theta : D \to D$ satisfying $\forall\, a_1, a_2 \in D$:

$$\begin{cases} \theta(a_1 \cdot a_2) = \theta(a_2) \cdot \theta(a_1), \\ \theta \circ \theta = \mathrm{id}_D. \end{cases} \quad (10)$$

Proposition 26. Let D be a k-algebra, M a right D-module and θ an involution of D, then we can define the left D-module $\widetilde{M}$, which is equal to M as a set and is endowed with the same addition as M, but with the following left action of D:

$$a\,m = m\,\theta(a), \quad m \in \widetilde{M}, \quad a \in D.$$

Example 27. (1) Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring. Then, $\theta = id_D$ is an involution of D.

(2) Let $A_n = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ be the Weyl algebra (see Ex. 2). An involution θ of A_n can be defined by:

$$x_i \mapsto x_i, \quad \partial_i \mapsto -\partial_i, \quad 1 \leq i \leq n.$$

(3) Let $S_h = k[t][\delta_h; \sigma_h, \delta]$ be as in Ex. 3. An involution θ of S_h can be defined by:

$$t \mapsto -t, \quad \delta_h \mapsto \delta_h.$$

(4) Let $H_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2][\tau_h; \sigma_3, \delta_3]$ be as in Ex. 4. An involution θ of H_h can be defined by:

$$t \mapsto t, \quad \partial \mapsto -\partial, \quad \delta_h \mapsto \tau_h, \quad \tau_h \mapsto \delta_h.$$

Definition 28. Let D be an Ore algebra with an involution θ, $R \in D^{q\times p}$ and $M = D^p / D^q\, R$ a left module. Then, the *transposed module* of M is the left D-module defined by:

$$N = D^q / D^p\, \theta(R). \quad (11)$$

The left D-module $N = D^q / D^p\, \theta(R)$ corresponds to the system $\theta(R)\, z = 0$, with $z = (z_1 : \ldots : z_q)^T$.

Example 29. (1) If D is a commutative ring (e.g. $D_h = \mathbb{R}(a, k, \zeta, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ defined in Ex. 12), then the involution θ is just the transposition of matrices, i.e. we have $\theta(R) = R^T$, and the transposed D-module is defined by $N = D^q / D^p\, R^T$.

(2) Let us consider the Ore algebra $H_h = k[t][\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2][\tau_h; \sigma_3, \delta_3]$ defined in Ex. 4 and $R = [t\,\partial : \ -t^2\,\delta_h] \in H_h^{1\times 2}$. Then, using 4 of Ex. 27, we obtain:

$$\theta(R) = \begin{pmatrix} -\partial\, t \\ -\tau_h\, t^2 \end{pmatrix} = \begin{pmatrix} -(t\,\partial + 1) \\ -(t + h)^2\,\tau_h \end{pmatrix}.$$

7. EXTENSION FUNCTOR

Definition 30. (Rotman, 1979) Let M be a finitely generated left D-module, S a left D-module and a free resolution $\ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$ of M. Then, the defects of exactness of the complex

$$\ldots \xleftarrow{d_2^*} \hom_D(F_1, S) \xleftarrow{d_1^*} \hom_D(F_0, S) \longleftarrow 0,$$

where, for $f \in \hom_D(F_{i-1}, S)$, $i \geq 1$, d_i^* is defined by $d_i^*(f) = f \circ d_i$, are given by:

$$\begin{cases} \mathrm{ext}_D^0(M,S) = \ker d_1^* = \hom_D(M,S), \\ \mathrm{ext}_D^i(M,S) = \ker d_{i+1}^*/\mathrm{im} d_i^*, \quad i \geq 1. \end{cases}$$

In the following, we shall only take $S = D$.

Proposition 31. (Rotman, 1979) The right D-module $\mathrm{ext}_D^i(M,D)$ only depends on M, i.e. one can choose any free resolution of M to compute $\mathrm{ext}_D^i(M,D)$, $i \in \mathbb{Z}_{\geq 0}$.

The next algorithm gives a description of a left D-module $\widetilde{\mathrm{ext}_D^1(M,D)}$, which corresponds to the right D-module $\mathrm{ext}_D^1(M,D)$ (see Prop. 26).

Algorithm 2. Input: Ore algebra D satisfying (2) with an involution θ and $R \in D^{q\times p}$.
Output: A list $L = [L_1,\, L_2]$ of matrices:
$L_1 \in D^{m\times q}$ is such that

$$\widetilde{\mathrm{ext}_D^1(M,D)} = D^m\, L_1/D^p\, \theta(R),$$

where $M = D^p/D^q\, R$,
$L_2 \in D^{q\times r}$ is such that $L_1 = \text{Syzygies}\,(L_2)$.

Pre-Ext1 (R)
  $R_2 \leftarrow \text{Syzygies}\,(R)$,
  $L_2 \leftarrow \theta(R_2)$,
  $L_1 \leftarrow \text{Syzygies}\,(L_2)$,
  $L \leftarrow [L_1,\, L_2]$.

