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Abstract: Vibration analysis aims at identifying potential failures of a rotating machinery from the monitoring of its vibration levels, i.e., by measuring the vibrations and comparing them to known failure vibration signals. For the diagnostic of gearboxes, new demodulation methods have recently been introduced in acoustic and signal processing. This new approach yields the problem of writing/factorizing a matrix M as $\sum_{i=1}^{r} D_i\, u\, v_i = (D_1\, u \;\; \ldots \;\; D_r\, u)\, \left(v_1^T \;\; \ldots \;\; v_r^T\right)^T$, where the D_i's are fixed matrices, u (resp., v_i) is a row (resp., column) vector to be determined and $i = 1, \ldots, r$. In this paper, using module theory and homological algebra, we study this rank factorization problem. More precisely, we characterize the general solutions of this family of polynomial systems. Finally, the results we develop are effective in the sense of computer algebra. Thus, they can be implemented in standard computer algebra systems handling polynomial systems and basic homological algebra methods (e.g., the `Singular` system, the `GAP` library `CapAndHomalg`, the `Maple` package OreModules).

Key-words: polynomial systems, rank factorization problem, centrohermitian matrix, module theory, homological algebra, demodulation problems, gearbox fault detection/surveillance, vibration analysis

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Sur les solutions générales d'un problème de factorisation relative au rang

Résumé : L'analyse vibratoire a pour but d'identifier de potentiels défauts d'une machine tournante grâce à la surveillance de ses niveaux de vibration, c'est-à-dire, grâce à la mesure de ses vibrations et à leur comparaison avec des signaux de défauts connus. Pour le diagnostic d'engrenages, de nouvelles méthodes de démodulation ont récemment été introduites en acoustique et en traitement du signal. Cette nouvelle approche a permis l'étude du problème consistant à écrire/factoriser une matrice M sous la forme de $\sum_{i=1}^{r} D_i\, u\, v_i = (D_1\, u \ \dots \ D_r\, u) \left(v_1^T \ \dots \ v_r^T\right)^T$, où les D_i sont des matrices fixées, u (resp., v_i) est un vecteur colonne (resp., un vecteur ligne) à déterminer et $i = 1, \dots, r$. Dans ce papier, en utilisant la théorie des modules et l'algèbre homologique, nous étudions ce problème de factorisation relative au rang. Plus précisément, nous caractérisons les solutions générales de cette famille de systèmes polynomiaux. Finalement, les résultats obtenus sont effectifs au sens du calcul formel. Ainsi, ils peuvent être implantés dans des systèmes standards de calcul formel permettant l'étude effective des systèmes polynomiaux et des méthodes élémentaires d'algèbre homologique (par exemple, le système `Singular`, la librairie `CapAndHomalg` de `GAP`, le package OreModules écrit en `Maple`).

Mots-clés : systèmes polynomiaux, factorisation relative au rang, matrice centrohermitiennes, théorie des modules, algèbre homologique, problèmes de démodulation, détection et surveillance des défauts d'engrenages, analyse vibratoire

1 Introduction

Before motivating and stating the main problem studied in this paper, let us first introduce a few standard notations.

Let $\mathbb{K}$ denote a field, $\mathcal{R}$ a commutative unital ring and $\mathcal{R}^{m\times n}$ the $\mathcal{R}$-module formed by all the $m \times n$ matrices with entries in $\mathcal{R}$. If $M \in \mathcal{R}^{m\times n}$, then we can consider the $\mathcal{R}$-homomorphisms $M. : \mathcal{R}^{n\times 1} \longrightarrow \mathcal{R}^{m\times 1}$ and $.M : \mathcal{R}^{1\times m} \longrightarrow \mathcal{R}^{1\times n}$ respectively defined by $(M.)(\eta) = M\,\eta$ for all $\eta \in \mathcal{R}^{n\times 1}$ and $(.M)(\lambda) = \lambda\,M$ for all $\lambda \in \mathcal{R}^{1\times m}$. Their kernels, images and cokernels $\mathcal{R}$-modules are respective denoted by $\ker_{\mathcal{R}}(M.)$, $\mathrm{im}_{\mathcal{R}}(M.)$, $\mathrm{coker}_{\mathcal{R}}(M.)$, and $\ker_{\mathcal{R}}(.M)$, $\mathrm{im}_{\mathcal{R}}(.M)$ and $\mathrm{coker}_{\mathcal{R}}(.M)$ [24]. A matrix M is said to have *full column rank* (resp., *full row rank*) if $\ker_{\mathcal{R}}(M.) = 0$ (resp., $\ker_{\mathcal{R}}(.M) = 0$). If $M \in \mathbb{K}^{m\times n}$, then the *rank* of M, i.e., $\dim_{\mathbb{K}}(\mathrm{im}_{\mathbb{K}}(M.))$, is denoted by $\mathrm{rank}_{\mathbb{K}}(M)$. Let I_n be the identity matrix, i.e., the $n \times n$ matrix with 1 on the first diagonal and 0 elsewhere, and J_n the exchange $n \times n$ matrix, i.e., the $n \times n$ matrix with 1 on the second diagonal and 0 elsewhere. The diagonal matrix with the elements of a list L on the first diagonal is denoted by $\mathrm{diag}(L)$. Finally, $M \in \mathbb{C}^{m\times n}$, then $\overline{M}$ (resp., $M^\star$) denotes the *conjugate matrix* (resp., the *adjoint matrix*, i.e., $\overline{M}^T \in \mathbb{C}^{n\times m}$).

Within the *frequency domain* (see, e.g., [23]), the *toothed gearbox vibration* [21] can be interpreted as a *modulation process* of a high-frequency carrier with a low-frequency modulation [3, 4, 13, 14]. For *gearbox fault surveillance*, one has to separate these two time-domain signals and compare them to known failure vibration signals. To solve this problem, *(amplitude, amplitude & phase) demodulation methods* [23] have naturally been introduced in [13, 14]. Within this approach, the toothed gearbox vibration is measured and Fourier coefficients of this periodic real-valued time signal are computed and stored into a so-called *centrohermitian matrix* $M \in \mathbb{C}^{m\times n}$ [12, 20], namely, a complex matrix which satisfies the identity $\overline{M} = J_m\,M\,J_n$. More precisely, if s is the T-periodic real-valued signal of the toothed gearbox vibration, then s can be expressed by its *Fourier series* $s(t) = \sum_{j\in\mathbb{Z}} c_j(s)\,e^{\frac{2\,\pi\,i\,j\,t}{T}}$, where the *Fourier coefficients* of s, defined by $c_j(s) = \frac{1}{T}\int_0^T s(t)\,e^{\frac{-2\,\pi\,i\,j\,t}{T}}dt$, $j \in \mathbb{Z}$, satisfy $\overline{c_j(s)} = c_{-j}(\overline{s}) = c_{-j}(s)$ for $j \in \mathbb{Z}$. The vectors $C_l = (c_{-l}(s)\ \ldots\ c_0(s)\ \ldots\ c_l(s))^T \in \mathbb{C}^{(2\,l+1)\times 1}$, $l \geq 0$, and the matrix

$$M = \begin{pmatrix} c_{q\,(2\,p+1)+p} & \cdots & c_p & \cdots & c_{-q\,(2\,p+1)+p} \\ \vdots & & \vdots & & \vdots \\ c_{q\,(2\,p+1)} & \cdots & c_0 & \cdots & c_{-q\,(2\,p+1)} \\ \vdots & & \vdots & & \vdots \\ c_{q\,(2\,p+1)-p} & \cdots & c_{-p} & \cdots & c_{-q\,(2\,p+1)-p} \end{pmatrix} \in \mathbb{C}^{(2p+1)\times(2q+1)}, \quad p,\, q \geq 0,$$

satisfy the relations $\overline{C_l} = J_{2\,l+1}\,C_l\,J_1$ (note that $J_1 = 1$) and $\overline{M} = J_{2\,p+1}\,M\,J_{2\,q+1}$, which shows that C_l and M are centrohermitian matrices. Given $r+1$ fixed centrohermitian matrices $D_1, \ldots, D_r \in \mathbb{C}^{(2\,p+1)\times(2\,p+1)}$ and $M \in \mathbb{C}^{(2p+1)\times(2q+1)}$, the corresponding demodulation problem then aims at determining if there exist a centrohermitian column vector $u \in \mathbb{C}^{(2\,p+1)\times 1}$ and r centrohermitian row vectors $v_1, \ldots, v_r \in \mathbb{C}^{1\times(2\,q+1)}$ satisfying the following equation:

$$M = \sum_{i=1}^{r} D_i\,u\,v_i. \tag{1}$$

For instance, the amplitude demodulation (resp., phase demodulation) problem corresponds to $r = 1$ and $D_1 = I_{2\,p+1}$ (resp., $r = 2$, $D_1 = I_{2\,p+1}$ and $D_2 = 2\,\pi\,i\,f_c\,\mathrm{diag}(-p, \ldots, 0, \ldots, p)$, where $f_c > 0$). For more details, see [13, 14].

Problem (1) can be generalized as follows. Given $M \in \mathbb{K}^{m\times n}$ and $D_1, \ldots, D_r \in \mathbb{K}^{m\times m}$, find a column vector $u \in \mathbb{K}^{m\times 1}$ and row vectors $v_i \in \mathbb{K}^{1\times n}$, $i = 1, \ldots, r$, satisfying (1). This problem will be called *rank factorization problem* since M can then be factorized as follows

$$M = (D_1\,u\ \ldots\ D_r\,u)\begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix},$$

where the rank of M must be less than or equal to r. The goal of this paper is to study the solutions of this rank factorization problem using algebraic and computer algebra methods.

A result due to Lee [20] shows that the set of centrohermitian matrices can be bijectively mapped onto the set of real matrices by a certain (*unitary*) transformation φ. Hence, the demodulation problem (1) can be transformed into a similar problem over $\mathbb{K} = \mathbb{R}$ for the transformed real matrices $\varphi(M), \varphi(D_1), \ldots, \varphi(D_r)$. The real solutions u_φ and $\{v_{i\varphi}\}_{i=1,\ldots,r}$, of the latter problem can then be transformed back to obtain the centrohermitian solutions $u = \varphi^{-1}(u_\varphi)$ and $\{v_i = \varphi^{-1}(v_{i,\varphi})\}_{i=1,\ldots,r}$ of (1). For more details, see [17, 18]. Hence, the demodulation problems can be reduced to solving the rank factorization problem for $\mathbb{K} = \mathbb{R}$.

For fixed matrices $M, D_1, \ldots, D_r$, (1) defines a system of $m\,n$ quadratic equations in $m+r\,n$ unknowns − the entries of the vectors u and v_i for $i = 1, \ldots, r$. Hence, different (effective) algebraic geometry methods can be used to study the rank factorization problem (1). For instance, see, e.g., [9, 11, 19, 25] and the references therein.

In this paper, we exploit the *bilinear structure* of (1) in u and $v = \left(v_1^T \ \ldots \ v_r^T\right)^T$ to effectively characterize the general solutions of (1). In particular, we find again a class of solutions − the ones with full row rank matrices v − characterized in [15, 16, 17, 18] using linear algebra methods. Our effective approach uses standard module theory, homological algebra and computer algebra methods [9, 11, 24]. The general solutions can be computed using standard computer algebra systems that contain both elimination theory for polynomial systems (e.g., *Gröbner* or *Janet basis techniques*) and basic homological methods such as `Singular` [11], the `GAP` library `CapAndHomalg` [1], or `Maple` package OreModules [5].

The paper is organized as follows. In this section, the notations, the context and the rank factorization problem are introduced. In Section 2, we first restrict the rank factorization problem to a particular class of solutions − those with full row rank v − and state again the results obtained in [15, 16, 17, 18] which only use linear algebra and module theory methods. In Section 3, using module theory and basic homological algebra, we explain how the results obtained in Section 2 can be extended to characterize the general solutions of the rank factorization problem. Explicit examples, computed with the `CapAndHomalg` [1], using the `Singular` computer algebra system [11], illustrate the main results of the paper. Finally, in Section 4, we end the paper by explaining problems that will be studied in the future.

2 Characterization of a particular set of solutions

2.1 A few remarks

In this section, we state preliminary remarks on the rank factorization problem (1). As stated in Section 1, the rank factorization problem (1) corresponds to a system of $m\,n$ quadratic equations in $m + r\,n$ unknowns, namely, the entries of the vectors $u \in \mathbb{K}^{m\times 1}$ and $v_i \in \mathbb{K}^{1\times n}$, $i = 1, \ldots, r$. Hence, this problem belongs to the realm of (effective) algebraic geometry (see, e.g., [9, 11, 19] and the references therein).

For $n = 1$, using $r \geq 1$, we note that $m + r > m$, which shows that (1) defines a system with more unknowns than equations. For $n \geq 2$, the sign of $m\,n - (m + r\,n)$ is the sign of the function $\Psi(m, n, r) = m - \left(1 + \frac{1}{n-1}\right) r$ and $m - 2\,r \leq \Psi(m, n, r) \leq m - r$. In particular, if $m \geq 2\,r$, then (1) defines a system with more equations than unknowns.

In what follows, we shall suppose that $M, D_1, \ldots, D_r$ are not 0 and we use the notations:

$$A(u) = (D_1\,u \ \ldots \ D_r\,u) \in \mathbb{K}^{m\times r}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \in \mathbb{K}^{r\times n}. \tag{2}$$

The rank factorization problem (1) can then be rewritten as follows:

$$A(u)\,v = M. \tag{3}$$

The bilinear structure in u and v is emphasized in (3). Under the form (3), (1) clearly corresponds to a factorization problem for the matrix M.

Remark 1. If $(u,\, v)$ is a solution of (3), then $(\lambda\, u,\, \lambda^{-1}\, v)$ is also a solution for all $\lambda \in \mathbb{K}^\times = \mathbb{K}\backslash\{0\}$. Hence, $\mathbb{K}^\times$ defines a *group action* on the solution space of (3) and the *orbit* $\mathcal{O}_{(u,\, v)} = \{(\lambda\, u,\, \lambda^{-1}\, v) \mid \lambda \in \mathbb{K}^\times\}$ can be considered instead of the solution $(u,\, v)$ only.

Remark 2. A natural approach is to transform (3) into a *multi-homogeneous polynomial system* over a *multi-projective space* [25]: Writing $v = (v_{\bullet 1} \;\ldots\; v_{\bullet n})$ (resp., $M = (M_{\bullet 1} \;\ldots\; M_{\bullet n})$), where $v_{\bullet i} \in \mathbb{K}^{r \times 1}$ (resp., $M_{\bullet i} \in \mathbb{K}^{m \times 1}$) denotes the i^{th} column of v (resp., M), (1) can be rewritten as $A(u)\, v_{i \bullet} = M_{\bullet i}$ for $i = 1, \ldots, n$. Introducing new variables u_0 and v_{0i} for $i = 1, \ldots, n$, and the matrix $M_h = u_0\, (v_{01}\, M_{\bullet 1} \;\ldots\; v_{0n}\, M_{\bullet n})$, the change of variables $u \longleftarrow u/u_0$ and $v_{\bullet i} \longleftarrow v_{\bullet i}/v_{0i}$, $i = 1, \ldots, n$, in the equations $A(u)\, v_{i \bullet} = M_{\bullet i}$ for $i = 1, \ldots, n$ then yields the following multi-homogeneous polynomial system

$$A(u)\, v = M_h \tag{4}$$

of degree $(1, 1)$ with respect to the partition $\{u_0, u_1, \ldots, u_m\} \cup \{v_{0j}, v_{1j}, \ldots, v_{rj}\}$ of the variables. Note $(u_0, u_1, \ldots, u_n)$ (resp., $(v_{0j}, v_{1j}, \ldots, v_{rj})$) is a point of the *projective space* $\mathbb{P}^n(\mathbb{K})$ (resp., $\mathbb{P}^r(\mathbb{K})$). Then, the solutions of (4) can be sought in the multi-projective space [25]:

$$\mathbb{P}^m(\mathbb{K}) \times \underbrace{\mathbb{P}^r(\mathbb{K}) \times \ldots \times \mathbb{P}^r(\mathbb{K})}_{n}.$$

In this paper, we shall not follow the approaches briefly described in Remarks 1 and 2. They will be studied elsewhere. In this paper, we follow an approach that we now explain.

2.2 Characterization of particular solutions

We briefly state again results obtained in [2, 15, 16, 17, 18] which characterize a particular class of solutions of (1). These results use linear algebra and module theory. In Section 3, using also homological algebra, this approach will be generalized to characterize the general solutions of (1).

First note that the existence of the vectors $u \in \mathbb{K}^{m \times 1}$ and $v_i \in \mathbb{K}^{1 \times n}$, $i = 1, \ldots, r$, satisfying (3) is equivalent to the existence of $u \in \mathbb{K}^{m \times 1}$ such that:

$$\operatorname{im}_{\mathbb{K}}(M.) \subseteq \operatorname{im}_{\mathbb{K}}(A(u).). \tag{5}$$

Indeed, (5) implies that the columns $M_{\bullet i}$'s of M belong to $\operatorname{im}_{\mathbb{K}}(A(u).)$, i.e., implies the existence of vectors $v_{\bullet i} \in \mathbb{K}^{r \times 1}$ satisfying $A(u)\, v_{\bullet i} = M_{\bullet i}$, $i = 1, \ldots, n$, which yields $A(u)\, v = M$, where $v = (v_{\bullet 1} \;\ldots\; v_{\bullet n}) \in \mathbb{K}^{r \times n}$. Conversely, (3) clearly yields (5). (5) then shows that a necessary condition on M for the solvability of the rank factorization problem (1) is:

$$\operatorname{rank}_{\mathbb{K}}(M) \leq \operatorname{rank}_{\mathbb{K}}(A(u)) \leq \min\{m, r\}. \tag{6}$$

In what follows, we shall note $l = \operatorname{rank}_{\mathbb{K}}(M)$. Hence, if l is not less than or equal to $\min\{m, r\}$, no solution of (3) exists. The name *rank factorization problem* comes from (3) and (6). Finally, note that $u \in \mathbb{K}^{m \times 1}$ has to be chosen so that not all the $l \times l$ minors of $A(u)$ vanish.

Let us now suppose that (3) has a solution. In what follows, we investigate when a solution $(u,\, v)$ exists with a full row rank matrix v (i.e., the rows of v are $\mathbb{K}$-linearly independent). Recall that the matrix v has full row rank if and only if it admits a right inverse $t \in \mathbb{K}^{n \times r}$, i.e., $v\, t = I_r$. Hence, if a solution $(u,\, v)$ of (3) exists with a full row rank matrix v, then (3) yields $A(u) = M\, t$, which shows that $\operatorname{im}_{\mathbb{K}}(A(u).) = \operatorname{im}_{\mathbb{K}}(M.)$, i.e., $u \in \mathbb{K}^{m \times 1}$ is such that:

1. $D_i\, u \in \operatorname{im}_{\mathbb{K}}(M.)$ for $i = 1, \ldots, r$.

2. $\operatorname{rank}_{\mathbb{K}}(A(u)) = \operatorname{rank}_{\mathbb{K}}(M)$, i.e., $\dim_{\mathbb{K}}(\operatorname{span}_{\mathbb{K}}\{D_i\, u\}_{i=1,\ldots,r}) = l$.

Let us study these two conditions and characterize the solutions of (3) satisfying them.

Let us first suppose that $\operatorname{im}_{\mathbb{K}}(M.) \neq \mathbb{K}^{m \times 1}$. Set $p = m - l > 0$. Let $L \in \mathbb{K}^{p \times m}$ be a full row rank matrix whose rows define a basis of $\ker_{\mathbb{K}}(.M)$, i.e., $\ker_{\mathbb{K}}(.M) = \operatorname{im}_{\mathbb{K}}(.L)$. Then, we get $L\, M = 0$, which shows that $\operatorname{im}_{\mathbb{K}}(M.) \subseteq \ker_{\mathbb{K}}(L.)$. Now, $\dim_{\mathbb{K}}(\ker_{\mathbb{K}}(L.)) = m - p = l = \operatorname{rank}_{\mathbb{K}}(M)$ yields $\ker_{\mathbb{K}}(L.) = \operatorname{im}_{\mathbb{K}}(M.)$.

Hence, the first above condition is equivalent to the system of linear equations $(L\,D_i)\,u = 0$, $i = 1, \ldots, r$, i.e., $u \in \ker_{\mathbb{K}}(N.)$, where N is given by:

$$N = \begin{pmatrix} L\,D_1 \\ \vdots \\ L\,D_r \end{pmatrix} \in \mathbb{K}^{p\,r \times m}. \tag{7}$$

Set $d = \dim_{\mathbb{K}}(\ker_{\mathbb{K}}(N.))$. Let $Z \in \mathbb{K}^{m \times d}$ be a full column rank (namely, the columns of Z are $\mathbb{K}$-linearly independent) whose columns define a basis of $\ker_{\mathbb{K}}(N.)$. In other words, we have $\ker_{\mathbb{K}}(N.) = \operatorname{im}_{\mathbb{K}}(Z.)$. Thus, the first above condition is equivalent to $u = Z\,\psi$ for all $\psi \in \mathbb{K}^{d \times 1}$. Substituting this expression into Condition 2 above, we are led to determining:

$$\mathcal{P} = \left\{\psi \in \mathbb{K}^{d \times 1} \mid \operatorname{rank}_{\mathbb{K}}(A(Z\,\psi)) = l\right\}.$$

If $d = 0$, then $Z = 0$ and $A(Z\,\psi) = A(0) = 0$, showing that $\operatorname{im}_{\mathbb{K}}(A(u).) = \operatorname{im}_{\mathbb{K}}(M.)$ is never satisfied, and thus, (3) has no solutions $(u,\, v)$ with full row rank matrices v.

Secondly, if $\operatorname{im}_{\mathbb{K}}(M.) = \mathbb{K}^{m \times 1}$, then Condition 1 is satisfied and we have $d = m$, $Z = I_m$.

