Introduction to symbolic methods for differential time-delay control systems

2nd WORKSHOP CNRS-NSF:

“Applications of Time-Delay systems”

NANTES, 13-15/09/04

Alban Quadrat

INRIA Sophia Antipolis,
CAFE Project,
2004 route des lucioles, BP 93,
06902 Sophia Antipolis cedex,
France.

Alban.Quadrat@sophia.inria.fr

www-sop.inria.fr/cafe/Alban.Quadrat/index.html

Linear systems over rings

Differential time-delay systems (Rouchaleau, Sontag):

Let $R = \mathbb{R}[\delta_{h_1}, \ldots \delta_{h_n}]$ be the commutative polynomial ring of incommensurable point-delays δ_{h_i},

$$\boxed{\delta_{h_i} z(t) = z(t - h_i), \quad h_i \in \mathbb{R}_+,}$$

$A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, $D \in R^{p \times m}$,

$$\left\{ \begin{array}{l} \dot{x}(t) = A(\underline{\delta})\, x(t) + B(\underline{\delta})\, u(t), \\ y(t) = C(\underline{\delta})\, x(t) + D(\underline{\delta})\, u(t), \end{array} \right. \qquad (\star).$$

$(\star)$ is **reachable**

$$\Leftrightarrow (B : \; A\, B : \ldots : A^{n-1}\, B)\, R^{n\, m} = R^n$$

$$\Leftrightarrow \exists\; S \in R[z]^{(n+m) \times n} : \; (z\, I_n - A : \; -B)\, S = I_n.$$

$(\star)$ is **internally stabilizable** w.r.t. a Hurwitz set $\mathcal{D}$

$$\Leftrightarrow \exists\; \pi \in \mathcal{D}, \; S \in R[z]^{(n+m) \times n} : \; (z\, I_n - A : \; -B)\, S = \pi\, I_n.$$

$\Rightarrow$ **Gröbner bases** (Habets' PhD thesis, 1994).

- **Geometric approach** (Conte-Perdon, Lafay, Assan. . .)

$\Rightarrow$ **CoCoA** (`http://cocoa.dima.unige.it`).

Definition

- **Notation**: $D = k[x_1, \ldots, x_n]$, $k = \mathbb{Q}$, $\mathbb{R}$.

- **Definition**: A **monomial ordering** $\succ$ is a total ordering on $\{x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} \,|\, \alpha \in \mathbb{N}^n\}$ satisfying:

$$\begin{cases} \forall\, x^\alpha \neq 1, \quad x^\alpha \succ 1, \\ x^\alpha \succ x^\beta \Rightarrow x^\gamma\, x^\alpha \succ x^\gamma\, x^\beta. \end{cases}$$

- **Example**: degree lexicographical ordering $\succ_{\text{lexdeg}}$:

$$x^\alpha \succ_{\text{lexdeg}} x^\beta \;\Leftrightarrow\; |\alpha| > |\beta| \quad \text{or} \quad |\alpha| = |\beta|,$$

$$\exists\, 1 \leq i \leq n : \; \alpha_1 = \beta_1, \ldots, \alpha_{i-1} = \beta_{i-1}, \; \alpha_i > \beta_i,$$

where $|\alpha| = \sum_{i=1}^n \alpha_i$ is the length of α.

- **Definition**: Let $P = a_\alpha\, x^\alpha + a_\beta\, x^\beta + \ldots + a_\gamma\, x^\gamma \in D$, where $x^\alpha \succ x^\beta \succ \ldots$ and $a_\alpha, a_\beta, \ldots, a_\gamma \in k$.

Then, $\text{lm}(P) = x^\alpha$ is the **leading monomial** of P.

- **Definition**: Let I be an ideal, i.e., $I = \sum_{i=1}^r D\, P_i$, for certain $P_i \in D$. Then,

$G = \{G_1, \ldots, G_s\} \subset A \backslash 0$ is a **Gröbner basis** of I

$\Leftrightarrow \forall\, 0 \neq Q \in I, \quad \exists\, 1 \leq i \leq s : \quad \text{lm}(G_i) \,|\, \text{lm}(Q).$

Properties

- **Example**: Let $D = k[x, y]$, $\succ_{\text{lexdef}}$ and the ideal $I = (x^2, x\,y - 1) = \{P_1\,x^2 + P_2(x\,y - 1),\ P_i \in D\}$.

$G = \{x^2,\ x\,y - 1\}$ is **not a Gröbner basis** of I as:

$$\begin{cases} S(x^2, x\,y - 1) = y\,(x^2) - x\,(x\,y - 1) = x \in I, \\ x^2 \nmid x, \quad x\,y \nmid x. \end{cases}$$

- **Theorem**: Every ideal $I \subset D \backslash 0$ admits a Gröbner basis G for a fixed monomial order. Moreover,

if $G = \{G_1, \ldots, G_r\}$, then $I = \sum_{i=1}^{r} D\,G_i$.