Example 32. Let us compute the $\mathrm{ext}_{D_h}^i(N, D_h)$ of the D_h-module $N = D_h^3/D_h^4\, R^T$ defined in Ex. 23. In Ex. 23, we have already computed the free resolution (9) of N. Thus, we have $\ker .R^T = D_h\, R_{-1}^T$, where R_{-1}^T is defined in Ex. 23. Then, using the fact that $D_h = \mathbb{R}(a,\, k,\, \zeta,\, \omega)[\partial; \sigma_1, \delta_1][\delta_h; \sigma_2, \delta_2]$ is a commutative polynomial ring, we obtain that $\theta(R_{-1}^T) = R_{-1}$ (see 1 of Ex. 27). Hence, we have the complex of D_h-modules

$$0 \longleftarrow D_h \overset{.R_{-1}}{\longleftarrow} D_h^4 \overset{.R}{\longleftarrow} D_h^3 \longleftarrow 0$$

and its defects of exactness are defined by:

$$\mathrm{ext}_{D_h}^1(N, D_h) = \ker .R_{-1}/D_h^3\, R, \qquad (12)$$

$$\mathrm{ext}_{D_h}^2(N, D_h) = D_h/D_h^4\, R_{-1}. \qquad (13)$$

Following Alg. 2, we need to compute the syzygy of $D_h^4\, R_{-1}$. Doing similarly as in Ex. 23, we obtain that the syzygy module of $D_h^4\, R_{-1}$ is defined by

$$L = \begin{pmatrix} 0 & \omega^2 & \partial + 2\,\zeta\,\omega & -\omega^2 \\ 0 & \partial & -1 & 0 \\ -\partial - a & k\,a\,\delta_h & 0 & 0 \\ \partial^2 + a\,\partial & 0 & -k\,a\,\delta_h & 0 \end{pmatrix}, \qquad (14)$$

and thus, we obtain $\mathrm{ext}_{D_h}^1(N, D_h) = D_h^4\, L/D_h^3\, R$. Finally, using (13), $\mathrm{ext}_{D_h}^2(N, D_h)$ corresponds to the system $R_{-1}\, z = 0$. Let us notice that $R_{-1}\, z = 0 \Rightarrow z \neq 0$ (this can be easily checked by inspecting the Gröbner basis used to compute the syzygy of $D_h^4\, R_{-1}$), and thus, $\mathrm{ext}_{D_h}^2(N, D_h) \neq 0$.

$\widetilde{\mathrm{ext}_D^1(M,D)} = D^m\, L_1/D^p\, \theta(R)$ can be computed using elimination techniques similar to Alg. 1.

Algorithm 3. Input: A matrix $R \in D^{q\times p}$ and $L_1 = (L_1^T : \ldots : L_m^T)^T \in D^{m\times q}$ computed by Pre-Ext1(R).
Output: A set S of generating equations satisfied by the residue class z_i of L_i^T in the left D-module $D^m\, L_1/D^p\, \theta(R)$.

Quotient (L_1, R)
  Compute $\theta(R)$.
  for $i = 1, \ldots, m$, do
  $L \leftarrow \{\sum_{j=1}^q L_{ij}\, \lambda_j - \mu_i\} \cup \{\sum_{j=1}^q \theta(R)_{kj}\, \lambda_j \mid k = 1, \ldots, p\}$
  in $D[\lambda_1, \ldots, \lambda_r, \mu_i]$, compute the Gröbner basis G_i of L w.r.t. an elimination order (eliminating the λ_j's).
  endfor
  $S \leftarrow \bigcup_{i=1}^m (G_i \cap D[\mu_i])$.

Example 33. Let us reconsider the D_h-module $N = D_h^3/D_h^4\, R^T$ defined in Ex. 23. In Ex. 32, we proved that $\mathrm{ext}_{D_h}^1(N, D_h) = D_h^4\, L/D_h^3\, R$, where R (resp. L) is defined by (5) (resp. (14)). If we denote

$$\begin{cases} R = (R_1^T : R_2^T : R_3^T)^T, \\ L = (L_1^T : L_2^T : L_3^T : L_4^T)^T, \end{cases}$$

then we check that we have $G_i \cap D[\mu_i] = \{\mu_i\}$, for $i = 1 \ldots 4$, because $L_1 = R_3$, $L_2 = R_2$, $L_3 = -R_1$ and $L_4 = \partial\, R_1 + k\, a\, \delta_h\, R_2$. Thus, $D_h^4\, L = D_h^3\, R$, which shows that we have $\mathrm{ext}_{D_h}^1(N, D_h) = 0$.

The next theorem gives some effective algorithms checking the module properties, and thus, the structural properties of the corresponding linear control system (see Section 4). This theorem is an extension for Ore algebras of results obtained in (Pommaret and Quadrat, 1999*a*; Pommaret and Quadrat, 1999*b*).