To explicitly characterize v, let $X \in \mathbb{K}^{m \times l}$ be a full column rank whose columns define a basis of $\operatorname{im}_{\mathbb{K}}(M.)$, i.e., $\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$. Hence, there exist $Y_{\bullet i} \in \mathbb{K}^{l \times 1}$, $i = 1, \ldots, n$, such that $M_{\bullet i} = X\,Y_{\bullet i}$ for $i = 1, \ldots, n$, which yields $M = X\,Y$, where $Y = (Y_{\bullet 1} \ \ldots \ Y_{\bullet n}) \in \mathbb{K}^{l \times n}$. Now, we have $D_i\,Z\,\psi \in \ker_{\mathbb{K}}(L.) = \operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$ for all $\psi \in \mathbb{K}^{d \times 1}$, which shows that there exists a unique matrix $W_i \in \mathbb{K}^{l \times d}$ such that $D_i\,Z = X\,W_i$ for $i = 1, \ldots, r$. We then get:

$$\forall\, \psi \in \mathbb{K}^{d \times 1}, \quad A(Z\,\psi) = (D_1\,Z\,\psi \ \ldots \ D_r\,Z\,\psi) = X\,(W_1\,\psi \ \ldots \ W_r\,\psi).$$

Let $B(\psi) = (W_1\,\psi \ \ldots \ W_r\,\psi) \in \mathbb{K}^{l \times r}$ for all $\psi \in \mathbb{K}^{d \times 1}$. Thus, we have $A(Z\,\psi) = X\,B(\psi)$ for all $\psi \in \mathbb{K}^{d \times 1}$. Since X has full column rank, we get:

$$\mathcal{P} = \left\{\psi \in \mathbb{K}^{d \times 1} \mid \operatorname{rank}_{\mathbb{K}}(B(\psi)) = l\right\}. \tag{8}$$

Note that $\mathcal{P}$ is not empty if at least one of the $l \times l$ minors of $B(\psi)$ is not 0. Since the columns $W_i\,\psi$ of the matrix $B(\psi)$ are linear forms in ψ, the $l \times l$ minors of $B(\psi)$ are either 0 or homogeneous polynomials in ψ of degree l. Hence, if $\psi \in \mathcal{P}$ then $\lambda\,\psi \in \mathcal{P}$ for all $\lambda \in \mathbb{K}^{\times}$. Moreover, using (6), i.e., $l \leq r$, $\mathcal{P}$ characterizes the ψ's which are so that $B(\psi)$ admits a right inverse $E_{\psi} \in \mathbb{K}^{r \times l}$, i.e., $B(\psi)\,E_{\psi} = I_l$. Using again that X has full column, we get:

$$\forall\, \psi \in \mathbb{K}^{d \times 1}, \quad A(Z\,\psi)\,v = M \iff X\,B(\psi)\,v = X\,Y \iff B(\psi)\,v = Y.$$

Hence, if $\psi \in \mathcal{P}$, then $v_{\star} = E_{\psi}\,Y \in \mathbb{K}^{r \times n}$ is a particular solution of the linear inhomogeneous system $B(\psi)\,v = Y$. Let $C_{\psi} \in \mathbb{K}^{r \times (r-l)}$ be a full column matrix whose columns define a basis of $\ker_{\mathbb{K}}(B(\psi).)$, i.e., $\ker_{\mathbb{K}}(B(\psi).) = \operatorname{im}_{\mathbb{K}}(C_{\psi}.)$. Then, we obtain the following solutions of (1):

$$\forall\, \psi \in \mathcal{P}, \quad \forall\, Y' \in \mathbb{K}^{(r-l) \times n}, \quad \begin{cases} u = Z\,\psi, \\ v = (E_{\psi} \quad C_{\psi}) \begin{pmatrix} Y \\ Y' \end{pmatrix}. \end{cases} \tag{9}$$

Note that the matrix $(E_{\psi} \quad C_{\psi}) \in \mathbb{K}^{r \times r}$ is invertible. Hence, v has full row rank if and only if so has the matrix $\left(Y^T \quad Y'^T\right)^T$. Let sum up the above results.

Theorem 1 ([15]). *Let $D_i \in \mathbb{K}^{m \times m}$ for $i = 1, \ldots, r$ and $M \in \mathbb{K}^{m \times n}$ be such that:*

$$l = \operatorname{rank}_{\mathbb{K}}(M) \leq \min\{m, r\}.$$

Let $L \in \mathbb{K}^{(m-l) \times m}$ be a full row matrix whose rows define a basis of $\ker_{\mathbb{K}}(.M)$ (with the convention that $L = 0$ if $l = m$), $N = \left((L\,D_1)^T \ \ldots \ (L\,D_r)^T\right)^T \in \mathbb{K}^{p\,r \times m}$ and $Z \in \mathbb{K}^{m \times d}$ be a full column matrix whose columns define a basis of $\ker_{\mathbb{K}}(N.)$, where $d = \dim_{\mathbb{K}}(\ker_{\mathbb{K}}(N.))$. In particular, $Z = I_m$ if $l = m$. Moreover, *let $X \in \mathbb{K}^{m \times l}$ be a full column matrix whose columns define a basis of $\mathrm{im}_{\mathbb{K}}(M.)$ and $Y \in \mathbb{K}^{l \times n}$ the unique matrix such that $M = X\,Y$. Finally, let $W_i \in \mathbb{K}^{l \times d}$ be the unique matrices such that $D_i\,Z = X\,W_i$ for $i = 1, \ldots, r$, and:*

$$\psi = (\psi_1 \ \ldots \ \psi_d)^T \in \mathbb{K}^{d \times 1}, \quad B = (W_1\,\psi \ \ldots \ D_r\,\psi) \in \mathbb{K}^{l \times r}.$$

If the linear cone $\mathcal{P} = \left\{\psi \in \mathbb{K}^{d \times 1} \mid \mathrm{rank}_{\mathbb{K}}(B(\psi)) = l\right\}$ is not empty, then (9) are solutions of (3), where $E_\psi \in \mathbb{K}^{r \times l}$ is a right inverse of $B(\psi)$ and $C_\psi \in \mathbb{K}^{r \times (r-l)}$ is a full column matrix whose columns define a basis of $\ker_{\mathbb{K}}(B(\psi).)$. If $l = r$, then $C_\psi = 0$ and (9) is unique.

The matrix v defined by (9) has full row rank if and only if the matrix $Y' \in \mathbb{K}^{(r-l) \times n}$ is chosen so that $\left(Y^T \quad {Y'}^T\right)^T \in \mathbb{K}^{r \times n}$ has full row rank.

Finally, the results do not depend on the choice of bases for the different $\mathbb{K}$-vector spaces.

Let us more precisely study $\mathcal{P}$. To do that, we first introduce a few notations. Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_d]$ be the commutative polynomial ring in $x_1, \ldots, x_d$ with coefficients in $\mathbb{K}$, $x = (x_1 \ \ldots \ x_d)^T$, and $B = (W_1\,x \ \ldots \ W_r\,x) \in \mathcal{R}^{l \times r}$. According to (6), we have $l \leq r$, i.e., B is a wide matrix. If $\mathcal{I}$ denotes the ideal of $\mathcal{R}$ defined by all the $l \times l$ minors of B, then either $\mathcal{I}$ is reduced to 0 or $\mathcal{I}$ can be generated by homogeneous polynomials $g_1, \ldots, g_t$ of degree l. If $V_{\mathbb{K}}(\mathcal{I}) = \left\{\psi \in \mathbb{K}^{d \times 1} \mid \forall\, P \in \mathcal{I} : P(\psi) = 0\right\}$ is the *affine algebraic set* associated with $\mathcal{I}$, then:

$$\mathcal{P} = \mathbb{K}^{d \times 1} \setminus V_{\mathbb{K}}(\mathcal{I}).$$

Let us consider the $\mathcal{R}$-module $\mathcal{B} = \mathrm{coker}_{\mathcal{R}}(B.) = \mathcal{R}^{l \times 1} / \left(B\,\mathcal{R}^{r \times 1}\right)$. Note that the 0^{th}-*Fitting ideal* $\mathrm{Fitt}_0(\mathcal{B})$ of $\mathcal{B}$ is the ideal of $\mathcal{R}$ generated by all the $l \times l$ minors of B [9, 22], i.e., $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B})$. If $\mathrm{ann}_{\mathcal{R}}(\mathcal{B}) = \{a \in \mathcal{R} \mid \forall\, b \in \mathcal{B} : a\,b = 0\}$ is the *annihilator* of $\mathcal{B}$, then we have

$$\mathrm{ann}_{\mathcal{R}}(\mathcal{B})^l \subseteq \mathrm{Fitt}_0(\mathcal{B}) \subseteq \mathrm{ann}_{\mathcal{R}}(\mathcal{B}) \implies \sqrt{\mathrm{ann}_{\mathcal{R}}(\mathcal{B})} = \sqrt{\mathrm{Fitt}_0(\mathcal{B})},$$

where $\sqrt{\mathcal{I}} = \left\{a \in \mathcal{R} \mid \exists\, k \in \mathbb{Z} : a^k \in I\right\}$ is the *radical* of $\mathcal{I}$. For more details, see, e.g., [9, 22]. In particular, if $\mathbb{K}$ is an *algebraically closed field* (e.g., $\mathbb{K} = \mathbb{C}$), then $V_{\mathbb{K}}(\mathcal{I}) = V_{\mathbb{K}}(\mathrm{ann}_{\mathcal{R}}(\mathcal{B}))$.

Corollary 1 ([2])**.** *Let $W_i \in \mathbb{K}^{l \times d}$, $i = 1, \ldots, r$, be the matrices defined in Theorem 1, $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_d]$, $x = (x_1 \ \ldots \ x_d)^T$, $B = (W_1\,x \ \ldots \ W_r\,x) \in \mathcal{R}^{l \times r}$, $\mathcal{B} = \mathrm{coker}_{\mathcal{R}}(B.)$, $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B})$, and $\mathrm{ann}_{\mathcal{R}}(\mathcal{B})$ the annihilator of $\mathcal{B}$. Then, the linear cone $\mathcal{P}$, defined in Theorem 1, is the complementary of the algebraic set $V_{\mathbb{K}}(\mathcal{I})$ in the affine space $\mathbb{K}^{d \times 1}$, and thus, $\mathcal{P}$ is a quasi-affine algebraic set. Finally, if $\mathbb{K}$ is an algebraically closed field, then $V_{\mathbb{K}}(\mathcal{I}) = V_{\mathbb{K}}(\mathrm{ann}_{\mathcal{R}}(\mathcal{B}))$.*

Example 1. Let us consider the following matrices:

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$D_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Then, we have $m = n = r = 4$. We can easily check that $l = \mathrm{rank}_{\mathbb{K}}(M) = 1$ and:

$$X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad Y = (1 \quad 0 \quad 0 \quad 1), \quad L = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$W_1 = -1, \quad W_2 = 0, \quad W_3 = 1, \quad W_4 = 0.$$

We have $d = 1$, $\psi = \psi_1 \in \mathbb{K}$, $\mathcal{R} = \mathbb{K}[x_1]$ and $B = (-x_1 \quad 0 \quad x_1 \quad 0) \in \mathcal{R}^{1 \times 4}$. Considering the $\mathcal{R}$-module $\mathcal{B} = \mathcal{R}/(B\,\mathcal{R}^{4 \times 1}) = \mathcal{R}/\mathcal{I}$, where $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \mathrm{ann}_{\mathcal{R}}(\mathcal{B}) = \langle x_1 \rangle$ denotes the ideal of $\mathcal{R}$ generated by x_1, $V_{\mathbb{K}}(\mathcal{I}) = \{0\}$ and $\mathcal{P} = \mathbb{K} \setminus \{0\}$. If we set $W = (-1 \quad 0 \quad 1 \quad 0)$, then $B = x_1\,W$, $F = 1/2\,(-1 \quad 0 \quad 1 \quad 0)^T$

is a right inverse of W, and thus, for $\psi \in \mathcal{P}$, $E_\psi = \psi_1^{-1} F$ is a right inverse of $B(\psi)$. Computing a basis of $\ker_{\mathbb{K}}(W.)$, we get $\ker_{\mathbb{K}}(W.) = \mathrm{im}_{\mathbb{K}}(C.)$, where

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{K}^{4\times 3}, \tag{10}$$

and thus, $\ker_{\mathbb{K}}(B(\psi).) = \mathrm{im}_{\mathbb{K}}(C.)$ for all $\psi \in \mathcal{P}$. Then, (9) defines solutions of (3), i.e.:

$$\forall\, \psi \in \mathcal{P} = \mathbb{K} \setminus \{0\}, \quad \forall\, Y' \in \mathbb{K}^{3\times 4}, \quad \begin{cases} u = Z\,\psi = \begin{pmatrix} -\psi \\ 0 \\ 0 \\ \psi \end{pmatrix}, \\ v = \dfrac{1}{2\,\psi} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y'. \end{cases} \tag{11}$$

Finally, all the solutions $(u,\, v)$ of (3) with full row rank matrices v can be written as (9) for all $Y' \in \mathbb{K}^{3\times 4}$ satisfying the following condition:

$$\det\left(\begin{pmatrix} Y \\ Y' \end{pmatrix}\right) \neq 0.$$

If $g_1, \ldots, g_t$ are homogeneous polynomials generating $\mathcal{I}$, i.e., $\mathcal{I} = \langle g_1, \ldots, g_t \rangle$, then we have $\mathcal{P} = \mathbb{K}^{d\times 1} \setminus V_{\mathbb{K}}(\langle g_1, \ldots, g_t \rangle) = \bigcup_{i=1}^{t} D(g_i)$, where $D(g_i) = \{\psi \in \mathbb{K}^{d\times 1} \mid g_i(\psi) \neq 0\}$ is the *distinguished open Zariski set* defined by g_i [9, 19]. For instance, if $l = r$, then $t = 1$, $g_1 = \det(B)$ and $\mathcal{P} = D(g_1)$. If $\mathcal{I} \neq \langle 0 \rangle$, then, as explained in [2], on each $D(g_i)$, regular closed-form solutions of (3) can be obtained using effective module theory (see, e.g., [1, 5, 11]).

The complete study of these closed-form solutions still need to be finalized since it is related to well-known open problems in module theory such that:

1. Testing if a *stably free* $\mathcal{R}_{g_i}$*-module* is *free*, where $\mathcal{R}_{g_i} = \{a/g_i^k \mid a \in \mathcal{R},\, k \in \mathbb{Z}_{\geq 0}\}$ is the *localization* of the polynomial ring $\mathcal{R}$ at the *multiplicatively closed set* $\{g_i^k\}_{k\in\mathbb{Z}_{\geq 0}}$ [9].

2. Effectively computing bases of finitely generated free $\mathcal{R}_{g_i}$-modules (i.e., possible extensions of the well-known *Quillen-Suslin theorem*) [9, 10].

3. Effective computation of minimal sets of generators of ideals [9].

The first two points are related to the characterization of the minimal integer $s \geq r - l$ such that a matrix $C \in \mathcal{R}_{g_i}^{r\times s}$ exists satisfying $\ker_{\mathcal{R}_{g_i}}(B.) = \mathrm{im}_{\mathcal{R}_{g_i}}(C.)$. Indeed, it can be shown that the $\mathcal{R}_{g_i}$-module $\ker_{\mathcal{R}_{g_i}}(B.)$ is stably free of rank $r - l$ [2]. In particular, the first two points can be effectively solved in the following particular cases:

- $g_i \in \mathbb{K} \setminus \{0\}$, i.e., $\mathcal{R}_{g_i} = \mathcal{R}$, by effective versions of the Quillen-Suslin theorem [9, 10].

- $g_i = x_i$ by an extension of the Quillen-Suslin theorem to *generalized Laurent polynomial ring.* See [10] and the references therein.

- $r = l + 1$ since stably free modules of rank 1 over a commutative ring are free [9, 10].

- $d = 1$ since $\mathcal{R} = \mathbb{K}[x_1]$ is a *principal ideal domain* and stably free $\mathcal{R}$-modules (e.g., $\ker_{\mathcal{R}}(B.)$) are free [24] and bases of free $\mathcal{R}$-modules can be computed using *Smith normal forms* [10].

- $d = 2$ since $\ker_{\mathcal{R}}(B.)$ is then a *projective* $\mathcal{R} = \mathbb{K}[x_1, x_2]$-module [9, 24], and thus, a free $\mathcal{R}$-module by the Quillen-Suslin theorem.

The third point is related to finding a minimal set of generators for $\mathcal{I}$, and thus, a minimal cover of $\mathbb{K}^{d\times 1}$ by distinguished open sets of the form $D(g_i)$. We have the following facts:

- If $\mu(\mathcal{I})$ denotes the number of elements of a minimal set of generators of $\mathcal{I}$, then we know that $\mu(\mathcal{I}) = \mu\left(\mathcal{I}/\mathcal{I}^2\right)$, where $\mathcal{I}/\mathcal{I}^2$ is the *conormal* $\mathcal{R}/\mathcal{I}$-module [2].

- $d = 1$ since $\mathcal{R} = \mathbb{K}[x_1]$ is principal and $\mathcal{I}$ can be generated by an element of $\mathcal{R}$.

For the connections between these issues and the rank factorization problem, see [2].

Finally, for the demodulation problems studied in vibration analysis, as explained in Section 1, the matrices M and D_i, $i = 1, \ldots, r$, are centrohermitian, and the solutions u and v_i, $i = 1, \ldots, r$ are also sought to be centrohermitian. We refer the readers to [17, 18] for extensions of the above results to this particular situation. See also [16].

3 General solutions

The goal of this section is to extend the approach explained in Section 2 to characterize the general solutions of the rank factorization problem (3). To do that, we shall use standard results on module theory and homological algebra [9, 11, 24].

3.1 Case where M has full row rank

Let us first consider the case where M has full row rank, i.e., $\operatorname{im}_{\mathbb{K}}(M.) = \mathbb{K}^{m\times 1}$. Then, we have $m \leq n$ and there exists $N \in \mathbb{K}^{n\times m}$ such that $M\,N = I_m$. Then, (3) yields $A(u)\,(v\,N) = I_m$, i.e., the matrix $A(u)$ has full row rank and $m \leq r$. Conversely, let us suppose that there exists $u \in \mathbb{K}^{m\times 1}$ such that $A(u) \in \mathbb{K}^{m\times r}$ has full row rank. Then, there exists $E_u \in \mathbb{K}^{r\times m}$ such that $A(u)\,E_u = I_m$, which yields $A(u)\,(E_u\,M) = M$ and shows that (3) has a solution. Thus, (3) is solvable if and only if $u \in \mathbb{K}^{m\times 1}$ exists such that $A(u)$ has full row rank. We have to characterize the following set:

$$\mathcal{U} = \left\{u \in \mathbb{K}^{m\times 1} \mid \operatorname{rank}_{\mathbb{K}}(A(u)) = m\right\}.$$

Remark 3. If M has full row rank, then following the approach of Section 2.2, we then have $d = m$, $Z = I_m$, $\psi = u = (u_1 \ \ldots \ u_m)^T$, $X = I_m$, $Y = M$ and $W_i = D_i$ for $i = 1, \ldots, r$, which shows that $B(\psi) = A(u)$ and $\mathcal{U} = \mathcal{P}$.

Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$ be the commutative polynomial ring in the variables $x_1, \ldots, x_m$ with coefficients in $\mathbb{K}$, $x = (x_1 \ \ldots \ x_m)^T$ and $A = (D_1\,x \ \ldots \ D_r\,x) \in \mathcal{R}^{m\times r}$. Moreover, let $\mathcal{I} = \langle m_i \rangle_{i=1,\ldots,r!/(m!\,(r-m)!)}$ be the ideal of $\mathcal{R}$ formed by all the $m \times m$ minors m_i's of A. Note that m_i is either 0 or a homogeneous polynomial of degree m. Then, we clearly have:

$$\mathcal{U} = \mathbb{K}^{m\times 1} \setminus V_{\mathbb{K}}(\mathcal{I}).$$

Hence, the rank factorization problem (3) has solutions if and only if $V_{\mathbb{K}}(\mathcal{I}) \neq \mathbb{K}^{m\times 1}$. If $\mathbb{K}$ is an infinite algebraically closed field, then the last condition is equivalent to $\mathcal{I} \neq \langle 0 \rangle$.

Example 2. Let us consider the following matrices:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -3 & -4 \\ 3 & 4 \end{pmatrix}.$$

Then, we have $m = n = r = 2$, $x = (x_1 \quad x_2)^T$, and:

$$A = \begin{pmatrix} -x_1 - 2\,x_2 & -3\,x_1 - 4\,x_2 \\ x_1 + 2\,x_2 & 3\,x_1 + 4\,x_2 \end{pmatrix} \implies \det(A) = 0 \implies \mathcal{I} = \langle 0 \rangle \implies \mathcal{U} = \emptyset.$$

Hence, the corresponding rank factorization problem (3) has no solutions.

Remark 4. Note that $\mathcal{U}$ is the complementary of the algebraic set $V_{\mathbb{K}}(\mathcal{I})$ in the affine space $\mathbb{K}^{m\times 1}$, i.e., $\mathcal{U}$ is a quasi-affine algebraic set.