- **Theorem**: A family $G = \{G_1, \ldots, G_r\}$ of generators of I is a Gröbner basis iff, for all $i \neq j$, the remainder of $S(G_i, G_j)$ by G is 0.

- **Example**: $\{x^2,\ x\,y - x\}$ **is a Gröbner basis** of the ideal $J = (x^2,\ x\,y - x)$ as:

$$\begin{cases} S(x^2, x\,y - x) = y\,(x^2) - x\,(x\,y - x) = x^2, \\ x^2 \mid S(x^2, x\,y - x). \end{cases}$$

- **Example**: We consider $\{x^2,\ x\,y - 1,\ x\}$ and

$S(x, x\,y - 1) = y\,(x) - (x\,y - 1) = 1 \in I \Rightarrow I = D$,

and $\{1\}$ is a Gröbner basis of $I = D$.

Extensions to modules

• **Extension to D-submodules of D^r** by extending the monomial ordering into a **module ordering** on:

$$x^\alpha\, e_i = (0 : \ldots : x^\alpha : \ldots 0)^T, \quad i = 1, \ldots, r.$$

$\Rightarrow$ **effective computations of syzygy modules**.

• **Example**: Consider $D = \mathbb{R}\left[\frac{d}{dt},\, \delta_h\right]$ and the **over-determined** differential time-delay linear system:

$$\begin{cases} \left(\frac{d}{dt} + a\right)\, \lambda_1 = \mu_1, \\ -k\, a\, \delta_h\, \lambda_1 + \frac{d}{dt}\, \lambda_2 + \omega^2\, \lambda_3 = \mu_2, \\ -\lambda_2 + \left(\frac{d}{dt} + 2\, \zeta\, \omega\right)\, \lambda_3 = \mu_3, \\ -\omega^2\, \lambda_3 = \mu_4. \end{cases} \qquad (\star).$$

$(\star)$ admits **compatibility conditions**, namely:

$$\omega^2\, k\, a\, \delta_h\, \mu_1 + \left(\omega^2\, \frac{d}{dt} - \omega^2\, a\right)\, \mu_2 + \left(\omega^2\, \frac{d^2}{dt^2} + \omega^2\, a\, \frac{d}{dt}\right)\, \mu_3$$
$$+ \left(\frac{d^3}{dt^3} + 2\, \zeta\, \omega\, \frac{d^2}{dt^2} + a\, \frac{d^2}{dt^2} + \omega^2\, \frac{d}{dt} + 2\, a\, \zeta\, \omega\, \frac{d}{dt} + a\, \omega^2\right)\, \mu_4 = 0.$$

Compatibility conditions can be obtained by means of a computation of a Gröbner basis for an elimination order.

Summary

• Gröbner bases (1965) **generalize**:

⋆ **Gauss elimination algorithm** in linear algebra.

⋆ **Hermite form** for matrices over $\mathbb{R}[x]$.

• Gröbner bases give

canonical forms for systems of polynomial equations w.r.t. a monomial ordering

and allow

effective computations on ideals and modules over commutative polynomial rings.

• The Gröbner bases **are implemented** in:

Maple, Mathematica, Singular, Macaulay...

• **Alternative methods**:

Janet bases (1920), characteristic sets (1950), standard bases (1970), involutive bases (1980)...

Parametrizability

• **Definition**: A system Σ is **parametrizable** if all solutions of Σ can be formally expressed by means of free functions and operators of the same classes as the system.

• **Example**:

$$\begin{cases} \dot{x}_1(t) - x_1(t-1) + 2\,x_1(t) + 2\,x_2(t) - 2\,u(t-1) = 0, \\ \dot{x}_1(t) + \dot{x}_2(t) - u(t-1) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = 2\,\dot{\xi}(t-1) - 2\,\xi(t-1), \\ x_2(t) = -\xi(t-2) - \dot{\xi}(t-1) + 2\,\xi(t-1), \\ u(t) = \ddot{\xi}(t) - \dot{\xi}(t-1). \end{cases}$$

• **Example**: Flexible rod (Mounier and co.):

$$(1) \quad \begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases}$$

$$\Leftarrow \begin{cases} y_1(t) = \xi(t) + \xi(t-2), \\ y_2(t) = 2\,\xi(t-1), \\ u(t) = \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases} \qquad (2)$$

$(1) \Rightarrow \theta(t) \triangleq 2\,y_1(t-1) - y_2(t) - y_2(t-2), \ \dot{\theta}(t) = 0.$

For e.g., if $0 \neq c \in \mathbb{R}$, then the solution of (1)

$$y_1(t) = -c/2, \ y_2(t) = -c, \ u(t) = 0,$$

is **not parametrized** by (2).

Ordinary differential & discrete systems

• **Example**: Let us consider the system

$$\ddot{y}(t) + \alpha(t)\,\dot{y}(t) + y(t) = \dot{u}(t) + \alpha(t)\,u(t),$$

where α is a **function of time**.