Theorem 34. Let $M = D^p/D^q\, R$ be a left D-module and $N = D^q/D^p\, \theta(R)$ the transposed module of M. Then, we have:

(1) $t(M) \cong \mathrm{ext}_D^1(N, D)$.
(2) M is a torsion-free left D-module if and only if $\mathrm{ext}_D^1(N, D) = 0$.
(3) The system $R\, y = 0$ is parametrizable if and only if $t(M) \cong \mathrm{ext}_D^1(N, D) = 0$. Then, the matrix L_2 in Pre-Ext1(R) is a parametrization of the system $R\, y = 0$.
(4) M is a projective left D-module if and only if $\mathrm{ext}_D^i(N, D) = 0$ for $1 \leq i \leq \mathrm{lgld}\, D$.

(5) If R has a full row rank, namely $S(D^q\, R) = 0$, then M is a projective left D-module iff $N = \mathrm{ext}^1_D(\widetilde{M}, D) = 0 \Leftrightarrow \exists\, S \in D^{p\times q}:\ R\,S = I_q$.

Example 35. Let us check whether or not the differential time-delay system (4) is controllable, and thus, parametrizable. By 3 of Prop. 18, we know that (4) is controllable iff the D_h-module $M = D_h^4/D_h^2\, R$ is torsion-free, where R is defined by (5). By 2 of Thm. 34, this is equivalent to check $\mathrm{ext}^1_{D_h}(N, D_h) = 0$, where $N = D_h^3/D_h^4\, R^T$ (see 1 of Ex. 29). Therefore, system (4) is controllable and, using 3 of Thm. 34, we deduce that a parametrization of (4) is given by the matrix R_{-1} defined in Ex. 23, i.e. we have:

$$\begin{cases} (\omega^2\, k\, a\, \delta_h)\, z(t) = x_1(t), \\ (\omega^2\, \partial + \omega^2\, a)\, z(t) = x_2(t), \\ (\omega^2\, \partial^2 + \omega^2\, a\, \partial)\, z(t) = x_3(t), \\ (\partial^3 + (2\,\zeta\,\omega + a)\,\partial^2 + \\ (\omega^2 + 2\,a\,\zeta\,\omega)\,\partial + a\,\omega^2)\, z(t) = u(t). \end{cases} \tag{15}$$

Finally, the fact that $\mathrm{ext}^2_{D_h}(N, D_h) \neq 0$ implies that $M = D_h^4/D_h^3\, R$ is not a projective, and thus, not a free D_h-module (see 2 of Thm. 16). Hence, by 5 of Prop. 18, (4) is not a flat differential time-delay system and z is not a flat output.

If R is a full row rank matrix, then 6 of Thm. 34 gives an economic way to check projectiveness.

Algorithm 4. Input: Ore algebra D satisfying (2) and a matrix $R \in D^{q\times p}$.
Output: A matrix $S \in D^{p\times q}$ satisfying $SR = I_p$ if it exists and [ ] otherwise.

Left-Inverse (R)
  $P \leftarrow \{\sum_{j=1}^p R_{ij}\,\lambda_j - \mu_i \mid i = 1, \ldots, q\}$,
  $G \leftarrow$ Gröbner basis of P in $\oplus_{i=1}^p D\lambda_i \oplus_{i=1}^q D\mu_i$
    by eliminating the λ_i's
  $L, M \leftarrow$ matrices such that the rows in
    $L\,(\lambda_1 : \ldots : \lambda_p)^T$ and
    $M\,(\mu_1 : \ldots : \mu_q)^T$ are equations in G.
  If L is invertible and $L^{-1}\, M \in D^{p\times q}$,
    then return $S = L^{-1}\, M$, else return [ ].

We can also compute a right-inverse $S \in D^{q\times p}$ of $R \in D^{q\times p}$ $(R\,S = I_q)$ by doing:

$$\text{Right-Inverse}(R) = \theta(\text{Left-Inverse}(\theta(R))).$$

Therefore, if $R \in D^{q\times p}$ has a full row rank, by 6 of Thm. 34, the left D-module $M = D^p/D^q\, R$ is projective iff Right-Inverse$(R) \neq [\,]$.

Example 36. Let us reconsider system (4). Applying Alg. 4 to $\theta(R) = R^T$, where R is defined by (5), we are led to the Gröbner basis G defined in Ex. 23. We easily check that G does not contain any relation of the form $\lambda_i - \sum_{j=1}^4 S_{ij}\,\mu_j$, where $S_{ij} \in D_h$, for $i = 1, 2$. Therefore, $M = D_h^4/D_h^3\, R$ is not a projective D_h-module, and thus, (4) is not a flat system (see also Ex. 35).

8. CONCLUSION

We hope to have convinced the reader that the simultaneous use of module theory, homological algebra and effective algebra allows us to study effectively the structural properties of linear non-commutative multidimensional systems. Particularly, in this mathematical framework, we presented effective algorithms checking controllability or flatness and computing the parametrizations/autonomous elements/flat outputs... Certain of these problems were open for linear differential time-delay systems (Fliess and Mounier, 1998).

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