Let us now suppose that $\mathcal{I} \neq \langle 0 \rangle$ and let $h_1, \ldots, h_s \in \mathcal{R}$ be such that $\mathcal{I} = \langle h_1, \ldots, h_s \rangle$. Then, we have $\mathcal{U} = \bigcup_{i=1}^{s} D(h_i)$, i.e., $\{D(h_i)\}_{i=1,\ldots,s}$ is a *cover* of $\mathcal{U}$ by distinguished open subsets. If $u \in D(h_i) = \{\psi \in \mathbb{K}^{m\times 1} \mid h_i(\psi) \neq 0\}$, then we have $\operatorname{rank}_{\mathbb{K}}(A(u)) = m$, which shows that there exists a right inverse $F_{h_i} \in \mathbb{K}^{r\times m}$ of $A(u)$ on $D(h_i)$. To compute such a right inverse, we can consider the localization $\mathcal{R}_{h_i} = \{a/h_i^k \mid a \in \mathcal{R},\, k \in \mathbb{Z}_{\geq 0}\}$ of the polynomial ring $\mathcal{R}$ and the $\mathcal{R}$-module $\mathcal{A} = \operatorname{coker}_{\mathcal{R}}(A.) = \mathcal{R}^{m\times 1}/(A\,\mathcal{R}^{r\times 1})$ finitely presented by the matrix A. Then, we have $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{A})$ and $\operatorname{ann}_{\mathcal{R}}(\mathcal{A})^m \subseteq \operatorname{Fitt}_0(\mathcal{A}) \subseteq \operatorname{ann}_{\mathcal{R}}(\mathcal{A})$ (see, e.g., [9, 19]). Thus, we have $h_i\,\mathcal{A} = 0$, and thus, $S_{h_i}^{-1}\mathcal{A} = 0$, where $S_{h_i}^{-1}\mathcal{A} = \{m/h_i^k \mid m \in \mathcal{A},\, k \in \mathbb{Z}\}$ denotes the localization of $\mathcal{A}$ at S_{h_i} (see, e.g., [9, 19, 24]). Applying the *right exact covariant functor* $\mathcal{R}_{h_i} \otimes_{\mathcal{R}} \cdot$ (since $\mathcal{R}_{h_i}$ is a *flat* $\mathcal{R}$-module; see, e.g., [9, 24]) to the *exact sequence* of $\mathcal{R}$-modules

$$\mathcal{R}^{r\times 1} \xrightarrow{A.} \mathcal{R}^{m\times 1} \xrightarrow{\sigma} \mathcal{A} \longrightarrow 0$$

we obtain the following *split exact sequence* of $\mathcal{R}_{h_i}$-modules (see, e.g., [9, 24]):

$$\mathcal{R}_{h_i}^{r\times 1} \xrightarrow{A.} \mathcal{R}_{h_i}^{m\times 1} \xrightarrow{\mathrm{id}\otimes\sigma} S_{h_i}^{-1}\mathcal{A} = 0.$$

Hence, there exists a right inverse $F_{h_i} \in \mathcal{R}_{h_i}^{r\times m}$ of the matrix A, i.e., $A\,F_{h_i} = I_m$. Using the fact that $\mathcal{R}_{h_i}$ is a *noetherian ring* (see, e.g., [9, 19, 24]), then $\ker_{\mathcal{R}_{h_i}}(A.)$ is a finitely generated $\mathcal{R}_{h_i}$-module. Hence, let $C_{h_i} \in \mathcal{R}_{h_i}^{r\times t}$ satisfy $\ker_{\mathcal{R}_{h_i}}(A.) = \operatorname{im}_{\mathcal{R}_{h_i}}(C_{h_i}.)$. Then, all the solutions of (3) on $D(h_i)$ are defined by:

$$\forall\, Y' \in \mathcal{R}_{h_i}^{t\times n}, \quad \begin{cases} u \in D(h_i), \\ v = F_{h_i}\,M + C_{h_i}\,Y'. \end{cases} \tag{12}$$

Using $\mathcal{U} = \bigcup_{i=1}^{s} D(h_i)$, we finally obtain an explicit characterization of all the solutions of (3): for $u \in \mathcal{U}$, there exists $i \in \{1, \ldots, s\}$ such that $u \in D(h_i)$ and (12) then defines all the solutions of (3) over the distinguished open set $D(h_i)$.

Remark 5. The matrix F_{h_i} can be computed by, for instance, the `LocalLeftInverse` command of the OreModules package [5] or the `PreInverse` command of the `CapAndHomalg` library [1]. The matrix C_{h_i} can be computed by, for instance, the `SyzygyModule` command of OreModules or the `WeakKernelEmbedding` command of `CapAndHomalg`.

Example 3. Let us consider the following matrices:

$$M = \begin{pmatrix} 15 & 14 & 13 \\ 24 & 20 & 16 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \quad D_3 = \begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}.$$

Then, we have $l = \operatorname{rank}_{\mathbb{K}}(M) = 2 = m < r = 3 = n$. In particular, M has full row rank. Let us consider $\mathcal{R} = \mathbb{K}[x_1, x_2]$ and the $\mathcal{R}$-module $\mathcal{A} = \mathcal{R}^{2\times 1}/(A\,\mathcal{R}^{3\times 1})$ finitely presented by:

$$A = (D_1\,x \quad D_2\,x \quad D_3\,x) = \begin{pmatrix} x_1 - x_2 & x_1 + 2\,x_2 & x_1 + 3\,x_2 \\ x_1 + x_2 & -x_1 + 2\,x_2 & 4\,x_1 + 3\,x_2 \end{pmatrix} \in \mathcal{R}^{2\times 3}.$$

We can check that $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{A}) = \langle x_1^2,\, x_1\,x_2,\, x_2^2 \rangle$. Hence, we have:

$$\mathcal{U} = \mathbb{K}^{2\times 1} \setminus V_{\mathbb{K}}(\mathcal{I}) = \mathbb{K}^{2\times 1} \setminus \{(0 \quad 0)^T\} = D\left(x_1^2\right) \cup D(x_1\,x_2) \cup D\left(x_2^2\right).$$

Moreover, we have:

$$F_{x_1^2} = \begin{pmatrix} \dfrac{66}{97\,x_1} & \dfrac{6}{97\,x_1} \\ \dfrac{38}{97\,x_1} + \dfrac{33\,x_2}{97\,x_1^2} & -\dfrac{23}{97\,x_1} + \dfrac{3\,x_2}{97\,x_1^2} \\ -\dfrac{7}{97\,x_1} - \dfrac{22\,x_2}{97\,x_1^2} & \dfrac{17}{97\,x_1} - \dfrac{2\,x_2}{97\,x_1^2} \end{pmatrix} \in \mathcal{R}_{x_1^2}^{3\times 2}, \quad R\,F_{x_1^2} = I_2,$$

$$F_{x_1 x_2} = \begin{pmatrix} -\dfrac{55}{194\, x_2} & -\dfrac{5}{194\, x_2} \\ \dfrac{33}{194\, x_2} + \dfrac{21}{194\, x_1} & \dfrac{3}{194\, x_2} - \dfrac{51}{194\, x_1} \\ \dfrac{11}{97\, x_2} - \dfrac{7}{97\, x_1} & \dfrac{1}{97\, x_2} + \dfrac{17}{97\, x_1} \end{pmatrix} \in \mathcal{R}_{x_1 x_2}^{3\times 2}, \quad R\, F_{x_1 x_2} = I_2,$$

$$F_{x_2^2} = \begin{pmatrix} -\dfrac{35\, x_1}{388\, x_2^2} - \dfrac{1}{2\, x_2} & \dfrac{85\, x_1}{388\, x_2^2} + \dfrac{1}{2\, x_2} \\ \dfrac{21\, x_1}{388\, x_2^2} + \dfrac{31}{388\, x_2} & -\dfrac{51\, x_1}{388\, x_2^2} + \dfrac{91}{388\, x_2} \\ \dfrac{7\, x_1}{194\, x_2^2} + \dfrac{11}{97\, x_2} & -\dfrac{17\, x_1}{194\, x_2^2} + \dfrac{1}{97\, x_2} \end{pmatrix} \in \mathcal{R}_{x_2^2}^{3\times 2}, \quad R\, F_{x_2^2} = I_2.$$

If we note

$$C = \begin{pmatrix} 5\, x_1^2 + 12\, x_1\, x_2 \\ -3\, x_1^2 + 5\, x_1\, x_2 + 6\, x_2^2 \\ -2\, x_1^2 - 4\, x_2^2 \end{pmatrix} \in \mathcal{R}^{3\times 1},$$

then we have $\ker_{\mathcal{R}_h^{-1}}(A.) = \operatorname{im}_{\mathcal{R}_h^{-1}}(C.)$ for $h = x_1^2$, $x_1\, x_2$ and x_2^2, which shows that all the solutions of (3) are of the form of (12):

$$\begin{array}{lll} \forall\, u \in D\left(x_1^2\right), & \forall\, Y'_{x_1^2} \in \mathcal{R}_{x_1^2}^{1\times 3}, & v = F_{x_1^2}\, M + C\, Y'_{x_1^2}, \\ \forall\, u \in D(x_1\, x_2), & \forall\, Y'_{x_1 x_2} \in \mathcal{R}_{x_1 x_2}^{1\times 3}, & v = F_{x_1 x_2}\, M + C\, Y'_{x_1 x_2}, \\ \forall\, u \in D\left(x_2^2\right), & \forall\, Y'_{x_2^2} \in \mathcal{R}_{x_2^2}^{1\times 3}, & v = F_{x_2^2}\, M + C\, Y'_{x_2^2}. \end{array} \tag{13}$$

The determinants of the matrices $U_h = (F_h \quad C)$, where $h = x_1^2$, $x_1\, x_2$ or x_2^2, are 1, which shows that U_h is invertible. Hence, the v's defined by of one of the three forms listed in (13) have full row rank if and only if so have $\left(M^T \quad Y_h'^T\right)^T$, where $h = x_1^2$, $x_1\, x_2$ or x_2^2.

For instance, if we consider the solution of (3) defined by $u = (1 \quad 1)^T$ and

$$v = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \tag{14}$$

then $u \in D\left(x_1^2\right) = (\mathbb{K} \setminus \{0\}) \times \mathbb{K}$, $u \in D(x_1\, x_2) = (\mathbb{K} \setminus \{0\})^2$, and $u \in D\left(x_2^2\right) = \mathbb{K} \times (\mathbb{K} \setminus \{0\})$, and we can check again that we have:

$$\begin{array}{ll} v = F_{x_1^2}\, M + C\, Y'_{x_1^2}, & Y'_{x_1^2} = \begin{pmatrix} -\dfrac{61}{97} & -\dfrac{50}{97} & -\dfrac{39}{97} \end{pmatrix}, \\ v = F_{x_1 x_2}\, M + C\, Y'_{x_1 x_2}, & Y'_{x_1^2} = \begin{pmatrix} -\dfrac{169}{388} & -\dfrac{47}{194} & -\dfrac{19}{388} \end{pmatrix}, \\ v = F_{x_2^2}\, M + C\, Y'_{x_2^2}, & Y'_{x_2^2} = \begin{pmatrix} \dfrac{67}{194} & \dfrac{37}{97} & \dfrac{81}{194} \end{pmatrix}. \end{array}$$

The matrix v defined by (14) has not full row rank ($\det v = 0$), which is consistent with:

$$\det\left(\begin{pmatrix} M \\ Y'_h \end{pmatrix}\right) = 0, \quad h = x_1^2,\, x_1\, x_2,\, x_2^2.$$

3.2 General case

Let us now consider the case where the matrix M has not full row rank, i.e., $\operatorname{im}_{\mathbb{K}}(M.) \neq \mathbb{K}^{m\times 1}$. Set $p = m - l > 0$, where $l = \operatorname{rank}_{\mathbb{K}}(M)$, and let $L \in \mathbb{K}^{p\times m}$ be a full column rank matrix whose rows define a basis of the $\mathbb{K}$-vector space $\ker_{\mathbb{K}}(.M)$.

Remark 6. The matrix L generates the *compatibility conditions* of the inhomogeneous linear system $M\,\eta = \zeta$, where ζ is a fixed vector of $\mathbb{K}^{m\times 1}$. Indeed, a necessary (and sufficient) condition on ζ for the solvability of $M\,\eta = \zeta$ is then $L\,\zeta = 0$.

3.2.1 Characterization of $\ker_S(Q.)$

Suppose that a solution $(u,\, v)$ of (3) exists and set $Q(u) = L\,A(u) \in \mathbb{K}^{p\times r}$. Then, (3) and $L\,M = 0$ yield $Q(u)\,v = L\,M = 0$, which shows that the columns of the matrix v belong to $\ker_{\mathbb{K}}(Q(u).)$. Hence, $u \in \mathbb{K}^{m\times 1}$ necessarily satisfies that $\ker_{\mathbb{K}}(Q(u).)$ is not reduced to 0.

In linear algebra, it is well-known that $\dim_{\mathbb{K}}(\operatorname{coker}_{\mathbb{K}}(Q(u).)) + r = \dim_{\mathbb{K}}(\ker_{\mathbb{K}}(Q(u).)) + p$, which yields $\dim_{\mathbb{K}}(\ker_{\mathbb{K}}(Q(u).)) \geq r - p$. Hence, if $r > p$, i.e., if the matrix $Q(u)$ is wide, then $\ker_{\mathbb{K}}(Q(u).)$ is not reduced to 0 for all $u \in \mathbb{K}^{m\times 1}$. But, if $r \leq p$, i.e., if the matrix $Q(u)$ is tall or square, then $\ker_{\mathbb{K}}(Q(u).)$ can be reduced to 0 for almost all $u \in \mathbb{K}^{m\times 1}$. The u's for which $\ker_{\mathbb{K}}(Q(u).)$ is not reduced to 0 are the common $\mathbb{K}$-zeros of all the $r \times r$ minors of $Q(u)$.

We can state again the general definition of the *Fitting ideals* Fitt_i's (see, e.g., [9, 22]).

Definition 1. *Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $M \in \mathcal{R}^{s\times r}$ and $\mathcal{M} = \mathcal{R}^{1\times r}/(\mathcal{R}^{1\times s}\,M)$ be the $\mathcal{R}$-module finitely presented by M. The Fitting ideals $\operatorname{Fitt}_i(\mathcal{M})$'s of $\mathcal{M}$ are defined by:*

- $\operatorname{Fitt}_i(\mathcal{M})$ *is the ideal of $\mathcal{R}$ generated by all the $(r-i)\times(r-i)$ minors of the matrix M for $1 \leq r-i \leq s$.*
- $\operatorname{Fitt}_i(\mathcal{M}) = \langle 0 \rangle$ *for $s < r - i$.*
- $\operatorname{Fitt}_i(\mathcal{M}) = \mathcal{R}$ *for $r - i \leq 0$.*

Let $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1 \;\ldots\; x_m)^T$, $A = (D_1\,x \;\ldots\; D_r\,x) \in \mathcal{R}^{m\times r}$, $Q = L\,A \in \mathcal{R}^{p\times r}$. Moreover, let $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q) = \mathcal{R}^{1\times r}/(\mathcal{R}^{1\times p}\,Q)$ be the $\mathcal{R}$-module finitely presented by Q. Hence, we have the following exact sequence of $\mathcal{R}$-modules:

$$\mathcal{R}^{1\times p} \xrightarrow{.Q} \mathcal{R}^{1\times r} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0. \tag{15}$$

Then, $\operatorname{Fitt}_i(\mathcal{Q})$ is the ideal of $\mathcal{R}$ generated by all the $(r-i)\times(r-i)$ minors of Q for $1 \leq r - i \leq p$. Thus, $\operatorname{Fitt}_0(\mathcal{Q})$ is either the ideal generated by all the $r \times r$ minors of $Q \in \mathcal{R}^{p\times r}$ if $r \leq p$, or $\langle 0 \rangle$ if $r > p$. Note that $\operatorname{Fitt}_0(\mathcal{Q})$ is generated by homogeneous polynomials of degree r if $r \leq p$, and is reduced to $\langle 0 \rangle$ if $r > p$. In the latter case, we then have $V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q})) = \mathbb{K}^{m\times 1}$. The fact that $\ker_{\mathbb{K}}(Q(u).)$ is not reduced to 0 is thus equivalent to $u \in V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q}))$.

Lemma 1. *With the above notations, a necessary condition for the solvability of (3) is $u \in V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q}))$, where $\operatorname{Fitt}_0(\mathcal{Q})$ is the ideal of $\mathcal{R}$ formed by the homogeneous polynomials of degree r defined by all the $r \times r$ minors of $Q \in \mathcal{R}^{(m-l)\times r}$ if $r \leq m - l$, or $\langle 0 \rangle$ if $r > m - l$.*

Remark 7. We can easily check again that following chain of Fitting ideals holds

$$\langle 0 \rangle \subseteq \operatorname{Fitt}_0(\mathcal{Q}) \subseteq \operatorname{Fitt}_1(\mathcal{Q}) \subseteq \ldots \subseteq \operatorname{Fitt}_{r-1}(\mathcal{Q}) \subseteq \mathcal{R}, \tag{16}$$

(see, e.g., [9, 11, 19]) which yields the following chain of affine algebraic sets of $\mathbb{K}^{m\times 1}$:

$$V_{\mathbb{K}}(\operatorname{Fitt}_{r-1}(\mathcal{Q})) \subseteq V_{\mathbb{K}}(\operatorname{Fitt}_{r-2}(\mathcal{Q})) \subseteq \ldots \subseteq V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q})) \subseteq V_{\mathbb{K}}(\langle 0 \rangle) = \mathbb{K}^{m\times 1}. \tag{17}$$

We note that $\operatorname{Fitt}_{r-1}(\mathcal{Q})$ is the ideal of $\mathcal{R}$ generated by all the 1×1 entries of the matrix $Q = (L\,D_1\,x \;\ldots\; L\,D_r\,x)$, i.e., $V_{\mathbb{K}}(\operatorname{Fitt}_{r-1}(\mathcal{Q})) = \left\{u \in \mathbb{K}^{m\times 1} \mid L\,D_i\,u = 0,\; i = 1, \ldots, r\right\}$, and thus, we get that $V_{\mathbb{K}}(\operatorname{Fitt}_{r-1}(\mathcal{Q})) = \ker_{\mathbb{K}}(N.)$, where the matrix N is defined by (7). Hence, the approach stated again in Section 2 corresponds to considering:

$$u \in V_{\mathbb{K}}(\operatorname{Fitt}_{r-1}(\mathcal{Q})) \subseteq V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q})).$$

Finally, $0 \in V_{\mathbb{K}}(\operatorname{Fitt}_{r-1}(\mathcal{Q}))$, which shows that $V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q})) \neq \emptyset$, and thus, $\operatorname{Fitt}_0(\mathcal{Q}) \neq \mathcal{R}$.

In what follows, we shall simply note $\mathrm{Fitt}_0(\mathcal{Q})$ by $\mathcal{J}$. To algebraically emulate the fact that u belongs to $V_{\mathbb{K}}(\mathcal{J})$, we shall work over the non trivial factor noetherian ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$ of $\mathcal{R}$. Let $\phi : \mathcal{R} \longrightarrow \mathcal{S}$ be the canonical epimorphism of $\mathbb{K}$-algebras which maps $r \in \mathcal{R}$ onto its residue class $\phi(r) \in \mathcal{S}$, simply denoted by $\overline{r}$. For $l \in \mathbb{Z}_{>0}$, we can define the $\mathcal{R}$-homomorphism:

$$\begin{array}{rcl} \mathrm{id}_l \otimes \phi : \mathcal{R}^{l\times 1} & \longrightarrow & \mathcal{S}^{l\times 1} \\ \eta = (\eta_1 \ \dots \ \eta_l)^T & \longmapsto & \overline{\eta} = (\overline{\eta_1} \ \dots \ \overline{\eta_l})^T . \end{array} \tag{18}$$

Let $\mathcal{T}(\mathcal{Q}) = \mathrm{coker}_{\mathcal{R}}(Q.) = \mathcal{R}^{p\times 1}/(Q\,\mathcal{R}^{r\times 1})$ be the so-called *Auslander transpose* of $\mathcal{Q}$ [9]. Applying the right exact covariant functor $\mathcal{S} \otimes_{\mathcal{R}} \cdot$ to the exact sequence of $\mathcal{R}$-modules

$$0 \longleftarrow \mathcal{T}(\mathcal{Q}) \xleftarrow{\kappa} \mathcal{R}^{p\times 1} \xleftarrow{Q.} \mathcal{R}^{r\times 1},$$

defining a finite presentation of $\mathcal{T}(\mathcal{Q})$, we obtain the following exact sequence of $\mathcal{S}$-modules

$$0 \longleftarrow \mathcal{S} \otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) \xleftarrow{\mathrm{id}_{\mathcal{S}} \otimes \kappa} \mathcal{S}^{p\times 1} \xleftarrow{\mathrm{id}_{\mathcal{S}} \otimes Q.} \mathcal{S}^{r\times 1},$$

with the notation $(\mathrm{id}_{\mathcal{S}} \otimes Q)(\overline{\eta}) = \overline{Q\,\eta}$ for all $\eta \in \mathcal{R}^{r\times 1}$. For more details, see, e.g., [9, 24]. We simply note $\mathrm{id}_{\mathcal{S}} \otimes Q.$ by $Q.$ so that we then have:

$$\forall\, \eta \in \mathcal{R}^{r\times 1}, \quad Q\,\overline{\eta} = \overline{Q\,\eta}. \tag{19}$$

Since $\mathcal{S}$ is a noetherian ring, $\ker_{\mathcal{S}}(Q.)$ is a finitely generated $\mathcal{S}$-module [9, 24]. Thus, if $\ker_{\mathcal{S}}(Q.) \neq 0$, then there exists $K \in \mathcal{S}^{r\times q}$ is such that $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}(K.)$. To prove that $\ker_{\mathcal{S}}(Q.) \neq 0$ (even when $\ker_{\mathcal{R}}(Q.) = 0$), we shall use *McCoy's theorem* stated again below.

Theorem 2 (Theorem 6, p. 63, [22])**.** *Let $Q \in \mathcal{R}^{p\times r}$ and $\mathcal{F}$ a non-zero $\mathcal{R}$-module. A necessary and sufficient for the existence of $0 \neq \eta \in \mathcal{F}^r$ satisfying $Q\,\eta = 0$ is that there exists a non-zero element ζ of $\mathcal{F}$ that is annihilated by the* determinantal ideal *$\mathfrak{U}_r(Q)$ defined by all the $r \times r$ minors of Q if $r \leq p$, or $\langle 0\rangle$ if $r > p$, i.e., $u\,\zeta = 0$ for all $u \in \mathfrak{U}_r(Q)$.*

Corollary 2. *Let $M \in \mathbb{K}^{m\times n}$ be such that $\mathrm{im}_{\mathbb{K}}(M.) \neq \mathbb{K}^{m\times 1}$ and $L \in \mathbb{K}^{p\times m}$ be a full row rank matrix satisfying $\ker_{\mathbb{K}}(.M) = \mathrm{im}_{\mathbb{K}}(.L)$, where $p = m - \mathrm{rank}_{\mathbb{K}}(M)$. Let $D_i \in \mathbb{K}^{m\times m}$ for $i = 1, \dots, r$, $\mathcal{R} = \mathbb{K}[x_1, \dots, x_m]$, $x = (x_1 \ \dots \ x_m)^T$, $A = (D_1\,x \ \dots \ D_r\,x) \in \mathcal{R}^{m\times r}$, $Q = L\,A \in \mathcal{R}^{p\times r}$, $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q)$, $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q})$ and $\mathcal{S} = \mathcal{R}/\mathcal{J}$. Then, we have $\ker_{\mathcal{S}}(Q.) \neq 0$, and thus, there exists a non-zero matrix $K \in \mathcal{S}^{r\times q}$ such that $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}(K.)$.*

Proof. Recall that $\mathcal{J} = \mathfrak{U}_r(Q)$ is a proper ideal of $\mathcal{R}$ so that $\mathcal{S} = \mathcal{R}/\mathcal{J}$ is a non-zero $\mathcal{R}$-module. McCoy's theorem, i.e., Theorem 2, shows that $\ker_{\mathcal{S}}(Q.) \neq 0$ if and only if there exists $0 \neq s \in \mathcal{S}$ such that $i\,s = 0$ for all $i \in \mathcal{J}$. If $r > p$, then $\mathcal{J} = \langle 0\rangle$ and $0 \neq 1 \in \mathcal{S}$ satisfies $0 \times 1 = 0$, which shows that $\ker_{\mathcal{S}}(Q.) \neq 0$ (see the comment before Lemma 1). Now, if $r \leq p$, then $\mathcal{J} = \langle m_i\rangle_{i=1,\dots,\alpha}$, where $\{m_i\}_{i=1,\dots,\alpha}$ denotes the set of all the $r \times r$ minors of Q and $\alpha = p!/(r!\,(p-r)!)$. The result holds since $0 \neq 1 \in \mathcal{S}$ satisfies $m_i \times 1 = 0$ for $i = 1, \dots, \alpha$. □

Remark 8. A matrix $K \in \mathcal{S}^{r\times q}$ satisfying $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}(K.)$ can be obtained using the `Ker` (resp., `WeakKernelEmbedding`) command of `Singular` [11] (resp., `CapAndHomalg` [1]).

Example 4. Let $\mathcal{R} = \mathbb{Q}[x]$ and $\mathcal{Q} = \mathrm{coker}_{\mathcal{R}}(.Q)$ be the $\mathcal{R}$-module finitely presented by:

$$Q = \begin{pmatrix} x+1 & 0 \\ 0 & x-1 \end{pmatrix} \in \mathcal{R}^{2\times 2}.$$

Since Q has full column rank, we have $\ker_{\mathcal{R}}(Q.) = 0$. Clearly, $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q}) = \langle x^2 - 1\rangle$ and if we set $\mathcal{S} = \mathcal{R}/\mathcal{J}$, then we can check again that $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}(K.)$, where K is defined by:

$$K = \begin{pmatrix} \overline{x} - 1 & 0 \\ 0 & \overline{x} + 1 \end{pmatrix} \in \mathcal{S}^{2\times 2}.$$

Note that K is the *cofactor matrix* of Q and we can check again that $Q\,K = \overline{\det(Q)}\,I_2 = 0$.

Remark 9. Considering $\mathcal{S}$ as a $\mathcal{R}$-module and applying the *left exact contravariant functor* $\hom_{\mathcal{R}}(\,\cdot\,,\mathcal{S})$ to (15), we obtain the exact sequence of $\mathcal{R}$-modules

$$\mathcal{S}^{p\times 1} \xleftarrow{Q.} \mathcal{S}^{r\times 1} \xleftarrow{\pi^\star} \hom_{\mathcal{R}}(\mathcal{Q},\mathcal{S}) \longleftarrow 0,$$

which shows that $\ker_{\mathcal{S}}(Q.) \cong \hom_{\mathcal{R}}(\mathcal{Q},\mathcal{S})$, where $\hom_{\mathcal{R}}(\mathcal{Q},\mathcal{S})$ inherits a $\mathcal{S}$-module structure defined by $(f\,s)(q) = f(q)\,s$ for all $s \in \mathcal{S}$, $f \in \hom_{\mathcal{R}}(\mathcal{Q},\mathcal{S})$ and $q \in \mathcal{Q}$ [9, 24]. Note that the finitely presented $\mathcal{R}$-module $\hom_{\mathcal{R}}(\mathcal{Q},\mathcal{S})$ can be effectively characterized (see, e.g., [11, 6]) and K can be computed using `Singular` [11], `CapAndHomalg` [1] or OreMorphisms [7].

Example 5. We consider again Example 4. We can check again that the $\mathcal{R}$-module $\hom_{\mathcal{R}}(\mathcal{Q},\mathcal{S})$ is generated by the two generators $g_1 = (0 \quad \overline{x}+1)^T$ and $g_2 = (\overline{x}-1 \quad 0)^T$ which satisfy the relations $(\overline{x}-1)\,g_1 = 0$ and $(\overline{x}+1)\,g_2 = 0$ [7]. Hence, the matrix $K = (g_1 \quad g_2)$ is such that $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}(K.)$. We find again the result obtained in Example 4.

Example 6. As explained above Lemma 1, if $r > p = m - l$, then $\mathcal{J} = \langle 0\rangle$, and $\ker_{\mathbb{K}}(Q(u).) \neq 0$ for all $u \in V_{\mathbb{K}}(\mathcal{J}) = \mathbb{K}^{m\times 1}$. Equivalently, we have $\mathcal{S} = \mathcal{R}$ and $1 \in \mathcal{R}$ is annihilated by the generator 0 of $\mathcal{J}$, which shows that $\ker_{\mathcal{R}}(Q.) \neq 0$ by Corollary 2. A matrix K can be computed by effective elimination theory (e.g., Gröbner basis methods) over the polynomial ring $\mathcal{R} = \mathbb{K}[x_1,\ldots,x_m]$. Finally, the fact that $\ker_{\mathcal{R}}(Q.) \neq 0$ can also be proved by considering the *Euler-Poincaré characteristic* [9, 24] of exact sequence of $\mathcal{R}$-modules

$$0 \longleftarrow \mathcal{T}(\mathcal{Q}) \xleftarrow{\kappa} \mathcal{R}^{p\times 1} \xleftarrow{Q.} \mathcal{R}^{r\times 1} \longleftarrow \ker_{\mathcal{S}}(Q.) \longleftarrow 0,$$

i.e., $\mathrm{rank}_{\mathcal{R}}(\ker_{\mathcal{R}}(Q.)) - r + p - \mathrm{rank}_{\mathcal{R}}(\mathcal{T}(\mathcal{Q})) = 0$, where $\mathrm{rank}_{\mathcal{R}}(\mathcal{M})$ is the *rank* of a finitely generated $\mathcal{R}$-module $\mathcal{M}$ defined as the dimension of the finite-dimensional $\mathbb{K}(x_1,\ldots,x_l)$-vector $\mathbb{K}(x_1,\ldots,x_l) \otimes_{\mathcal{R}} \mathcal{M}$ [9, 24], and $\mathbb{K}(x_1,\ldots,x_l)$ is the *field of fractions* of $\mathcal{R}$, i.e., the field of rational functions in $x_1,\ldots,x_m$ with coefficients in $\mathbb{K}$. Thus, $\mathrm{rank}_{\mathcal{R}}(\ker_{\mathcal{R}}(Q.)) \geq r - p > 0$, which shows again that $\ker_{\mathcal{R}}(Q.) \neq 0$ and the existence of $K \in \mathcal{R}^{r\times q}$, where $q \geq r - p > 0$, satisfying $\ker_{\mathcal{R}}(Q.) = \mathrm{im}_{\mathcal{R}}(K.)$.

Remark 10. If $0 \neq \overline{\eta} \in \ker_{\mathcal{S}}(Q.)$ (e.g., η is a column of K), i.e., if $\overline{Q\,\eta} = 0$ and $\overline{\eta} \neq 0$, then $L\,A\,\eta = Q\,\eta \in \mathcal{J}^{p\times 1}$ and $\eta \notin \mathcal{J}^{r\times 1}$. If $u \in V_{\mathbb{K}}(\mathcal{J})$, then $L\,A(u)\,\eta(u) = 0$, where $A(u)$ denotes the substitution of $x = u$ in A, which shows again that $0 \neq \eta(u) \in \ker_{\mathbb{K}}(Q(u).)$.

Remark 11. Considering the ascending chain formed by the Fitting ideals (16) and using the notations $\mathcal{J}_k = \mathrm{Fitt}_k(Q)$ for $k = 0,\ldots,r-1$, and $\mathcal{J} = \mathcal{J}_0$, then, for $k = 1,\ldots,r-1$, we have the following standard short exact sequences of $\mathbb{K}$-algebras:

$$0 \longrightarrow \mathcal{J}_k/\mathcal{J} \longrightarrow \mathcal{R}/\mathcal{J} \xrightarrow{\varphi_k} \mathcal{R}/\mathcal{J}_k \longrightarrow 0.$$

Noting $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$ for $k = 0,\ldots,r-1$, with $\mathcal{S} = \mathcal{S}_0$, we have the ring epimorphisms $\varphi_k : \mathcal{S} \longrightarrow \mathcal{S}_k$. Thus, $\mathcal{S}_k$ inherits a $\mathcal{S}$-module structure defined by $s\,.\,s_k = \varphi_k(s)\,s_k$ for all $s \in \mathcal{S}$ and for all $s_k \in \mathcal{S}_k$. If $\mathcal{J}_k \neq \mathcal{R}$, i.e., if $\mathcal{S}_k$ is not the trivial ring, then $\ker_{\mathcal{S}_k}(Q.) \neq 0$ since all the $(r-k)\times(r-k)$ minors of Q vanish in $\mathcal{S}_k$, and thus, so do all the $r \times r$ minors of Q. Equivalently, it is a consequence of Theorem 2 by considering the fact that $1 \in \mathcal{S}_k$ satisfies $m_i \times 1 = 0$ for $i = 1,\ldots,\alpha$, where $\mathcal{J} = \langle m_1,\ldots,m_\alpha\rangle \subseteq \mathcal{J}_k$, i.e., $\mathcal{J} \subseteq \mathcal{J}_k = \mathrm{ann}_{\mathcal{R}}(\mathcal{S}_k)$. Therefore, if we note $\varphi_k(Q) = (\varphi_k(Q_{ij}))_{1\leq r, 1\leq j\leq p} \in \mathcal{S}_k^{p\times r}$, then there exists a non-zero matrix $K_k \in \mathcal{S}_k^{r\times q_k}$ such that $\ker_{\mathcal{S}_k}(\varphi_k(Q).) = \mathrm{im}_{\mathcal{S}_k}(K_k.)$. For instance, we can consider $k = r-1$, which corresponds to the approach developed in Section 2.2 (see Remark 7).

3.2.2 Construction of the matrix B as a pullback

As explained in Section 2 , the full row matrix $L \in \mathbb{K}^{p\times m}$, where $p = m - l$, is such that $\ker_{\mathbb{K}}(L.) = \mathrm{im}_{\mathbb{K}}(M.)$. Hence, we have the following exact sequence of $\mathbb{K}$-vector spaces:

$$0 \longleftarrow \mathbb{K}^{p\times 1} \xleftarrow{L.} \mathbb{K}^{m\times 1} \xleftarrow{M.} \mathbb{K}^{n\times 1}.$$

By definition of the matrices $X \in \mathbb{K}^{m\times l}$ and $Y \in \mathbb{K}^{l\times n}$ (see Section 2), we have $M = X\,Y$, where the columns of X define a basis of $\mathrm{im}_{\mathbb{K}}(M.)$, i.e., $\mathrm{im}_{\mathbb{K}}(M.) = \mathrm{im}_{\mathbb{K}}(X.)$ and $\ker_{\mathbb{K}}(X.) = 0$. Hence, we have

$\ker_{\mathbb{K}}(M.) = \ker_{\mathbb{K}}(Y.)$. Moreover, since $\mathrm{im}_{\mathbb{K}}(X.) \subseteq \mathrm{im}_{\mathbb{K}}(M.)$, there exists $H \in \mathbb{K}^{n\times l}$ such that $X = M\,H$, which yields $X = X\,Y\,H$, i.e., $Y\,H = I_l$ since X has full column rank. Thus, $\mathrm{im}_{\mathbb{K}}(Y.) = \mathbb{K}^{l\times 1}$ and Y has full row rank. Hence, we have the *commutative exact diagram* of finite-dimensional $\mathbb{K}$-vector spaces:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathbb{K}^{p\times 1} & \xleftarrow{L.} & \mathbb{K}^{m\times 1} & \xleftarrow{M.} & \mathbb{K}^{n\times 1} & & \\
 & & \| & & \| & & \downarrow Y. & & \\
0 & \longleftarrow & \mathbb{K}^{p\times 1} & \xleftarrow{L.} & \mathbb{K}^{m\times 1} & \xleftarrow{X.} & \mathbb{K}^{l\times 1} & \longleftarrow & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

The second horizontal short exact sequence of finite-dimensional $\mathbb{K}$-vector spaces of the above commutative exact diagram splits [24], i.e., there exist $U \in \mathbb{K}^{m\times p}$ and $V \in \mathbb{K}^{l\times m}$ such that:

$$(U \quad X) \begin{pmatrix} L \\ V \end{pmatrix} = I_m. \tag{20}$$

Since $\mathbb{K}$ is a field and the matrices $(U \quad X)$ and $\left(L^T \quad V^T\right)^T$ are square, (20) is equivalent to:

$$\begin{pmatrix} L \\ V \end{pmatrix} (U \quad X) = \begin{pmatrix} I_p & 0 \\ 0 & I_l \end{pmatrix} = I_m. \tag{21}$$

Using the fact that $\mathcal{S}$ is a $\mathbb{K}$-vector space and applying the *exact functor* $\mathcal{S} \otimes_{\mathbb{K}} \cdot$ [9, 24] to the above diagram, we obtain the following commutative exact diagram of $\mathcal{S}$-modules:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{S}^{p\times 1} & \xleftarrow{L.} & \mathcal{S}^{m\times 1} & \xleftarrow{M.} & \mathcal{S}^{n\times 1} & & \\
 & & \| & & \| & & \downarrow Y. & & \\
0 & \longleftarrow & \mathcal{S}^{p\times 1} & \xleftarrow{L.} & \mathcal{S}^{m\times 1} & \xleftarrow{X.} & \mathcal{S}^{l\times 1} & \longleftarrow & 0. \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

Using $Q = L\,A$, $Q\,K = 0$ and $\ker_{\mathcal{S}}(X.) = 0$, $L\,(A\,K) = 0$, i.e., $\mathrm{im}_{\mathcal{S}}((A\,K).) \subseteq \ker_{\mathcal{S}}(L.) = \mathrm{im}_{\mathcal{S}}(X.)$, which shows that there exists a unique matrix $B \in \mathcal{S}^{l\times q}$ satisfying:

$$A\,K = X\,B. \tag{22}$$

Hence, we have the following commutative exact diagram of S-modules:

$$\begin{array}{ccccccccccc}
 & & 0 & \longleftarrow & \mathcal{S}^{p\times 1} & \xleftarrow{L.} & \mathcal{S}^{m\times 1} & \xleftarrow{M.} & \mathcal{S}^{n\times 1} & & \\
 & & & & \| & & \| & & \downarrow Y. & & \\
 & & 0 & \longleftarrow & \mathcal{S}^{p\times 1} & \xleftarrow{L.} & \mathcal{S}^{m\times 1} & \xleftarrow{X.} & \mathcal{S}^{l\times 1} & \longleftarrow & 0. \\
 & & & & \| & & \uparrow A. & & \uparrow B. & & \\
0 & \longleftarrow & \mathcal{S} \otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) & \xleftarrow{\mathrm{id}_{\mathcal{S}} \otimes \kappa} & \mathcal{S}^{p\times 1} & \xleftarrow{Q.} & \mathcal{S}^{r\times 1} & \xleftarrow{K.} & \mathcal{S}^{q\times 1} & &
\end{array} \tag{23}$$

Using the identity $V\,X = I_l$ coming from (21), (22) then yields:

$$B = V\,A\,K. \tag{24}$$

Since B is unique, it does not depend on a chosen left inverse V of X.

Remark 12. This result can be checked again by considering a second left inverse V' of X, i.e., $V' X = I_l$. Then, we have $(V' - V) X = 0$, which shows that the rows of $V' - V$ belong to $\ker_{\mathbb{K}}(.X) = \operatorname{im}_{\mathbb{K}}(.L)$, which shows that there exists $L' \in \mathbb{K}^{l\times p}$ such that $V' = V + L' L$, which, using $Q = L A$ and $Q K = 0$, finally yields $V' A K = V A K + L' (L A) K = V A K$.

Note also that using $Q K = L A K = 0$ and (20), i.e., $X V = I_m - U L$, we get again:

$$X B = X (V A K) = A K - U (L A K) = A K.$$

Remark 13. Note that the construction of the commutative exact diagram (23) corresponds to finding an $\mathcal{R}$-algebra $\mathcal{S}$ for which the *pullback* [9, 24] of the $\mathcal{S}$-homomorphisms $A. : \mathcal{S}^{r\times 1} \longrightarrow \mathcal{S}^{m\times 1}$ and $X. : \mathcal{S}^{l\times 1} \longrightarrow \mathcal{S}^{m\times 1}$ is non-trivial, i.e., $\ker_{\mathcal{S}}((A \quad X).) \neq 0$.

Remark 14. Using the notations defined in Remark 11, let us explain the connections between the matrices $K \in \mathcal{S}^{r\times q}$ and $K_k \in \mathcal{S}_k^{r\times q_k}$ for $k = 1, \ldots, r-1$, respectively defined by $\ker_{\mathcal{S}}(Q.) = \operatorname{im}_{\mathcal{S}}(K.)$ and $\ker_{\mathcal{S}_k}(\varphi_k(Q).) = \operatorname{im}_{\mathcal{S}_k}(K_k.)$ for $k = 1, \ldots, r-1$, as well as the connections between the matrices $B_k = V \varphi_k(A) K_k \in \mathcal{S}_k^{l\times q_k}$ for $k = 0, \ldots, r-1$, with the notation $q_0 = q$. We first note that $\varphi_{r-1}(Q) = 0$, which yields $K_{r-1} = I_r$, and thus, $A = X B$ by (22). This identity corresponds to the identity $A(Z \psi) = X B(\psi)$ for all $\psi \in \mathbb{K}^{d\times 1}$ obtained in Section 2.2. Now, considering $\mathcal{S}_k$ as a $\mathcal{S}$-module (see Remark 11) and applying the covariant functor $\mathcal{S}_k \otimes_{\mathcal{S}} \cdot$ to the exact sequence of $\mathcal{S}$-modules $\mathcal{S}^{p\times 1} \xleftarrow{Q.} \mathcal{S}^{r\times 1} \xleftarrow{K.} \mathcal{S}^{q\times 1}$, we get the *complex* of $\mathcal{S}_k$-modules $\mathcal{S}_k^{p\times 1} \xleftarrow{\varphi_k(Q).} \mathcal{S}_k^{r\times 1} \xleftarrow{\varphi_k(K).} \mathcal{S}_k^{q\times 1}$, namely, $\varphi_k(Q)\, \varphi_k(K) = 0$, with the notations $\varphi_k(K) = (\varphi_k(K_{ij}))_{1\leq r, 1\leq j\leq p} \in \mathcal{S}_k^{r\times q}$ for $k = 1, \ldots, r-1$. Hence, we have $\operatorname{im}_{\mathcal{S}_k}(\varphi_k(K).) \subseteq \ker_{\mathcal{S}_k}(\varphi_k(Q).) = \operatorname{im}_{\mathcal{S}_k}(K_k.)$, and thus, there exists $L_k \in \mathcal{S}_k^{q_k\times q}$ such that:

$$\varphi_k(K) = K_k L_k, \quad k = 1, \ldots, r-1.$$

Repeating the same arguments as above with $\mathcal{S}_k$ instead of $\mathcal{S}$, we obtain the existence of a matrix $B_k = V \varphi_k(A) K_k \in \mathcal{S}_k^{r\times q_k}$ making commuting the exact diagram (23) where $\mathcal{S}$ is now replaced by $\mathcal{S}_k$. Hence, we finally obtain:

$$\varphi_k(B) = V \varphi_k(A)\, \varphi_k(K) = V \varphi_k(A) K_k L_k = B_k L_k, \quad k = 1, \ldots, r-1. \tag{25}$$

3.2.3 Characterization of the existence of a right inverse for the matrix B

Since $\ker_{\mathcal{S}}(Q.) = \operatorname{im}_{\mathcal{S}}(K.)$, $u \in V_{\mathbb{K}}(\mathcal{J})$ and $v \in \ker_{\mathbb{K}}(Q(u).)$, we now have to characterize the existence of a matrix T of size $q \times n$ such that $v = K T$ satisfies $A v = M$. See Remark 10.

Using (22) and the fact that X has full column rank, we have:

$$A v = (A K) T = X (B T) = M = X Y \iff B T = Y. \tag{26}$$

Using the fact that Y has a right inverse $H \in \mathbb{K}^{n\times l}$, i.e., $Y H = I_l$, (26) then implies that $B (T H) = I_l$, i.e., B has a right inverse. Conversely, if B has a right inverse E, then $T = E Y$ satisfies $B T = Y$. Hence, $B T = Y$ is equivalent to the existence of a right inverse E of B. Note that this right inverse E can have entries in a localization $\mathcal{T}$ of the ring S.

To characterize the solutions of the rank factorization problem (3), we have to determine when $B \in \mathcal{S}^{l\times q}$, defined by (24), has a right inverse with entries in a ring $\mathcal{T}$ containing $\mathcal{S}$.

Let $\mathcal{B} = \operatorname{coker}_{\mathcal{S}}(B.) = \mathcal{S}^{l\times 1}/(B\, \mathcal{S}^{q\times 1})$ be the $\mathcal{S}$-module finitely presented by $B.$, i.e., defined by the following finite presentation:

$$\mathcal{S}^{q\times 1} \xrightarrow{B.} \mathcal{S}^{l\times 1} \xrightarrow{\sigma} \mathcal{B} \longrightarrow 0. \tag{27}$$

Remark 15. Let us show that, up to isomorphism, the $\mathcal{S}$-module $\mathcal{B}$ does not depend on the choice of the bases defining L and X, and on the choice of a generating set of $\ker_{\mathcal{S}}(Q.)$ defining K. Let $L' \in \mathbb{K}^{p\times m}$ be a matrix defined by another basis of $\ker_{\mathbb{K}}(.M)$. Hence, there exists an invertible matrix $\Theta \in \mathbb{K}^{p\times p}$ such that $L' = \Theta L$. Similarly, let $X' \in \mathbb{K}^{m\times l}$ be a matrix defined by another basis of $\operatorname{im}_{\mathbb{K}}(M.)$. Hence, there exists an invertible matrix $\Gamma \in \mathbb{K}^{l\times l}$ such that $X' = X\,\Gamma$. Finally, let $Q' = L'\,A = \Theta\,Q$ and $K' \in \mathcal{S}^{r\times q'}$ be such that $\ker_{\mathcal{S}}(Q'.) = \mathrm{im}_{\mathcal{S}}(K'.)$. We clearly have $\ker_{\mathcal{S}}(Q'.) = \ker_{\mathcal{S}}(Q.)$, and thus, $\mathrm{im}_{\mathcal{S}}(K'.) = \mathrm{im}_{\mathcal{S}}(K.)$, which shows that there exists $\Lambda \in \mathcal{S}^{q\times q'}$ such that $K' = K\,\Lambda$. Let $B' \in \mathcal{S}^{l\times q'}$ be such that $A\,K' = X'\,B'$. Using $A\,K = X\,B$ and the fact that X has full column rank, we then get:

$$X\,\Gamma\,B' = X'\,B' = A\,K' = A\,K\,\Lambda = X\,B\,\Lambda \implies \Gamma\,B' = B\,\Lambda.$$

If $\mathcal{B}' = \mathrm{coker}_{\mathcal{S}}(B'.)$, then we have the following commutative exact diagram of $\mathcal{S}$-modules

$$\begin{array}{ccccccc}
\mathcal{S}^{q'\times 1} & \xrightarrow{B'.} & \mathcal{S}^{l\times 1} & \xrightarrow{\sigma'} & \mathcal{B}' & \longrightarrow & 0 \\
\downarrow{\scriptstyle \Lambda.} & & \downarrow{\scriptstyle \Gamma.} & & \downarrow{\scriptstyle \gamma} & & \\
\mathcal{S}^{q\times 1} & \xrightarrow{B.} & \mathcal{S}^{l\times 1} & \xrightarrow{\sigma} & \mathcal{B} & \longrightarrow & 0,
\end{array}$$

where $\gamma : \mathcal{B}' \longrightarrow \mathcal{B}$ is the $\mathcal{S}$-homomorphism defined by $\gamma(\sigma'(\mu)) = \sigma(\Gamma\,\mu)$ for all $\mu \in \mathcal{S}^{l\times 1}$. Since $\Gamma \in \mathbb{K}^{l\times l}$ is an invertible matrix, γ is an isomorphism and $\gamma^{-1}(\sigma(\nu)) = \sigma'(\Gamma^{-1}\,\nu)$ for all $\nu \in \mathcal{S}^{l\times 1}$. Hence, the ideals $\mathrm{Fitt}_i(\mathcal{B})$'s only depend on the matrices M and D_i's, and not on particular choices of bases or generating sets defining the matrices L, X and K. Finally, using Definition 1, we note that we have:

$$\mathrm{Fitt}_i(\mathcal{B}) = \mathrm{Fitt}_i\left(\mathrm{coker}_{\mathcal{S}}\left(.B^T\right)\right), \quad i = 0, \ldots, l. \tag{28}$$

Let $\mathcal{T}$ be a ring containing $\mathcal{S}$. Applying the right exact covariant functor $\mathcal{T} \otimes_{\mathcal{S}} \cdot$ to the above exact sequence [9, 24], we obtain the following exact sequence of $\mathcal{T}$-modules:

$$\mathcal{T}^{q\times 1} \xrightarrow{B.} \mathcal{T}^{l\times 1} \xrightarrow{\mathrm{id}_{\mathcal{T}} \otimes \sigma} \mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \longrightarrow 0. \tag{29}$$

Note that $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong \mathrm{coker}_{\mathcal{T}}(B.) = 0$, i.e., $B\,\mathcal{T}^{q\times 1} = \mathcal{T}^{l\times 1}$, if and only if there exists a matrix $E \in \mathcal{T}^{q\times l}$ satisfying $B\,E = I_l$, i.e., if and only if the matrix B has a right inverse with entries in $\mathcal{T}$. Therefore, we now have to investigate when the $\mathcal{T}$-module $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B}$ is reduced to 0.

We state again that a *prime ideal* $\mathfrak{p}$ of $\mathcal{T}$ is such that the factor ring $\mathcal{T}/\mathfrak{p}$ is an integral domain (i.e., has no non-trivial zero divisors). A $\mathcal{T}$-module $\mathcal{M}$ is said to be *projective of constant rank t* if the $\mathcal{T}_{\mathfrak{p}} = \{t/s \mid t \in \mathcal{T},\, s \notin \mathfrak{p}\}$-module $\mathcal{M}_{\mathfrak{p}} = \{m/s \mid m \in \mathcal{M},\, s \notin \mathfrak{p}\}$ (i.e., the localization of $\mathcal{M}$ at the multiplicatively closed set $S = \mathcal{T} \setminus \mathfrak{p}$) is a free module of rank r, i.e., $\mathcal{M}_{\mathfrak{p}} \cong \mathcal{T}_{\mathfrak{p}}^t$, for all prime ideals $\mathfrak{p}$ of $\mathcal{T}$. For more details, see [9, 19].

Proposition 1 (Proposition 20.7, [9])**.** *A finitely presented $\mathcal{T}$-module $\mathcal{M}$ is projective of constant rank r if and only if* $\mathrm{Fitt}_r(\mathcal{M}) = \mathcal{T}$ *and* $\mathrm{Fitt}_{r-1}(\mathcal{M}) = \langle 0\rangle$*.*

Note that 0 is a *projective* $\mathcal{T}$-module of rank 0. Hence, Proposition 1 shows that we have $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong \mathrm{coker}_{\mathcal{T}}(B.) = 0$ if and only if $\mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{T}}(B.)) = \mathcal{T}$.

The next proposition is a direct consequence of the right exact covariant functor $\mathcal{T} \otimes_{\mathcal{S}} \cdot$.

Proposition 2 (Corollary 20.5 of [9])**.** *Let $\mathcal{L}$ be a finitely presented $\mathcal{S}$-module and $\mathcal{T}$ a ring containing $\mathcal{S}$. Then, we have* $\mathrm{Fitt}_i(\mathcal{T} \otimes_{\mathcal{S}} \mathcal{L}) = \mathcal{T} \otimes_{\mathcal{S}} \mathrm{Fitt}_i(\mathcal{L})$*, for all $i \geq 0$, where* $\mathcal{T} \otimes_{\mathcal{S}} \mathrm{Fitt}_i(\mathcal{L})$ *denotes the ideal of $\mathcal{T}$ generated by the ideal* $\mathrm{Fitt}_i(\mathcal{L})$ *of $\mathcal{S}$.*

Using Proposition 2, we obtain that $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} \cong \mathrm{coker}_{\mathcal{T}}(B.) = 0$ if and only if we have:

$$\mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{T}}(B.)) = \mathcal{T} \otimes_{\mathcal{S}} \mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{S}}(B.)) = \mathcal{T} \otimes_{\mathcal{S}} \mathrm{Fitt}_0(\mathcal{B}) = \mathcal{T}.$$

Corollary 3. *With the above notations, the matrix $B \in \mathcal{S}^{l\times q}$ admits a right inverse with entries in a ring $\mathcal{T}$ containing $\mathcal{S}$ if and only if the ideal of $\mathcal{T}$ generated by* $\mathrm{Fitt}_0(\mathcal{B})$ *is $\mathcal{T}$.*

Let $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \langle h_1, \ldots, h_\beta\rangle_{\mathcal{S}}$, where $\{h_i\}_{i=1,\ldots,\beta}$ is a set of generators of $\mathcal{I}$.

If $\mathcal{I} = \langle 0\rangle$, i.e., $h_1 = \ldots = h_\beta = 0$, then no right inverse of B exists, and thus, Problem (3) has no solutions. Now, suppose that $\mathcal{I} \neq \langle 0\rangle$ and $h_i \neq 0$ for $i = 1, \ldots, \beta$. According to Corollary 3, the matrix B has a right inverse with entries in $\mathcal{T}$ if and only if there exist $t_i \in \mathcal{T}$, $i = 1, \ldots, \beta$, satisfying the following Bézout equation:

$$\sum_{i=1}^{\beta} t_i\,h_i = 1. \tag{30}$$

Remark 16. Note that $\{h_i\}_{i=1,\ldots,\beta}$ defines a *partition of unity* of the *spectrum* $\mathrm{Spec}(\mathcal{T})$ formed by all the prime ideals $\mathfrak{p}$ of $\mathcal{T}$ and endowed with the Zariski topology, and $\mathcal{T} \otimes_{\mathcal{S}} \mathcal{B} = 0$ if and only if $\mathcal{T}_{h_i} \otimes_{\mathcal{S}} \mathcal{B} = 0$ for $i = 1, \ldots, \beta$, where $\mathcal{T}_{h_i}$ denotes the localization of $\mathcal{T}$ at the closed multiplicatively closed $\{h_i^k \mid k \in \mathbb{Z}_{\geq 0}\}$ (see Exercises 2.19 and 2.20 of [9]).

Remark 17. As recalled after Theorem 1 (see [9, 22]), we have:

$$\mathrm{ann}_{\mathcal{S}}(\mathcal{B})^l \subseteq \mathrm{Fitt}_0(\mathcal{B}) \subseteq \mathrm{ann}_{\mathcal{S}}(\mathcal{B}) = \{s \in \mathcal{S} \mid \forall\, b \in \mathcal{B} : s\, b = 0\} \implies \sqrt{\mathrm{ann}_{\mathcal{R}}(\mathcal{B})} = \sqrt{\mathrm{Fitt}_0(\mathcal{B})}.$$

In particular, we have $h_i\, \mathcal{B} = 0$ for $i = 1, \ldots, \beta$, and $\mathrm{ann}_{\mathcal{S}}(\mathcal{B}) = \langle 0 \rangle$ if and only if $\mathrm{Fitt}_0(\mathcal{B}) = \langle 0 \rangle$ since $\mathrm{ann}_{\mathcal{R}}(\mathcal{B}) \subseteq \sqrt{\mathrm{ann}_R(\mathcal{B})}$.

Finally, generators of $\mathrm{ann}_{\mathcal{S}}(\mathcal{B})$ can be computed by, e.g., `Singular` and `CapAndHomalg`.

We suppose that h_i is not a *nilpotent* element of $\mathcal{S}$, namely, $h_i^k \neq 0$ for all $k \in \mathbb{Z}_{\geq 0}$. Then, we can consider $t_i = h_i^{-1}$ and $t_j = 0$ for $i \neq j$ in (30). Let $\mathcal{S}_{h_i} = \{s/h_i^k \mid s \in \mathcal{S},\ k \in \mathbb{Z}_{\geq 0}\}$ be the localization of the ring $\mathcal{S}$ at the multiplicatively closed set $\{h_i^k\}_{k \in \mathbb{Z}_{\geq 0}}$, and $\mathcal{S}_{h_i}^{-1}\mathcal{B} = \{b/h_i^k \mid b \in \mathcal{B},\ k \in \mathbb{Z}_{\geq 0}\}$ the $\mathcal{S}_{h_i}$-module defined by the localization of $\mathcal{B}$ at $\{h_i^k\}_{k \in \mathbb{Z}_{\geq 0}}$ [9, 24]. Using $h_i\, \mathcal{B} = 0$ (see Remark 17), we have $\mathcal{S}_{h_i}^{-1}\mathcal{B} = 0$. Localizing (27), i.e., applying the exact functor $\mathcal{S}_{h_i} \otimes_{\mathcal{S}} \cdot$ to (27) ($\mathcal{S}_{h_i}$ is a flat $\mathcal{S}$-module), we get the following split exact sequence of $\mathcal{S}_{h_i}$-modules [9, 24]

$$\mathcal{S}_{h_i}^{t \times 1} \underset{F_{h_i}.}{\overset{C_{h_i}.}{\rightleftarrows}} \mathcal{S}_{h_i}^{q \times 1} \underset{E_{h_i}.}{\overset{B.}{\rightleftarrows}} \mathcal{S}_{h_i}^{l \times 1} \longrightarrow \mathcal{S}_{h_i}^{-1}\mathcal{B} = 0,$$

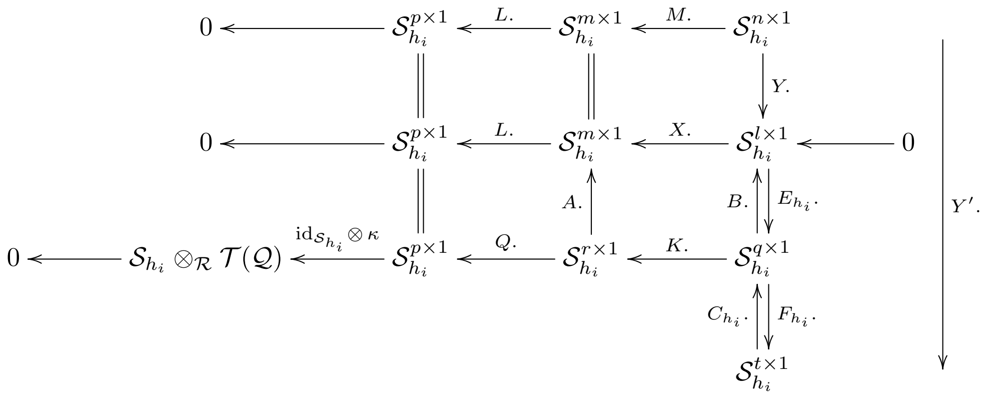
where the matrices $C_{h_i} \in \mathcal{S}_{h_i}^{q \times t}$ is such that $\ker_{\mathcal{S}_{h_i}}(B.) = \mathrm{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$, $E_{h_i} \in \mathcal{S}_{h_i}^{q \times l}$ is a right inverse of B, i.e., $B\, E_{h_i} = I_l$, and $F_{h_i} \in \mathcal{S}_{h_i}^{t \times q}$ is such that $C_{h_i}\, F_{h_i} + E_{h_i}\, B = I_q$.

Remark 18. The matrices C_{h_i}, E_{h_i} and F_{h_i} can be computed by `CapAndHomalg` [1].

We note that $T_p = E_{h_i}\, Y \in \mathcal{S}_{h_i}^{q \times n}$ is a particular solution of $B\, T = Y$. If $T \in \mathcal{S}_{h_i}^{q \times n}$ is another solution of $B\, T = Y$, then $B\, (T - T_p) = 0$, and thus, there exists $Y' \in \mathcal{S}_{h_i}^{t \times n}$ such that $T = E_{h_i}\, Y + C_{h_i}\, Y'$. Conversely, $E_{h_i}\, Y + C_{h_i}\, Y'$ are solutions of $B\, T = Y$ for all $Y' \in \mathcal{S}_{h_i}^{t \times n}$. Hence, $T(Y') = E_{h_i}\, Y + C_{h_i}\, Y'$ defines the general solution of $B\, T = Y$ for all $Y' \in \mathcal{S}_{h_i}^{t \times n}$. Thus, all the solutions $v \in \mathcal{S}_{h_i}^{r \times n}$ of $A\, v = M$ are of the form:

$$\forall\, Y' \in \mathcal{S}_{h_i}^{t \times n}, \quad v_{h_i}(Y') = K\, (E_{h_i}\, Y + C_{h_i}\, Y').$$

We have the following commutative exact diagram of $\mathcal{S}_{h_i}$-modules



(31)

up to the fact that the homomorphisms $Y.$ and $E_{h_i}.$ do not form a complex.

Using that $\mathcal{S} = \mathcal{R}/\mathcal{J}$, with the notation (18), we can write $h_i = \overline{g_i}$, where $g_i \in \mathcal{R} \setminus \mathcal{J}$.

Remark 19. If $\mathcal{R}_{g_i} = \{a/g_i^k \mid a \in \mathcal{R},\, k \in \mathbb{Z}_{\geq 0}\}$ is the localization of $\mathcal{R}$ at the multiplicatively closed set $\{g_i^k\}_{k \in \mathbb{Z}_{\geq 0}}$ and $\mathcal{J}_{g_i}$ is the ideal of $\mathcal{R}_{g_i}$ generated by $\mathcal{J}$, then we can check that $\rho : \mathcal{S}_{h_i} \longrightarrow \mathcal{R}_{g_i}/\mathcal{J}_{g_i}$ defined by

$$\forall\, s = r + \mathcal{J} \in \mathcal{S}, \quad r \in \mathcal{R}, \quad \rho\left(\frac{s}{h_i^k}\right) = \rho\left(\frac{r + \mathcal{J}}{g_i^k + \mathcal{J}}\right) = \frac{r}{g_k^k} + \mathcal{J}_{g_i}$$

is a ring isomorphism. For more details, see, e.g., Rule 4.16 on page 83 of [19].

The u's corresponding to v_{h_i} are then the elements of $V_{\mathbb{K}}(\mathcal{J})$ which are such that $g_i(u) \neq 0$, i.e., $u \in V_{\mathbb{K}}(\mathcal{J}) \setminus (V_{\mathbb{K}}(\mathcal{J}) \cap V_{\mathbb{K}}(\langle g_i \rangle))$, i.e., $u \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle)$, where $\mathcal{J} + \langle g_i \rangle$ denotes the ideal of $\mathcal{R}$ generated by $\mathcal{J}$ and g_i. Note that if $h_i = \overline{g_i'}$, where $g_i' \in \mathcal{R}$, then $\mathcal{J} + \langle g_i \rangle = \mathcal{J} + \langle g_i' \rangle$, and thus, $V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle)$ does not depend on the choice of the representative g_i of h_i. Finally, we note also that $g_i \notin \sqrt{\mathcal{J}}$ since h_i is not a nilpotent element of $\mathcal{S}$.

Remark 20. If h_i is an nilpotent element of $\mathcal{S}$, then $h_i^k = 0$ for a certain integer k, which yields $g_i^k \in \mathcal{J}$, i.e., $g_i \in \sqrt{\mathcal{J}}$. Thus, we have $V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle) = V_{\mathbb{K}}(\mathcal{J})$, $V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle) = \emptyset$, i.e., there is no solution, which is consistent with the fact that $\mathcal{S}_{h_i}$ is then the trivial ring 0.

Hence, the solutions of (3) are then defined by:

$$\begin{cases} u = x \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle), \\ v_{h_i}(Y') = K\,(E_{h_i}\,Y + C_{h_i}\,Y'), \ \forall\, Y' \in \mathcal{S}_{h_i}^{t \times n}, \quad i = 1, \ldots, \beta. \end{cases}$$

Remark 21. If $\mathbb{K}$ is an algebraic closed field, then the smallest affine algebraic set containing the quasi-affine algebraic set $V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle)$, namely, its *Zariski closure* is defined by

$$\overline{V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle)} = V_{\mathbb{K}}(\mathcal{J} : (\mathcal{J} + \langle g_i \rangle)^{\infty}),$$

where $\mathcal{J} : (\mathcal{J} + \langle g_i \rangle)^{\infty}$ denotes the *saturation* of $\mathcal{J}$ with respect to $\mathcal{J} + \langle g_i \rangle$ defined by:

$$\begin{aligned} \mathcal{J} : (\mathcal{J} + \langle g_i \rangle)^{\infty} &= \left\{ a \in \mathcal{R} \mid \exists\, k \in \mathbb{Z}_{\geq 0} : \, a\,(\mathcal{J} + \langle g_i \rangle)^k \subseteq \mathcal{J} \right\} \\ &= \left\{ a \in \mathcal{R} \mid \exists\, k \in \mathbb{Z}_{\geq 0} : \, a\,g_i^k \in \mathcal{J} \right\}. \end{aligned}$$

See, e.g., [9, 19]. This saturation can be computed with, e.g., `Singular` or `CapAndHomalg`. Finally, we note that $g_i^k \notin \mathcal{J}$ for all $k \in \mathbb{Z}_{\geq 0}$ since $g_i \notin \sqrt{\mathcal{J}}$, and thus, $1 \notin \mathcal{J} : (\mathcal{J} + \langle g_i \rangle)^{\infty}$, i.e., $\mathcal{J} : (\mathcal{J} + \langle g_i \rangle)^{\infty}$ is not equal to $\mathcal{R}$, which shows that $\overline{V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle)} \neq \emptyset$.

Let us note $\mathcal{I}_{\mathcal{R}} = \langle g_1, \ldots, g_\beta \rangle_{\mathcal{R}}$ the ideal of $\mathcal{R}$ generated by the g_i's. Hence, we have:

$$\begin{aligned} \mathcal{U} = V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \mathcal{I}_{\mathcal{R}}) &= V_{\mathbb{K}}(\mathcal{J}) \setminus (V_{\mathbb{K}}(\mathcal{J}) \cap V_{\mathbb{K}}(\mathcal{I}_{\mathcal{R}})) = V_{\mathbb{K}}(\mathcal{J}) \setminus \bigcap_{i=1}^{\beta} (V_{\mathbb{K}}(\mathcal{J}) \cap V_{\mathbb{K}}(\langle g_i \rangle)) \\ &= V_{\mathbb{K}}(\mathcal{J}) \setminus \bigcap_{i=1}^{\beta} V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle) = \bigcup_{i=1}^{\beta} V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle) = \bigcup_{i \in I} V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle), \end{aligned}$$

where I denotes the set formed by the g_i's for $i = 1, \ldots, \beta$ which are such that $g_i \notin \sqrt{\mathcal{J}}$ since, otherwise, we have $V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle) = \emptyset$ by Remark 20.

Let us now sum up the results obtained above in the following theorem.

Theorem 3. *Let $D_i \in \mathbb{K}^{m \times m}$ for $i = 1, \ldots, r$ and $M \in \mathbb{K}^{m \times n}$ be such that:*

$$l = \operatorname{rank}_{\mathbb{K}}(M) \leq \min\{m, r\}.$$

Let $X \in \mathbb{K}^{m \times l}$ be a full column rank matrix such that $\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$, i.e., the columns of X define a basis of $\operatorname{im}_{\mathbb{K}}(M.)$. Let $V \in \mathbb{K}^{l \times m}$ be any left inverse of X. Moreover, let $Y \in \mathbb{K}^{l \times n}$ be such that $M = X\,Y$. Hence, $\ker_{\mathbb{K}}(M.) = \ker_{\mathbb{K}}(Y.)$ and Y has full row rank.

Let $L \in \mathbb{K}^{(m-l) \times m}$ be a full row matrix whose rows define a basis of $\ker_{\mathbb{K}}(.M)$ (with the convention that $L = 0$ if $l = m$), $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1 \ \ldots \ x_m)^T$, $A = (D_1\,x \ \ldots \ D_r\,x) \in \mathcal{R}^{m \times r}$, $Q = L\,A \in R^{p \times r}$, $\mathcal{Q} = \mathcal{R}^{1 \times r}/(\mathcal{R}^{1 \times p}\,Q)$, $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q})$ the 0^{th} Fitting ideal of the $\mathcal{R}$-module $\mathcal{Q}$ ($\mathcal{J} = \langle 0 \rangle$ if $r > p$ or if $l = m$), and $\mathcal{S} = \mathcal{R}/\mathcal{J}$. Then, we have $\ker_{\mathcal{S}}(Q.) \neq 0$.

Let $K \in \mathcal{S}^{r \times q}$ be such that $\ker_{\mathcal{S}}(Q.) = \operatorname{im}_{\mathcal{S}}(K.)$, $B = V\,A\,K \in \mathcal{S}^{l \times q}$, $\mathcal{B} = \mathcal{S}^{l \times 1}/(B\,\mathcal{S}^{q \times 1})$ be the $\mathcal{S}$-module finitely presented by B and $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B})$ defined by (28). Then, we have:

1. *If $\mathcal{I} = \langle 0 \rangle$, then Problem (3) has no solution.*

2. *If $\mathcal{I} = \langle h_1, \ldots, h_\beta \rangle_{\mathcal{S}} \neq \langle 0 \rangle$, then for all the h_i's, $i = 1, \ldots, \beta$, which are not nilpotent, we have $\mathcal{S}_{h_i} \otimes_{\mathcal{S}} \mathcal{B} = 0$, where $\mathcal{S}_{h_i} = \{s/h_i^k \mid s \in S, \, k \in \mathbb{Z}_{\geq 0}\}$ is the localization of $\mathcal{S}$ at the multiplicatively set $\{h_i^k \mid k \in \mathbb{Z}_{\geq 0}\}$, i.e., B has a right inverse $E_{h_i} \in \mathcal{S}_{h_i}^{q \times l}$, namely, $B \, E_{h_i} = I_l$. Let $C_{h_i} \in \mathcal{S}_{h_i}^{q \times t}$ be such that $\ker_{\mathcal{S}_{h_i}}(B.) = \operatorname{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$. Let $g_i \in \mathcal{R} \setminus \mathcal{J}$ be such that its residue class $\overline{g_i}$ in $\mathcal{S} = \mathcal{R}/\mathcal{J}$ satisfies $\overline{g_i} = h_i$. Then, the commutative exact diagram (31) holds and solutions of Problem (3) are of the form:*

$$\begin{cases} u = x \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle), \\ v_{h_i}(Y') = K \, (E_{h_i} \, Y + C_{h_i} \, Y'), \; \forall \, Y' \in \mathcal{S}_{h_i}^{t \times n}. \end{cases} \tag{32}$$

Moreover, if we denote by $\mathcal{I}_{\mathcal{R}} = \langle g_1, \ldots, g_\beta \rangle_{\mathcal{R}}$ the ideal of $\mathcal{R}$ generated by the g_i's, then the solution exists on the following quasi-affine algebraic set

$$\mathcal{U} = V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \mathcal{I}_{\mathcal{R}}) = \bigcup_{i \in I} V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_i \rangle),$$

where I denotes the set formed by the g_i's for $i = 1, \ldots, \beta$ which are such that $g_i \notin \sqrt{\mathcal{J}}$. Finally, the rank factorization problem has no solution if and only if $I = \emptyset$.

Remark 22. The results of Section 3.1 obtained for a full column rank matrix M can be seen as a particular case of Theorem 3. Indeed, if M has full row rank, then $L = 0$ and $X = I_m$, which yields $Q = L \, A = 0$ and $\ker_{\mathcal{R}}(Q.) = \mathcal{R}^{r \times 1}$, i.e., $K = I_r$, and thus, $B = A$, $\mathcal{B} = \mathcal{A}$, $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q) = \mathcal{R}^{1 \times r}$, $\operatorname{ann}_{\mathcal{R}}(\mathcal{Q}) = \langle 0 \rangle$, which, using since $\operatorname{Fitt}_0(\mathcal{Q}) \subseteq \operatorname{ann}_{\mathcal{R}}(\mathcal{Q})$, implies that $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q}) = \langle 0 \rangle$, and thus, $\mathcal{S} = \mathcal{R}$ and:

$$\mathcal{U} = V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{I} + \mathcal{J}) = \mathbb{K}^{m \times 1} \setminus V_{\mathbb{K}}(\mathcal{I}), \quad \mathcal{I} = \mathcal{I}_{\mathcal{R}} = \operatorname{Fitt}_0(\mathcal{A}) = \langle h_1, \ldots, h_s \rangle.$$

Thus, Problem (3) has a solution if and only if $V_{\mathbb{K}}(\mathcal{I}) \neq \mathbb{K}^{m \times 1}$, which, in the case of an infinite algebraic closed field $\mathbb{K}$, is equivalent to $\mathcal{I} \neq \langle 0 \rangle$.

In Algorithm 1, we present a pseudocode summarizing Theorem 3. It can easily be implemented in standard computer algebra systems such as, for instance, `Singular` or `CapAndHomalg`. For an explicit example handled with `CapAndHomalg`, see Appendix of [8].

Example 7. Let us consider the following matrices:

$$M = \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then, we have $m = n = 2$, $l = 1 < r = 2$, $p = m - l = 1 < r$, and:

$$L = (3 \quad -2), \quad X = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & \frac{3}{2} \end{pmatrix}.$$

For all $u = (u_1 \quad u_2)^T \in \mathbb{K}^{2 \times 1}$, we have:

$$A(u) = (D_1 \, u \quad D_2 \, u) = \begin{pmatrix} u_1 & u_1 \\ u_2 & 2 \, u_2 \end{pmatrix}, \quad L \, A(u) = (3 \, u_1 - 2 \, u_2 \quad 3 \, u_1 - 4 \, u_2).$$

Now, $L \, A(u) = 0$ yields $u = 0$, which shows that no solutions $(u, \, v)$ of Problem (3) exist where v have full row rank. Hence, the results stated in Section Section 2 cannot be used.

Let us characterize the general solutions of Problem (3). Let us note $\mathcal{R} = \mathbb{Q}[x_1, x_2]$, $x = (x_1 \quad x_2)^T$, $A = (D_1 \, x \quad D_2 \, x) \in \mathcal{R}^{2 \times 2}$, $Q = L \, A \in \mathcal{R}^{1 \times 2}$, and $\mathcal{Q} = \mathcal{R}^{1 \times 2}/(\mathcal{R} \, Q)$. We can check again that $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q}) = \langle 0 \rangle$ and $\operatorname{Fitt}_1(\mathcal{Q}) = \langle 3 \, x_1 - 2 \, x_2, 3 \, x_1 - 4 \, x_2 \rangle = \mathcal{R}$. Hence,we have $V_{\mathbb{K}}(\operatorname{Fitt}_0(\mathcal{Q})) = \mathbb{K}^{2 \times 1}$ and $\mathcal{S} = \mathcal{R}/\mathcal{J} = \mathcal{R}$. Now, we can check again that $\ker_{\mathcal{R}}(Q.) = \operatorname{im}_{\mathcal{R}}(K.)$, where $K \in \mathcal{R}^{2 \times 1}$ is defined by:

$$K = \begin{pmatrix} -3 \, x_1 + 4 \, x_2 \\ 3 \, x_1 - 2 \, x_2 \end{pmatrix}.$$

Algorithm 1 RankFactorizationProblem

procedure RankFactorizationProblem($D_1 \in \mathbb{K}^{m\times m}, \ldots, D_r \in \mathbb{K}^{m\times m}, M \in \mathbb{K}^{m\times n}$)
  Define $\mathcal{R} = \mathbb{K}[x_1, \ldots, x_m]$, $x = (x_1 \ \ldots \ x_m)^T$, $A = (D_1\, x \ \ldots \ D_r\, x) \in \mathcal{R}^{m\times r}$
  Compute a basis of $\ker_{\mathbb{K}}(.M)$ to get a full row rank matrix $L \in \mathbb{K}^{p\times m}$ satisfying

$$\ker_{\mathbb{K}}(.M) = \operatorname{im}_{\mathbb{K}}(.L)$$

  Define $Q = L\, A \in \mathcal{R}^{p\times r}$ and the finitely presented $\mathcal{R}$-module $\mathcal{Q} = \operatorname{coker}_{\mathcal{R}}(.Q)$
  Compute the ideal $\mathcal{J} = \operatorname{Fitt}_0(\mathcal{Q})$ and define the ring $\mathcal{S} = \mathcal{R}/\mathcal{J}$
  Compute $K \in \mathcal{S}^{r\times q}$ such that $\ker_{\mathcal{S}}(Q.) = \operatorname{im}_{\mathcal{S}}(K.)$
  Compute a basis of $\operatorname{im}_{\mathbb{K}}(M.)$ to get a full column rank matrix $X \in \mathbb{K}^{m\times l}$ satisfying

$$\operatorname{im}_{\mathbb{K}}(M.) = \operatorname{im}_{\mathbb{K}}(X.)$$

  Compute a full row rank matrix $Y \in \mathbb{K}^{l\times n}$ such that $M = X\, Y$
  Compute a left inverse $V \in \mathbb{K}^{l\times m}$ of X
  Define $B = V\, A\, K \in \mathcal{S}^{l\times q}$ and the finitely presented $\mathcal{S}$-module $\mathcal{B} = \operatorname{coker}_{\mathcal{S}}(B.)$
  Compute the ideal $\mathcal{I} = \operatorname{Fitt}_0(\mathcal{B}) = \langle h_1, \ldots, h_\beta \rangle$, where $h_1, \ldots, h_\beta \in \mathcal{S}$ (see (28))
  if $\mathcal{I} = \langle 0 \rangle$ **then** the rank factorization problem has no solution
  else
    $I = \emptyset$
    for $i \leftarrow 1, \ldots, \beta$ **do**
      if h_i is nilpotent **then** $i := i + 1$
      else
        Define the localization $\mathcal{S}_{h_i}$ of $\mathcal{S}$ at the multiplicatively closed set $\{h_i^k\}_{k \in \mathbb{Z}_{\geq 0}}$
        Compute a right inverse $E_{h_i} \in \mathcal{S}_{h_i}^{q\times l}$ of $\mathcal{B}$
        Compute $C_{h_i} \in \mathcal{S}_{h_i}^{q\times t}$ such that $\ker_{\mathcal{S}_{h_i}}(B.) = \operatorname{im}_{\mathcal{S}_{h_i}}(C_{h_i}.)$
        Define $g_i \in \mathcal{R}$ to be an element in the residue class of $h_i \in \mathcal{S}$ and $I := I \cup \{g_i\}$
      end if
    end for
  end if
  return $\mathcal{I}$, $\mathcal{J}$, K, Y, I, $\{E_{h_i}\}_{i\in I}$ and $\{C_{h_i}\}_{i\in I}$ defining the solutions (32)
end procedure

Using the left inverse $V = (1/4 \quad 0)$ of X, we then obtain:

$$A\,K = \left(\begin{array}{c} 2\,x_1\,x_2 \\ 3\,x_1\,x_2 \end{array}\right) \Longrightarrow B = V\,A\,K = \frac{1}{2}\,x_1\,x_2.$$

Hence, we get the following commutative exact diagram of $\mathcal{R}$-modules:

$$\begin{array}{ccccccccc}
 & & 0 & \longleftarrow & \mathcal{R} & \xleftarrow{L.} & \mathcal{R}^{2\times 1} & \xleftarrow{M.} & \mathcal{R}^{2\times 1} & & \\
 & & & & \| & & \| & & \downarrow Y. & & \\
 & & 0 & \longleftarrow & \mathcal{R} & \xleftarrow{L.} & \mathcal{R}^{2\times 1} & \xleftarrow{X.} & \mathcal{R} & \longleftarrow & 0 \\
 & & & & \| & & A.\uparrow & & B.\uparrow & & \\
0 & \longleftarrow & \mathcal{T}(\mathcal{Q}) & \longleftarrow & \mathcal{R} & \xleftarrow{Q.} & \mathcal{R}^{2\times 1} & \xleftarrow{K.} & \mathcal{R} & \longleftarrow & 0.
\end{array}$$

Let us now consider the *cyclic* $\mathcal{R}$-module $\mathcal{B} = \mathcal{R}/\langle B\rangle = \mathbb{K}[x_1, x_2]/\langle x_1\,x_2\rangle$. Then, we get $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \langle B\rangle = \langle x_1\,x_2\rangle$, which shows that $h_1 = x_1\,x_2$ and $E_{h_1} = B^{-1} = 1/(x_1\,x_2)$. Since $\ker_{\mathcal{R}}(B.) = 0$, the solutions of Problem (3) are of the form:

$$\left\{\begin{array}{l} u = (x_1 \quad x_2)^T \in \mathcal{U} = \mathbb{K}^{2\times 1} \setminus V_{\mathbb{K}}(\langle x_1\,x_2\rangle) = (\mathbb{K} \times \mathbb{K}^{\times}) \times (\mathbb{K}^{\times} \times \mathbb{K}), \\ v = K\,E_{h_1}\,Y = \dfrac{1}{2\,x_1\,x_2}\left(\begin{array}{cc} 2\,(-3\,x_1 + 4\,x_2) & 3\,(-3\,x_1 + 4\,x_2) \\ 2\,(3\,x_1 - 2\,x_2) & 3\,(3\,x_1 - 2\,x_2) \end{array}\right). \end{array}\right.$$

We can verify that $(3\,x_1 - 2\,x_2 \quad 3\,x_1 - 4\,x_2)\,v = 0$, which shows again that all the solutions (u, v) of Problem (3) are such that the v's have not full row rank. Finally, we note that $\overline{\mathcal{U}} = V_{\mathbb{K}}(\langle 0\rangle : \langle x_1\,x_2\rangle^{\infty})$, where $\langle 0\rangle : \langle x_1\,x_2\rangle^{\infty} = \langle 0\rangle$, and thus, we find again that $\overline{\mathcal{U}} = \mathbb{K}^{2\times 1}$.

Example 8. Let us consider the following matrices:

$$D_1 = \left(\begin{array}{cccc} 0 & 0 & 0 & 2 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right), \; D_2 = \left(\begin{array}{cccc} 5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 3 & 2 & 0 \end{array}\right), \; M = \left(\begin{array}{ccc} 30 & 0 & 0 \\ 0 & 0 & 0 \\ 12 & 0 & 0 \\ 12 & 0 & 0 \end{array}\right).$$

We have $m = 4$, $n = 3$, $l = 1 \leq r = 2$ and $p = m - l = 3 > r$. Moreover, we get:

$$L = \left(\begin{array}{cccc} 2 & 0 & 0 & -5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right), \quad X = \left(\begin{array}{c} 5 \\ 0 \\ 2 \\ 2 \end{array}\right), \quad Y = (6 \quad 0 \quad 0).$$

Now, set $\mathcal{R} = \mathbb{Q}[x_1, x_2, x_3, x_4]$, $x = (x_1 \; \ldots \; x_4)^T$,

$$A = (D_1\,x \quad D_2\,x) = \left(\begin{array}{cc} 2\,x_4 & 5\,x_1 + 3\,x_2 \\ 3\,x_1 + x_4 & 0 \\ 0 & 5\,x_2 + 2\,x_3 \\ 2\,x_4 & 3\,x_2 + 2\,x_3 \end{array}\right) \in R^{4\times 2},$$

$$Q = L\,A = \left(\begin{array}{cc} -6\,x_4 & 10\,x_1 - 9\,x_2 - 10\,x_3 \\ 3\,x_1 + x_4 & 0 \\ -2\,x_4 & 2\,x_2 \end{array}\right) \in R^{3\times 2},$$

and let $\mathcal{Q} = \mathcal{R}^{1\times 2}/(\mathcal{R}^{1\times 3}\,Q)$ be the $\mathcal{R}$-module finitely presented by Q. Then, we have:

$$\begin{array}{l} \mathrm{Fitt}_0(\mathcal{Q}) = \langle (10\,x_1 - 9\,x_2 - 10\,x_3)\,(3\,x_1 + x_4), (2\,x_1 - 3\,x_2 - 2\,x_3)\,x_4, (3\,x_1 + x_4)\,x_2\rangle, \\ \mathrm{Fitt}_1(\mathcal{Q}) = \langle x_1, \, x_2, \, x_3, \, x_4\rangle. \end{array}$$

A necessary condition for the existence of a solution of Problem (3) is then $u \in V_{\mathbb{K}}(\mathrm{Fitt}_0(\mathcal{Q}))$. For instance, we can take $u = (u_1 \quad u_2 \quad u_1 - 3\,u_2/2 \quad -3\,u_1)^T$, $u = (u_1 \quad 0 \quad u_1 \quad u_4)^T$ or $u = (0 \quad u_2 \quad u_3 \quad 0)^T$. Let us note $\mathcal{J} = \mathrm{Fitt}_0(\mathcal{Q})$, $\mathcal{S} = \mathcal{R}/\mathcal{J}$ and $\overline{x_i}$ the residue class of x_i in $\mathcal{S}$ for $i = 1, \ldots, 4$. Then, we can check again that the following matrix

$$K = \begin{pmatrix} \overline{x_2} & 0 & 2\,\overline{x_1} - 2\,\overline{x_3} \\ \overline{x_4} & 3\,\overline{x_1} + \overline{x_4} & 3\,\overline{x_4} \end{pmatrix} \in \mathcal{S}^{2\times 3}$$

is such that $\ker_{\mathcal{S}}(Q.) = \mathrm{im}_{\mathcal{S}}(K.)$. We then get the identity $A\,K = X\,B$, where:

$$B = \begin{pmatrix} \frac{5}{2}\,\overline{x_2}\,\overline{x_4} + \overline{x_3}\,\overline{x_4} & 3\,\overline{x_1}\,\overline{x_3} + \overline{x_3}\,\overline{x_4} & \frac{15}{2}\,\overline{x_2}\,\overline{x_4} + 3\,\overline{x_3}\,\overline{x_4} \end{pmatrix} \in \mathcal{S}^{1\times 3}.$$

Let us denote by $\mathcal{B} = \mathcal{S}/(B\,\mathcal{S}^{3\times 1}) = \mathcal{S}/\langle B_1, B_2, B_3\rangle_{\mathcal{S}}$ the $\mathcal{S}$-module finitely presented by B, where B_i stands for the i^{th} entry of B. Then, using $B_3 = 3\,B_1$, we have $\mathcal{I} = \mathrm{Fitt}_0(\mathcal{B}) = \langle B_1,\, B_2\rangle_{\mathcal{S}}$. Thus, we have to consider the following two cases:

- If $h_1 = B_1$, then $E_{h_1} = (h_1^{-1} \quad 0 \quad 0)^T \in \mathcal{S}_{h_1}^{3\times 1}$, where $\mathcal{S}_{h_1} = \{s/h_1^k \mid s \in \mathcal{S},\, k \in \mathbb{Z}_{\geq 0}\}$, satisfies $B\,E_{h_1} = 1$. If $g_1 = x_4\,(5\,x_2/2 + x_3) \in \mathcal{R}$, i.e., $B_1 = \overline{g_1}$, then

$$v = K\,E_{h_1}\,Y = \begin{pmatrix} \dfrac{6\,\overline{x_2}}{h_1} & 0 & 0 \\ \dfrac{6\,\overline{x_4}}{h_1} & 0 & 0 \end{pmatrix}$$

  is such that $u \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_1\rangle)$ and v is a solution of Problem (3). More generally, if we consider the following matrix

$$C_{h_1} = \begin{pmatrix} 3 & 0 \\ 0 & -6\,\overline{x_4} \\ -1 & 9\,\overline{x_2} + 6\,\overline{x_3} + 2\,\overline{x_4} \end{pmatrix} \in \mathcal{S}_{h_1}^{3\times 2}$$

  which satisfies $\ker_{\mathcal{S}_{h_1}}(B.) = \mathrm{im}_{\mathcal{S}_{h_1}}(C_{h_1}.)$, then a solution $(u,\, v)$ of Problem (3) is defined by $u \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_1\rangle)$ and $v = K\,(E_{h_1}\,Y + C_{h_1}\,Y')$ for all $Y' \in \mathcal{S}_{h_1}^{2\times 3}$.

- If $h_2 = B_2$, then $E_{h_2} = (0 \quad h_2^{-1} \quad 0)^T \in \mathcal{S}_{h_2}^{3\times 1}$, where $\mathcal{S}_{h_2} = \{s/h_2^k \mid s \in \mathcal{S},\, k \in \mathbb{Z}_{\geq 0}\}$, satisfies $B\,E_{h_2} = 1$. If $g_2 = x_3\,(3\,x_1 + x_4)$, i.e., $B_2 = \overline{g_2}$, then

$$v = K\,E_{h_2}\,Y = \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{6\,(3\,\overline{x_3} + \overline{x_4})}{h_2} & 0 & 0 \end{pmatrix}$$

  is such that $u = x \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_2\rangle)$ and v form a solution of Problem (3). More generally, if we consider the following matrix

$$C_{h_2} = \begin{pmatrix} 3 & 0 \\ 0 & -3\,\overline{x_4} \\ -1 & 3\,\overline{x_3} + \overline{x_4} \end{pmatrix} \in \mathcal{S}_{h_2}^{3\times 2}$$

  which satisfies $\ker_{\mathcal{S}_{h_2}}(B.) = \mathrm{im}_{\mathcal{S}_{h_2}}(C_{h_2}.)$, then a solution $(u,\, v)$ of Problem (3) is defined by $u = x \in V_{\mathbb{K}}(\mathcal{J}) \setminus V_{\mathbb{K}}(\mathcal{J} + \langle g_2\rangle)$ and $v = K\,(E_{h_2}\,Y + C_{h_2}\,Y')$ for all $Y' \in \mathcal{S}_{h_2}^{2\times 3}$.

Moreover, we have the following commutative diagram of S_{h_i}-modules:

$$\begin{array}{ccccccccccc}
 & & 0 & \longleftarrow & \mathcal{S}_{h_i}^{3\times 1} & \xleftarrow{L.} & \mathcal{S}_{h_i}^{4\times 1} & \xleftarrow{M.} & \mathcal{S}_{h_i}^{3\times 1} & & \\
 & & & & \| & & \| & & \downarrow Y. & & \\
 & & 0 & \longleftarrow & \mathcal{S}_{h_i}^{3\times 1} & \xleftarrow{L.} & \mathcal{S}_{h_i}^{4\times 1} & \xleftarrow{X.} & \mathcal{S}_{h_i} & \longleftarrow & 0 \\
 & & & & \| & & A. \uparrow & & B. \uparrow\downarrow E_{h_i}. & & \\
0 & \longleftarrow & \mathcal{S}_{h_i} \otimes_{\mathcal{R}} \mathcal{T}(\mathcal{Q}) & \longleftarrow & \mathcal{S}_{h_i}^{3\times 1} & \xleftarrow{Q.} & \mathcal{S}^{2\times 1} & \xleftarrow{K.} & \mathcal{S}_{h_i}^{3\times 1} & \longleftarrow & 0.
\end{array}$$

Finally, note that $\mathrm{Fitt}_1(\mathcal{Q}) = \langle x_1, x_2, x_3, x_4 \rangle$, i.e., $V_{\mathbb{K}}(\mathrm{Fitt}_1(\mathcal{Q})) = \ker_{\mathbb{K}}(N.) = 0$, which shows that all the solutions of Problem (3) are such that the v's have not full row rank.

Remark 23. We continue Remarks 11 and 14. Let $\mathcal{B}_k = \mathrm{coker}_{\mathcal{S}_k}(B_k.)$ be the $\mathcal{S}_k$-module finitely presented by B_k. Applying the right exact covariant functor $\mathcal{S}_k \otimes_{\mathcal{S}} \cdot$ to the exact sequence (27) of $\mathcal{S}$-modules, we obtain the following exact sequence of $\mathcal{S}_k$-modules

$$\mathcal{S}_k^{q\times 1} \xrightarrow{\varphi_k(B).} \mathcal{S}_k^{l\times 1} \xrightarrow{\mathrm{id}_{\mathcal{S}_k} \otimes \sigma} \mathcal{S}_k \otimes_{\mathcal{S}} \mathcal{B} \longrightarrow 0,$$

i.e., $\mathcal{S}_k \otimes_{\mathcal{S}} \mathcal{B} \cong \mathrm{coker}_{\mathcal{S}_k}(\varphi_k(B).)$. Using Proposition 2, we then have:

$$\mathcal{S}_k \otimes_{\mathcal{S}} \mathrm{Fitt}_0(\mathcal{B}) = \mathrm{Fitt}_0(\mathcal{S}_k \otimes_{\mathcal{S}} \mathcal{B}) = \mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{S}_k}(\varphi_k(B).)).$$

Using (25), i.e., $\varphi_k(B) = B_k\, L_k$, and *the Laplace expansion theorem* for computing the determinant of a product of two matrices, we obtain:

$$\mathcal{S}_k \otimes_{\mathcal{S}} \mathrm{Fitt}_0(\mathcal{B}) = \mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{S}_k}(\varphi_k(B).)) \subseteq \mathrm{Fitt}_0(\mathcal{B}_k) \cap \mathrm{Fitt}_0(\mathrm{coker}_{\mathcal{S}_k}(L_k.)) \subseteq \mathrm{Fitt}_0(\mathcal{B}_k).$$

If we note $\mathcal{I}_k = \mathrm{Fitt}_0(\mathcal{B}_k)$ for $k = 1, \ldots, r-1$, then we have $\mathcal{S}_k \otimes_{\mathcal{S}} \mathcal{I} \subseteq \mathcal{I}_k$ for $k = 1, \ldots, r-1$.

Example 9. We continue Example 1. Let $x = (x_1 \ \ldots \ x_4)^T$, $\mathcal{R} = \mathbb{Q}[x_1, \ldots, x_4]$,

$$A = (D_1\, x \ \ldots \ D_r\, x) = \begin{pmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_2 & 0 & x_3 \\ 0 & -x_3 & 0 & -x_2 \\ -x_4 & 0 & -x_1 & 0 \end{pmatrix} \in \mathcal{R}^{4\times 4},$$

$\mathcal{Q} = \mathcal{R}^{1\times 4}/(\mathcal{R}^{1\times 3}\, Q)$ be the $\mathcal{R}$ finitely presented by the following matrix

$$Q = L\, A = \begin{pmatrix} 0 & -x_3 & 0 & -x_2 \\ 0 & x_2 & 0 & x_3 \\ x_1 + x_4 & 0 & x_1 + x_4 & 0 \end{pmatrix} \in \mathcal{R}^{3\times 4},$$

the Fitting ideals of $\mathcal{Q}$ defined by:

$$\begin{cases} \mathcal{J} = \mathcal{J}_0 = \mathrm{Fitt}_0(\mathcal{Q}) = \langle 0 \rangle, \\ \mathcal{J}_1 = \mathrm{Fitt}_1(\mathcal{Q}) = \langle (x_1 + x_4)\,(x_2^2 - x_3^2) \rangle, \\ \mathcal{J}_2 = \mathrm{Fitt}_2(\mathcal{Q}) = \langle x_2^2 - x_3^2,\, x_3\,(x_1 + x_4),\, x_2\,(x_1 + x_4) \rangle, \\ \mathcal{J}_3 = \mathrm{Fitt}_3(\mathcal{Q}) = \langle x_2,\, x_3,\, x_1 + x_4 \rangle. \end{cases}$$

Then, the rank of Q on $V_{\mathbb{K}}(\mathcal{J}_k)$ is less than or equal to $3 - k$ for $k = 0, \ldots, 3$.

Let us consider the rings $\mathcal{S}_k = \mathcal{R}/\mathcal{J}_k$ for $k = 0, \ldots, 3$, with the notation $\mathcal{S}_0 = \mathcal{S} = \mathcal{R}$, and $\varphi_k : \mathcal{S} \longrightarrow \mathcal{S}_k$ the canonical ring epimorphisms for $k = 1, 2, 3$. In what follows, the residue class $\overline{x_i}$ (resp., $\varphi_k(\overline{x_i})$) of x_i in $\mathcal{S}$ (resp., $\mathcal{S}_k$) will simply be denoted by x_i. Let K_k be the matrices satisfying $\ker_{\mathcal{S}_k}(\varphi_k(Q).) = \mathrm{im}_{\mathcal{S}_k}(K_k.)$

for $k=0,\ldots,3$, where $Q_0=Q$ and:

$$K=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}\in\mathcal{S}^{4\times 1},\ K_1=\begin{pmatrix}1&0&0&0\\0&0&-x_3\,(x_1+x_4)&x_2\,(x_1+x_4)\\-1&x_2^2-x_3^2&0&0\\0&0&x_2\,(x_1+x_4)&-x_3\,(x_1+x_4)\end{pmatrix}\in\mathcal{S}_1^{4\times 4},$$

$$K_2=\begin{pmatrix}1&0&0&0&0&0&0\\0&0&-x_3&0&x_2&0&x_1+x_4\\-1&x_3&0&x_2&0&0&0\\0&0&x_2&0&-x_3&x_1+x_4&0\end{pmatrix}\in\mathcal{S}_2^{4\times 7},\quad K_3=I_4.$$

Then, we have the identity $\varphi_k(K)=K_k\,L_k$, where:

$$L_1=(1\quad 0\quad 0\quad 0)^T,\quad L_2=(1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0)^T,\quad L_3=(1\quad 0\quad -1\quad 0)^T.$$

Using the left inverse $V=(0\quad 0\quad 0\quad 1)$ of X, then we get:

$$\begin{cases}B=V\,A\,K=x_1-x_4\in\mathcal{S},\\ B_1=V\,\varphi_1(A)\,K_1=\big(x_1-x_4\quad x_4\,(x_2^2-x_3^2)\quad 0\quad 0\big)\in\mathcal{S}_1^{1\times 4},\\ B_2=V\,\varphi_2(A)\,K_2=(x_1-x_4\quad x_3\,x_4\quad 0\quad x_2\,x_4\quad 0\quad 0\quad 0)\in\mathcal{S}_2^{1\times 7},\\ B_3=V\,\varphi_3(A)\,K_3=(-x_4\quad 0\quad x_4\quad 0)\in\mathcal{S}_3^{1\times 4}.\end{cases}$$

We can check again that $\varphi_k(B)=B_k\,L_k$ for $k=1,2,3$. Introducing the finitely presented $\mathcal{S}_k$-modules $\mathcal{B}_k=\operatorname{coker}_{\mathcal{S}_k}(B_k.)$ for $k=1,2,3$, we then have:

$$\begin{cases}\mathcal{I}=\operatorname{Fitt}_0(\mathcal{B})=\langle x_1-x_4\rangle_{\mathcal{S}},\\ \mathcal{I}_1=\operatorname{Fitt}_0(\mathcal{B}_1)=\big\langle x_1-x_4,\,x_4\,(x_2^2-x_3^2)\big\rangle_{\mathcal{S}_1}=\langle x_1-x_4\rangle_{\mathcal{S}_1},\\ \mathcal{I}_2=\operatorname{Fitt}_0(\mathcal{B}_2)=\langle x_1-x_4,\,x_3\,x_4,\,x_2\,x_4\rangle_{\mathcal{S}_2}=\langle x_1-x_4\rangle_{\mathcal{S}_2},\\ \mathcal{I}_3=\operatorname{Fitt}_0(\mathcal{B}_3)=\langle x_1,x_4\rangle_{\mathcal{S}_3}=\langle x_4\rangle_{\mathcal{S}_3}.\end{cases}$$

Then, we can check again that $\mathcal{S}_k\otimes_{\mathcal{S}}\mathcal{I}=\langle x_1-x_4\rangle_{\mathcal{S}_k}=\mathcal{I}_k$ for $k=1,2,3$.

We first consider $\mathcal{I}$. Let $g=x_1-x_4$ and consider the localization $\mathcal{S}_g=\mathcal{S}[g^{-1}]$ of $\mathcal{S}$. Then, B has the (right) inverse $E_g=g^{-1}$ and $C_g=0$ since $\ker_{\mathcal{S}_g}(B.)=0$. Hence, the corresponding solutions of Problem (3) are defined by:

$$\begin{cases}u=(x_1\ \ldots\ x_4)^T\in\mathcal{U}=\mathbb{K}^{4\times 1}\setminus V_{\mathbb{K}}(\langle x_1-x_4\rangle),\\ v=K\,E_g\,Y=\dfrac{1}{x_1-x_4}\begin{pmatrix}1&0&0&1\\0&0&0&0\\-1&0&0&-1\\0&0&0&0\end{pmatrix}.\end{cases}$$

Now, we consider the ideal $\mathcal{I}_1$ of $\mathcal{S}_1$, $g=x_1-x_4$ and the localization $\mathcal{S}_{1g}=\mathcal{S}_1[g^{-1}]$ of $\mathcal{S}_1$. Then, B_1 has a right inverse $E_{1g}=g^{-1}\,(1\quad 0\quad 0\quad 0)^T\in{\mathcal{S}_{1g}}^{4\times 1}$ and

$$C_{1g}=\begin{pmatrix}0&0&x_2^2-x_3^2\\0&0&2\\0&1&0\\1&0&0\end{pmatrix}\in{\mathcal{S}_{1g}}^{4\times 3}$$

satisfies $\ker_{\mathcal{S}_{1g}}(B_1.)=\operatorname{im}_{\mathcal{S}_{1g}}(C_{1g}.)$. Let us define the following quasi-affine algebraic set:

$$\begin{aligned}\mathcal{U}_1&=V_{\mathbb{K}}(\langle(x_1+x_4)\,(x_2^2-x_3^2)\rangle)\setminus V_{\mathbb{K}}(\langle(x_1+x_4)\,(x_2^2-x_3^2),\,x_1-x_4\rangle)\\&=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\-x_1\end{pmatrix},\begin{pmatrix}x_1\\x_2\\x_2\\x_4\end{pmatrix},\begin{pmatrix}x_1\\x_2\\-x_2\\x_4\end{pmatrix}\right\}\setminus\left\{\begin{pmatrix}0\\x_2\\x_3\\0\end{pmatrix},\begin{pmatrix}x_1\\x_2\\x_2\\x_1\end{pmatrix},\begin{pmatrix}x_1\\x_2\\-x_2\\x_1\end{pmatrix}\right\}.\end{aligned}$$

Then, the corresponding solutions of Problem (3) are defined by:

$$\left\{ \begin{array}{ll} u & = (x_1 \ \ldots \ x_4)^T \in \mathcal{U}_1, \\ v & = K_1\,(E_{1g}\,Y + C_{1g}\,Y_1') \\ & = \dfrac{1}{x_1 - x_4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & x_2^2 - x_3^2 \\ x_2\,(x_1 + x_4) & -x_3\,(x_1 + x_4) & 0 \\ 0 & 0 & x_2^2 - x_3^2 \\ -x_3\,(x_1 + x_4) & x_2\,(x_1 + x_4) & 0 \end{pmatrix} Y_1', \\ & \hfill \forall\, Y_1' \in {\mathcal{S}_{1g}}^{4 \times 4}. \end{array} \right.$$

Consider the ideal $\mathcal{I}_2$ of $\mathcal{S}_2$, $g = x_1 - x_4$ and the localization $\mathcal{S}_{2g} = \mathcal{S}_2[g^{-1}]$ of $\mathcal{S}_2$. Then, B_2 has a right inverse $E_{2g} = g^{-1}\,(1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)^T \in {\mathcal{S}_{2g}}^{7 \times 1}$ and

$$C_{2g} = \begin{pmatrix} 0 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in {\mathcal{S}_{2g}}^{7 \times 6}$$

satisfies $\ker_{\mathcal{S}_{2g}}(B_2.) = \mathrm{im}_{\mathcal{S}_{2g}}(C_{2g}.)$. Let us define the following quasi-affine algebraic set:

$$\begin{array}{c} \mathcal{U}_2 = \\ V_{\mathbb{K}}(\langle x_2^2 - x_3^2,\, x_3\,(x_1 + x_4),\, x_2\,(x_1 + x_4)\rangle) \setminus V_{\mathbb{K}}(\langle x_2^2 - x_3^2,\, x_3\,(x_1 + x_4),\, x_2\,(x_1 + x_4),\, x_1 - x_4\rangle) \\ = \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_2 \\ -x_1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ -x_2 \\ -x_1 \end{pmatrix} \right\} \setminus \left\{ \begin{pmatrix} x_1 \\ 0 \\ 0 \\ x_1 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 \\ -x_2 \\ 0 \end{pmatrix} \right\}. \end{array}$$

Then, the corresponding solutions of Problem (3) are:

$$\left\{ \begin{array}{ll} u & = (x_1 \ \ldots \ x_4)^T \in \mathcal{U}_2, \\ v & = K_2\,(E_{2g}\,Y + C_{2g}\,Y_2') \\ & = \dfrac{1}{x_1 - x_4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & x_2 & x_3 \\ x_1 + x_4 & 0 & x_2 & -x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 & x_2 \\ 0 & x_1 + x_4 & -x_3 & x_2 & 0 & 0 \end{pmatrix} Y_2', \\ & \hfill \forall\, Y_2' \in {\mathcal{S}_{2g}}^{7 \times 4}. \end{array} \right.$$

Finally, the solutions obtained in Example 1 correspond to $\mathcal{S}_3 = \mathcal{R}/\mathcal{J}_3$, $\mathcal{S}_{3x_4} = \mathcal{S}_3[x_4^{-1}]$, $E_{3x_4} = (2\,x_4)^{-1}\,(-1 \quad 0 \quad 1 \quad 0)^T \in \mathcal{S}_{3x_4}^{4 \times 1}$, $\ker_{\mathcal{S}_{3x_4}}(B_3.) = \mathrm{im}_{\mathcal{S}_{3x_4}}(C.)$, where C is defined by (10). Hence, we find again (11) since we have:

$$\mathcal{U}_3 = V_{\mathbb{K}}(\langle x_2,\, x_3,\, x_1 + x_4\rangle) \setminus V_{\mathbb{K}}(\langle x_2,\, x_3,\, x_1 + x_4,\, x_4\rangle) = \left\{ \begin{pmatrix} -x_4 \\ 0 \\ 0 \\ x_4 \end{pmatrix} \mid x_4 \neq 0 \right\}.$$

4 Conclusion

In this paper, we have studied the general solutions of a rank factorization problem appearing as a demodulation problem in gearbox vibration analysis [13, 14]. More precisely, using module theory, homological and computer algebra methods, we have shown how to characterize the general solutions of this rank factorization problem. The results obtained in [15, 16, 17, 18], characterizing a special class of solutions of the problem, can be found again as particular cases. The characterization of the general solutions obtained in this paper is effective and can be obtained by modern computer algebra systems. In particular, the `GAP` library `CapAndHomalg` [1] was used to handle the examples studied in this paper.

Many important issues still have to be investigated in the future such as the algebraic (projective) geometric interpretation of the rank factorization problem and of its solutions. Moreover, symbolic-numeric methods based on, e.g., *the rational univariate representation* (RUR) and *root isolation*, will be used in the future to get certified numerical solutions.

In applications to vibration analysis, as explained in Section 1, the complex matrices D_i's and M are centrohermitian and the rank factorization problem can be transformed into an equivalent rank factorization problem for real matrices $\varphi(D_i)$'s and $\varphi(M)$. In practice, the matrices D_i's are explicitly known contrary to M which is only measured. The matrix M is usually corrupted by perturbations and noise. Hence, in the non-exact case, i.e., when M is not supposed to be exactly known, one usually prefers to consider the optimization problem

$$\underset{u \in \mathrm{CH}_{n,1}(\mathbb{C}),\, v_i \in \mathrm{CH}_{1,m}(\mathbb{C})}{\arg\min} \left\| \sum_{i=1}^{r} D_i\, u\, v_i - M \right\|_{\mathrm{Frob}}, \tag{33}$$

where the *Frobenius norm* of a complex matrix A is defined by:

$$\|A\|_{\mathrm{Frob}} = \sqrt{\mathrm{trace}(A^{\star} A)}.$$

Using the fact that the transformation φ can be defined by means of unitary matrices and the Frobenius norm is invariant by unitary transformations, we thus obtain:

$$\begin{array}{c} \min_{u \in \mathrm{CH}_{n,1}(\mathbb{C}),\, v_i \in \mathrm{CH}_{1,m}(\mathbb{C})} \|\sum_{i=1}^{r} D_i\, u\, v_i - M\|_{\mathrm{Frob}} \\ = \min_{u_\varphi \in \mathbb{R}^{n \times 1},\, v_{i\varphi} \in \mathbb{R}^{1 \times m}} \|\sum_{i=1}^{r} \varphi(D_i)\, u_\varphi\, v_{i\varphi} - \varphi(M)\|_{\mathrm{Frob}}. \end{array}$$

For more details, see [18]. Thus, the optimization problem (33) is reduced to a real polynomial optimization problem. This problem will be studied in a future work using symbolic-numeric methods. The results obtained here on the structure of the solution space of the rank factorization problem are good assets to investigate this real polynomial optimization problem (e.g., obtaining good initial conditions for the optimization algorithms).

Finally, in the future, we also want to analyze the continuity of the method proposed in this paper with respect to certain variations of the matrix M observed in practice. This study can yield an interesting alternative to the previous real polynomial optimization problem.

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5 A worked example with `CapAndHomalg`

In this Appendix, we demonstrate how the `GAP` library `CapAndHomalg` [1] can be used to effectively solve the rank factorization problem studied in this paper. We consider again Example 9. Using the `notebook` of `IJulia`, we display below the different command lines and the corresponding results. For more examples, we refer the reader to the webpage dedicated to the effective aspects of the rank factorization problem:

`https://who.paris.inria.fr/Alban.Quadrat/RankFactorizationProblem.html`

```
[1]: using CapAndHomalg
```

```
GAP 4.11.0 of 29-Feb-2020
GAP     https://www.gap-system.org
 Architecture: x86_64-apple-darwin19.6.0-julia64-kv7-v1.5
  Configuration:  gmp 6.1.2, Julia GC, Julia 1.5.3, readline
 Loading the library and packages ...
 Packages:   AClib 1.3.2, Alnuth 3.1.2, AtlasRep 2.1.0, AutoDoc 2019.09.04,
             AutPGrp 1.10.2, CRISP 1.4.5, Cryst 4.1.23, CrystCat 1.1.9,
             CTblLib 1.2.2, FactInt 1.6.3, FGA 1.4.0, Forms 1.2.5,
             GAPDoc 1.6.3, genss 1.6.6, IO 4.7.0, IRREDSOL 1.4, LAGUNA 3.9.3,
             orb 4.8.3, Polenta 1.3.9, Polycyclic 2.15.1, PrimGrp 3.4.0,
             RadiRoot 2.8, recog 1.3.2, ResClasses 4.7.2, SmallGrp 1.4.1,
             Sophus 1.24, SpinSym 1.5.2, TomLib 1.2.9, TransGrp 2.0.5,
             utils 0.69
 Try '??help' for help. See also '?copyright', '?cite' and '?authors'
CapAndHomalg v1.0.3
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information
```

```
[2]: LoadPackage( "IntrinsicModules" )
```

```
[3]: ℚ = HomalgFieldOfRationalsInSingular()
```

```
[3]: GAP: ℚ
```

```
[4]: R = ℚ["x1..4"]
```

```
[4]: GAP: Q[x1,x2,x3,x4]
```

```
[5]: Mmat = HomalgMatrix( "[1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1]", 4, 4, R␣
     ↪)
```

```
[5]: GAP: <A 4 x 4 matrix over an external ring>
```

```
[6]: Display( Mmat )
```

```
1,0,0,1,
0,0,0,0,
0,0,0,0,
1,0,0,1
```

```
[7]: D1 = HomalgMatrix( "[1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1]", 4, 4, R )
```

```
[7]: GAP: <A 4 x 4 matrix over an external ring>
```

```
[8]: Display( D1 )
```

```
1,0,0,0,
0,0,0,0,
0,0,0,0,
0,0,0,-1
```

[9]:
```
D2 = HomalgMatrix( "[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0]", 4, 4, R )
```

[9]: GAP: <A 4 x 4 matrix over an external ring>

[10]:
```
Display( D2 )
```

```
0,0,0, 0,
0,1,0, 0,
0,0,-1,0,
0,0,0, 0
```

[11]:
```
D3 = HomalgMatrix( "[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0]", 4, 4, R )
```

[11]: GAP: <A 4 x 4 matrix over an external ring>

[12]:
```
Display( D3 )
```

```
0, 0,0,1,
0, 0,0,0,
0, 0,0,0,
-1,0,0,0
```

[13]:
```
D4 = HomalgMatrix( "[0, 0, 0, 0, 0, 0, 1, 0, 0, -1, 0, 0, 0, 0, 0, 0]", 4, 4, R )
```

[13]: GAP: <A 4 x 4 matrix over an external ring>

[14]:
```
Display( D4 )
```

```
0,0, 0,0,
0,0, 1,0,
0,-1,0,0,
0,0, 0,0
```

[15]:
```
n = NrRows( Mmat )
```

[15]: 4

[16]:
```
x = HomalgMatrix( "[x1, x2, x3, x4]", n, 1, R )
```

[16]: GAP: <A 4 x 1 matrix over an external ring>

[17]:
```
Amat =UnionOfColumns( D1 * x, D2 * x, D3 * x, D4 * x )
```

[17]: GAP: <An unevaluated 4 x 4 matrix over an external ring>

[18]:
```
Display( Amat )
```

```
x1, 0,  x4, 0,
0,  x2, 0,  x3,
0,  -x3,0,  -x2,
-x4,0,  -x1,0
```

[19]:
```
ColsR = CategoryOfColumns( R )
```

[19]: GAP: Columns(Q[x1,x2,x3,x4])

[20]:
```
M =  Mmat / ColsR
```

[20]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[21]:
```
L = WeakCokernelProjection( M )
```

[21]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[22]:
```
Display( L )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 4

Matrix:
0,0,1,0,
0,1,0,0,
1,0,0,-1

Range:
A column module over Q[x1,x2,x3,x4] of rank 3

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[23]:
```
X = WeakKernelEmbedding( L )
```

[23]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[24]:
```
Display( X )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 1

Matrix:
1,
0,
```

```
0,
1

Range:
A column module over Q[x1,x2,x3,x4] of rank 4

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[25]:
```
Y = Lift( M, X )
```

[25]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[26]:
```
Display( Y )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 4

Matrix:
1,0,0,1

Range:
A column module over Q[x1,x2,x3,x4] of rank 1

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[27]:
```
A = Amat / ColsR
```

[27]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[28]:
```
KA = WeakCokernelProjection( A )
```

[28]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[29]:
```
Display( KA )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 4

Matrix:
(an empty 0 x 4 matrix)

Range:
A column module over Q[x1,x2,x3,x4] of rank 0

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[30]:
```
Q = PreCompose( A, L )
```

[30]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[31]:
```
Display( Q )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 4

Matrix:
0,    -x3,0,    -x2,
0,    x2, 0,    x3,
x1+x4,0,  x1+x4,0

Range:
A column module over Q[x1,x2,x3,x4] of rank 3

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[32]:
```
J = FittingIdeal( 0,  LeftPresentation( UnderlyingMatrix( Q ) ) )
```

[32]: GAP: <A zero (left) ideal>

[33]:
```
ColsS = CategoryOfColumns( R )
```

[33]: GAP: Columns(Q[x1,x2,x3,x4])

[34]:
```
KS = WeakKernelEmbedding( Q )
```

[34]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[35]:
```
Display( KS )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 1

Matrix:
1,
0,
-1,
0

Range:
A column module over Q[x1,x2,x3,x4] of rank 4

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[36]:
```
BS = Lift( PreCompose( KS, A ), X )
```

[36]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[37]:
```
Display( BS )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 1

Matrix:
x1-x4

Range:
A column module over Q[x1,x2,x3,x4] of rank 1

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[38]:
```
I = FittingIdeal( 0,  LeftPresentation( UnderlyingMatrix( BS ) ) )
```

[38]: GAP: <A principal torsion-free (left) ideal given by a cyclic generator>

[39]:
```
Display( I )
```

```
x1-x4

A (left) ideal generated by the entry of the above matrix
```

[40]:
```
C = WeakKernelEmbedding( BS )
```

[40]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4])>

[41]:
```
Display( C )
```

```
Source:
A column module over Q[x1,x2,x3,x4] of rank 0

Matrix:
(an empty 1 x 0 matrix)

Range:
A column module over Q[x1,x2,x3,x4] of rank 1

A morphism in Columns( Q[x1,x2,x3,x4] )
```

[42]:
```
T = R["t"] / g"t*(x1-x4) - 1"
```

[42]: GAP: Q[x1,x2,x3,x4][t]/(x1*t-x4*t-1)

[43]:
```
ColsT = CategoryOfColumns( T )
```

```
[43]: GAP: Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )

[44]: BT = BS / ColsT

[44]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )>

[45]: ET = PreInverse( BT )

[45]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )>

[46]: Display( ET )

      Source:
      A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) of rank 1

      Matrix:
      t

      modulo [ x1*t-x4*t-1 ]

      Range:
      A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) of rank 1

      A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )

[47]: YT = Y / ColsT

[47]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )>

[48]: KT = KS / ColsT

[48]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )>

[49]: SolT = PreCompose( YT, PreCompose( ET, KT ) )

[49]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )>

[50]: Display( SolT )

      Source:
      A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) of rank 4

      Matrix:
      t, 0,0,t,
      0, 0,0,0,
      -t,0,0,-t,
      0, 0,0,0
```

```
modulo [ x1*t-x4*t-1 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )
```

[51]:
```
MT = M / ColsT
```

[51]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x1*t-x4*t-1))>

[52]:
```
AT = A / ColsT
```

[52]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x1*t-x4*t-1))>

[53]:
```
PreCompose( SolT, AT ) == MT
```

[53]: true

[54]:
```
CT = WeakKernelEmbedding( BT )
```

[54]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x1*t-x4*t-1))>

[55]:
```
Display( CT )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) of rank 0

Matrix:
(an empty 1 x 0 matrix)

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) of rank 1

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1 ) )
```

[56]:
```
J1 = FittingIdeal( 1,  LeftPresentation( UnderlyingMatrix( Q ) ) )
```

[56]: GAP: <A torsion-free (left) ideal given by 4 generators>

[57]:
```
Display( J1 )
```

```
0,
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
0,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4

A (left) ideal generated by the 4 entries of the above matrix
```

[58]:
```
S1 = R / J1
```

[58]:
```
GAP: Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 )
```

[59]:
```
ColsS1 = CategoryOfColumns( S1 )
```

[59]:
```
GAP: Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )
```

[60]:
```
QS1 = Q / ColsS1
```

[60]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )>
```

[61]:
```
KS1 = WeakKernelEmbedding( QS1 )
```

[61]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )>
```

[62]:
```
Display( KS1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4]/( x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) of rank
4

Matrix:
1, 0,        0,           0,
0, 0,        -x1*x3-x3*x4,x1*x2+x2*x4,
-1,x2^2-x3^2,0,           0,
0, 0,        x1*x2+x2*x4, -x1*x3-x3*x4

modulo [ x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ]

Range:
A column module over Q[x1,x2,x3,x4]/( x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) of rank
4

A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )
```

[63]:
```
AS1 = A / ColsS1
```

[63]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )>
```

[64]:
```
XS1 = X / ColsS1
```

```
[64]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
      x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )>
```

```
[65]: BS1 = Lift( PreCompose( KS1, AS1 ), XS1 )
```

```
[65]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
      x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )>
```

```
[66]: Display( BS1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4]/( x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) of rank
4

Matrix:
x1-x4,x2^2*x4-x3^2*x4,0,0

modulo [ x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ]

Range:
A column module over Q[x1,x2,x3,x4]/( x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) of rank
1

A morphism in Columns( Q[x1,x2,x3,x4]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ) )
```

```
[67]: I1 = FittingIdeal( 0,  RightPresentation( UnderlyingMatrix( BS1 ) ) )
```

```
[67]: GAP: <A principal torsion-free (right) ideal given by a cyclic generator>
```

```
[68]: Display( I1 )
```

```
x1-x4

modulo [ x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ]

A (right) ideal generated by the entry of the above matrix
```

```
[69]: ann1 = Annihilator( RightPresentation( UnderlyingMatrix( BS1 ) ) )
```

```
[69]: GAP: <A non-zero principal torsion-free (right) ideal given by a cyclic
      generator>
```

```
[70]: Display( ann1 )
```

```
x1-x4

modulo [ x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4 ]
```

A (right) ideal generated by the entry of the above matrix

[71]:
```
TS1 = S1["t"] / g"t*(x1-x4) - 1"
```

[71]:
```
GAP: Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 )
```

[72]:
```
ColsTS1 = CategoryOfColumns( TS1 )
```

[72]:
```
GAP: Columns( Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 ) )
```

[73]:
```
BTS1 = BS1 / ColsTS1
```

[73]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 )
)>
```

[74]:
```
ETS1 = PreInverse( BTS1 )
```

[74]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 )
)>
```

[75]:
```
Display( ETS1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 1

Matrix:
t,
0,
0,
0

modulo [ x1*t-x4*t-1, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4,
2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 ) )
```

[76]:
```
YTS1 = Y / ColsTS1
```

[76]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[77]:
```
KTS1 = KS1 / ColsTS1
```

[77]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[78]:
```
SolTS1 = PreCompose( YTS1, PreCompose( ETS1, KTS1 ) )
```

[78]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[79]:
```
Display( SolTS1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 4

Matrix:
t, 0,0,t,
0, 0,0,0,
-t,0,0,-t,
0, 0,0,0

modulo [ x1*t-x4*t-1, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4,
2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 ) )
```

[80]:
```
MTS1 = M / ColsTS1
```

[80]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[81]:
```
ATS1 = A / ColsTS1
```

[81]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[82]:
```
PreCompose( SolTS1, ATS1 ) == MTS1
```

[82]: true

[83]:
```
CTS1 = WeakKernelEmbedding( BTS1 )
```

[83]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[84]:
```
Display( CTS1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 3

Matrix:
0,0,x2^2-x3^2,
0,0,2,
0,1,0,
1,0,0

modulo [ x1*t-x4*t-1, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4,
2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 ) )
```

[85]:
```
solTS1 = PreCompose( CTS1, KTS1 )
```

[85]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1)
)>

[86]:
```
Display( solTS1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 3
```

```
Matrix:
0,           0,           x2^2-x3^2,
x1*x2+x2*x4, -x1*x3-x3*x4,0,
0,           0,           x2^2-x3^2,
-x1*x3-x3*x4,x1*x2+x2*x4, 0

modulo [ x1*t-x4*t-1, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4,
2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 ) )
```

```
[87]: check1 =  PreCompose( solTS1, ATS1 )
```

```
[87]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/(
-x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 )
)>
```

```
[88]: Display( check1 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 3

Matrix:
0,0,0,
0,0,0,
0,0,0,
0,0,0

modulo [ x1*t-x4*t-1, x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4,
2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, 2*x2^2*x4*t-2*x3^2*x4*t+x2^2-x3^2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x1*x2^2+x1*x3^2-x2^2*x4+x3^2*x4,
x1*x2^2-x1*x3^2+x2^2*x4-x3^2*x4, x1*t-x4*t-1 ) )
```

```
[89]: J2 = FittingIdeal( 2,  LeftPresentation( UnderlyingMatrix( Q ) ) )
```

```
[89]: GAP: <A torsion-free (left) ideal given by 18 generators>
```

[90]:
```
Display( J2 )
```

```
0,
0,
0,
0,
-x2^2+x3^2,
0,
-x1*x3-x3*x4,
0,
-x1*x2-x2*x4,
x1*x3+x3*x4,
0,
-x1*x2-x2*x4,
-x1*x2-x2*x4,
0,
-x1*x3-x3*x4,
x1*x2+x2*x4,
0,
-x1*x3-x3*x4

A (left) ideal generated by the 18 entries of the above matrix
```

[91]:
```
S2 = R / J2
```

[91]:
```
GAP: Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4, -x1*x2-x2*x4, x1*x3+x3*x4,
-x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4, x1*x2+x2*x4, -x1*x3-x3*x4 )
```

[92]:
```
ColsS2 = CategoryOfColumns( S2 )
```

[92]:
```
GAP: Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4, -x1*x2-x2*x4,
x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4, x1*x2+x2*x4, -x1*x3-x3*x4
) )
```

[93]:
```
QS2 = Q / ColsS2
```

[93]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4 ) )>
```

[94]:
```
KS2 = WeakKernelEmbedding( QS2 )
```

[94]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4 ) )>
```

[95]:
```
Display( KS2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4]/( x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ) of
rank 7

Matrix:
1, 0, 0,  0, 0,  0,    0,
0, 0, -x3,0, x2, 0,    x1+x4,
-1,x3,0,  x2,0,  0,    0,
0, 0, x2, 0, -x3,x1+x4,0

modulo [ x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ]

Range:
A column module over Q[x1,x2,x3,x4]/( x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ) of
rank 4

A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4, -x1*x2-x2*x4,
x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4, x1*x2+x2*x4, -x1*x3-x3*x4
) )
```

[96]:
```
AS2 = A / ColsS2
```

[96]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4 ) )>
```

[97]:
```
XS2 = X / ColsS2
```

[97]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4 ) )>
```

[98]:
```
BS2 = Lift( PreCompose( KS2, AS2 ), XS2 )
```

[98]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4 ) )>
```

[99]:
```
Display( BS2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4]/( x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ) of
rank 7

Matrix:
x1-x4,x3*x4,0,x2*x4,0,0,0

modulo [ x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ]
```

```
Range:
A column module over Q[x1,x2,x3,x4]/( x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ) of
rank 1

A morphism in Columns( Q[x1,x2,x3,x4]/( -x2^2+x3^2, -x1*x3-x3*x4, -x1*x2-x2*x4,
x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4, x1*x2+x2*x4, -x1*x3-x3*x4
) )
```

[100]:
```
I2 = FittingIdeal( 0,  RightPresentation( UnderlyingMatrix( BS2 ) ) )
```

[100]:
```
GAP: <A principal torsion-free (right) ideal given by a cyclic generator>
```

[101]:
```
Display( I2 )
```

```
x1-x4

modulo [ x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ]

A (right) ideal generated by the entry of the above matrix
```

[102]:
```
ann2 = Annihilator( RightPresentation( UnderlyingMatrix( BS2 ) ) )
```

[102]:
```
GAP: <A non-zero principal torsion-free (right) ideal given by a cyclic
generator>
```

[103]:
```
Display( ann2 )
```

```
x1-x4

modulo [ x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4 ]

A (right) ideal generated by the entry of the above matrix
```

[104]:
```
TS2 = S2["t"] / g"t*(x1-x4) - 1"
```

[104]:
```
GAP: Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4, -x1*x2-x2*x4, x1*x3+x3*x4,
-x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4, x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1
)
```

[105]:
```
ColsTS2 = CategoryOfColumns( TS2 )
```

[105]:
```
GAP: Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4, -x1*x2-x2*x4,
x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4, x1*x2+x2*x4,
-x1*x3-x3*x4, x1*t-x4*t-1 ) )
```

[106]:
```
BTS2 = BS2 / ColsTS2
```

```
[106]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
       -x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
       x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

```
[107]: ETS2 = PreInverse( BTS2 )
```

```
[107]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
       -x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
       x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

```
[108]: Display( ETS2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 1

Matrix:
t,
0,
0,
0,
0,
0,
0

modulo [ x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4, 2*x3*x4*t+x3,
2*x2*x4*t+x2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 7

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )
```

```
[109]: YTS2 = Y / ColsTS2
```

```
[109]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
       -x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
       x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

```
[110]: KTS2 = KS2 / ColsTS2
```

```
[110]: GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
       -x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
       x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

[111]:
```
SolTS2 = PreCompose( YTS2, PreCompose( ETS2, KTS2 ) )
```

[111]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

[112]:
```
Display( SolTS2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 4

Matrix:
t, 0,0,t,
0, 0,0,0,
-t,0,0,-t,
0, 0,0,0

modulo [ x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4, 2*x3*x4*t+x3,
2*x2*x4*t+x2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )
```

[113]:
```
MTS2 = M / ColsTS2
```

[113]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

[114]:
```
ATS2 = A / ColsTS2
```

[114]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

[115]:
```
PreCompose( SolTS2, ATS2 ) == MTS2
```

[115]:
```
true
```

[116]:
```
CTS2 = WeakKernelEmbedding( BTS2 )
```

[116]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(-x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1))>

[117]:
```
Display( CTS2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 6

Matrix:
0,0,0,0,x3,x2,
0,0,0,0,2, 0,
0,0,0,1,0, 0,
0,0,0,0,0, 2,
0,0,1,0,0, 0,
0,1,0,0,0, 0,
1,0,0,0,0, 0

modulo [ x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4, 2*x3*x4*t+x3,
2*x2*x4*t+x2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 7

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )
```

[118]:
```
solTS2 = PreCompose( CTS2, KTS2 )
```

[118]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(-x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1))>

[119]:
```
Display( solTS2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 6

Matrix:
0,     0,     0,  0,  x3,x2,
x1+x4,0,     x2, -x3,0,  0,
0,     0,     0,  0,  x3,x2,
0,     x1+x4,-x3,x2, 0,  0
```

```
modulo [ x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4, 2*x3*x4*t+x3,
2*x2*x4*t+x2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )
```

[120]:
```
check2 =  PreCompose( solTS2, ATS2 )
```

[120]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )>
```

[121]:
```
Display( check2 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 6

Matrix:
0,0,0,0,0,0,
0,0,0,0,0,0,
0,0,0,0,0,0,
0,0,0,0,0,0

modulo [ x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2, x1*x2+x2*x4, 2*x3*x4*t+x3,
2*x2*x4*t+x2 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x1*t-x4*t-1, x1*x3+x3*x4, x2^2-x3^2,
x1*x2+x2*x4, 2*x3*x4*t+x3, 2*x2*x4*t+x2 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( -x2^2+x3^2, -x1*x3-x3*x4,
-x1*x2-x2*x4, x1*x3+x3*x4, -x1*x2-x2*x4, -x1*x2-x2*x4, -x1*x3-x3*x4,
x1*x2+x2*x4, -x1*x3-x3*x4, x1*t-x4*t-1 ) )
```

[122]:
```
J3 = FittingIdeal( 3,  LeftPresentation( UnderlyingMatrix( Q ) ) )
```

[122]:
```
GAP: <A torsion-free (left) ideal given by 12 generators>
```

[123]:
```
Display( J3 )
```

```
0,
x3,
```

```
0,
x2,
0,
x2,
0,
x3,
x1+x4,
0,
x1+x4,
0

A (left) ideal generated by the 12 entries of the above matrix
```

[124]:
```
S3 = R / J3
```

[124]: GAP: Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4)

[125]:
```
ColsS3 = CategoryOfColumns( S3 )
```

[125]: GAP: Columns(Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4))

[126]:
```
QS3 = Q / ColsS3
```

[126]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4))>

[127]:
```
KS3 = WeakKernelEmbedding( QS3 )
```

[127]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4))>

[128]:
```
Display( KS3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4]/( x3, x2, x1+x4 ) of rank 4

Matrix:
1,0,0,0,
0,1,0,0,
0,0,1,0,
0,0,0,1

modulo [ x3, x2, x1+x4 ]

Range:
A column module over Q[x1,x2,x3,x4]/( x3, x2, x1+x4 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4]/( x3, x2, x2, x3, x1+x4, x1+x4 ) )
```

[129]:
```
AS3 = A / ColsS3
```

[129]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4))>

[130]:
```
XS3 = X / ColsS3
```

[130]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4))>

[131]:
```
BS3 = Lift( PreCompose( KS3, AS3 ), XS3 )
```

[131]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4]/(x3, x2, x2, x3, x1+x4, x1+x4))>

[132]:
```
Display( BS3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4]/( x3, x2, x1+x4 ) of rank 4

Matrix:
-x4,0,x4,0

modulo [ x3, x2, x1+x4 ]

Range:
A column module over Q[x1,x2,x3,x4]/( x3, x2, x1+x4 ) of rank 1

A morphism in Columns( Q[x1,x2,x3,x4]/( x3, x2, x2, x3, x1+x4, x1+x4 ) )
```

[133]:
```
I3 = FittingIdeal( 0,  RightPresentation( UnderlyingMatrix( BS3 ) ) )
```

[133]: GAP: <A principal torsion-free (right) ideal given by a cyclic generator>

[134]:
```
Display( I3 )
```

```
x4

modulo [ x3, x2, x1+x4 ]

A (right) ideal generated by the entry of the above matrix
```

[135]:
```
ann3 = Annihilator( RightPresentation( UnderlyingMatrix( BS3 ) ) )
```

[135]: GAP: <A non-zero principal torsion-free (right) ideal given by a cyclic generator>

[136]:
```
Display( ann3 )
```

```
-x4

modulo [ x3, x2, x1+x4 ]
```

```
A (right) ideal generated by the entry of the above matrix
```

[137]:
```
TS3 = S3["t"] / g"t*x4 - 1"
```

[137]:
```
GAP: Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1 )
```

[138]:
```
ColsTS3 = CategoryOfColumns( TS3 )
```

[138]:
```
GAP: Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1 ) )
```

[139]:
```
BTS3 = BS3 / ColsTS3
```

[139]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4,
x4*t-1 ) )>
```

[140]:
```
ETS3 = PreInverse( BTS3 )
```

[140]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4,
x4*t-1 ) )>
```

[141]:
```
Display( ETS3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 1

Matrix:
0,
0,
t,
0

modulo [ x3, x2, x1+x4, x4*t-1 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1
) )
```

[142]:
```
YTS3 = Y / ColsTS3
```

[142]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4,
x4*t-1 ) )>
```

[143]:
```
KTS3 = KS3 / ColsTS3
```

[143]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1))>

[144]:
```
SolTS3 = PreCompose( YTS3, PreCompose( ETS3, KTS3 ) )
```

[144]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1))>

[145]:
```
Display( SolTS3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 4

Matrix:
0,0,0,0,
0,0,0,0,
t,0,0,t,
0,0,0,0

modulo [ x3, x2, x1+x4, x4*t-1 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1
) )
```

[146]:
```
MTS3 = M / ColsTS3
```

[146]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1))>

[147]:
```
ATS3 = A / ColsTS3
```

[147]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1))>

[148]:
```
PreCompose( SolTS3, ATS3 ) == MTS3
```

[148]: true

[149]:
```
CTS3 = WeakKernelEmbedding( BTS3 )
```

[149]: GAP: <A morphism in Columns(Q[x1,x2,x3,x4][t]/(x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1))>

[150]:
```
Display( CTS3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 3

Matrix:
0,0,1,
0,1,0,
0,0,1,
1,0,0

modulo [ x3, x2, x1+x4, x4*t-1 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1
) )
```

[151]:
```
solTS3 = PreCompose( CTS3, KTS3 )
```

[151]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4,
x4*t-1 ) )>
```

[152]:
```
Display( solTS3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 3

Matrix:
0,0,1,
0,1,0,
0,0,1,
1,0,0

modulo [ x3, x2, x1+x4, x4*t-1 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1
) )
```

[153]:
```
check3 = PreCompose( solTS3, ATS3 )
```

[153]:
```
GAP: <A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4,
x4*t-1 ) )>
```

[154]:
```
Display( check3 )
```

```
Source:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 3

Matrix:
0,0,0,
0,0,0,
0,0,0,
0,0,0

modulo [ x3, x2, x1+x4, x4*t-1 ]

Range:
A column module over Q[x1,x2,x3,x4][t]/( x3, x2, x1+x4, x4*t-1 ) of rank 4

A morphism in Columns( Q[x1,x2,x3,x4][t]/( x3, x2, x2, x3, x1+x4, x1+x4, x4*t-1
) )
```

[ ]:

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