A **parametrization** of the system is defined by:

$$\begin{cases} y(t) = \dot{\xi}(t) + \alpha(t)\,\xi(t), \\ u(t) = \ddot{\xi}(t) + \alpha(t)\,\dot{\xi}(t) + (1 + \dot{\alpha}(t))\,\xi(t). \end{cases}$$

• **Example**: Wave equation (Curtain-Zwart)

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(t, x) - \frac{\partial^2 z}{\partial x^2}(t, x) = 0, \quad (\star) \\ \frac{\partial z}{\partial x}(t, 0) = 0, \\ \frac{\partial z}{\partial x}(t, 1) = u(t), \\ y(t) = \frac{\partial z}{\partial t}(1, t). \end{cases}$$

$$\Rightarrow y(t) - y(t-2) = u(t) + u(t-2). \; (\star\star)$$

The **parametrization** of the system $(\star\star)$ is given by:

$$\begin{cases} y(t) = \xi(t) + \xi(t-2), \\ u(t) = \xi(t) - \xi(t-2). \end{cases}$$

Electromagnetism

- **First set of Maxwell equations**:

$$\left\{ \begin{array}{l} \frac{\partial \vec{B}}{\partial t} + \nabla \wedge \vec{E} = \vec{0}, \\ \vec{\nabla} . \vec{B} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \vec{\nabla} \wedge \vec{A}, \end{array} \right.$$

where $(\vec{A}, V)$ is called the **quadri-potential**.

Gauge transformation:

$$\forall f : (\vec{A}, V) \longmapsto \left(\vec{A} + \vec{\nabla} f, V + \frac{\partial f}{\partial t} \right)$$

because $\left\{ \begin{array}{l} \vec{\nabla} V + \frac{\partial \vec{A}}{\partial t} = 0, \\ \vec{\nabla} \wedge \vec{A} = \vec{0}, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} V = \frac{\partial f}{\partial t}, \\ \vec{A} = \vec{\nabla} f. \end{array} \right.$

- **Second set of Maxwell equations** (duality):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} . \vec{\jmath} = 0 \Leftrightarrow \left\{ \begin{array}{l} \vec{\jmath} = \vec{\nabla} \wedge \vec{H} - \frac{\partial \vec{D}}{\partial t}, \\ \rho = \vec{\nabla} . \vec{D}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{\partial \vec{D}}{\partial t} + \vec{\nabla} \wedge \vec{H} = 0, \\ \vec{\nabla} . \vec{D} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{H} = \vec{\nabla} \phi_1 + \frac{\partial \vec{\phi_2}}{\partial t}, \\ \vec{D} = \vec{\nabla} \wedge \vec{\phi_2}. \end{array} \right.$$

$$\left\{ \begin{array}{l} \vec{\nabla} \phi_1 + \frac{\partial \vec{\phi_2}}{\partial t} = 0, \\ \vec{\nabla} \wedge \vec{\phi_2} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \phi_1 = \frac{\partial \psi}{\partial t}, \\ \vec{\phi_2} = \vec{\nabla} \psi. \end{array} \right.$$

$$\vec{B}/\mu_0 = \vec{H}, \quad \epsilon_0 \vec{E} = \vec{D}.$$

Flat linear systems

- **Definition**: (Fliess-Mounier) A linear system

$$R\left(\frac{d}{dt},\underline{\delta}\right)\begin{pmatrix} x(t)\\ y(t)\\ u(t)\end{pmatrix} = 0 \ \ (\star)$$

is **flat** if ($\underline{\delta} = (\delta_{h_1},\ldots,\delta_{h_r})$ incommensurable):

1. The system $(\star)$ is **parametrizable**:

$$(\star) \Leftrightarrow \begin{pmatrix} x(t)\\ y(t)\\ u(t)\end{pmatrix} = P\left(\frac{d}{dt},\underline{\delta}\right)\xi(t).$$

2. The **parameter** ξ is an element of the system i.e.

$$\xi(t) = S\left(\frac{d}{dt},\underline{\delta}\right)\begin{pmatrix} x(t)\\ y(t)\\ u(t)\end{pmatrix},$$

$\Leftrightarrow$ the matrix P admits a **left-inverse**:

$$S\left(\frac{d}{dt},\underline{\delta}\right) P\left(\frac{d}{dt},\underline{\delta}\right) = I.$$

- **Example**: The following system is **flat**

$$\dot{y}(t-h)+y(t) = u(t-h) \Leftrightarrow \begin{cases} y(t) = \xi(t-h),\\ u(t) = \dot{\xi}(t-h)+\xi(t),\end{cases}$$

because $\xi(t) = -\dot{y}(t)+u(t)$ (called **flat output**).

π-free linear systems

- **Definition**: (Fliess-Mounier) The system

$$R\left(\frac{d}{dt}, \underline{\delta}\right) \left(\begin{array}{c} x(t) \\ y(t) \\ u(t) \end{array}\right) = 0 \;\; (\star)$$

is π-**free** if:

1. The system $(\star)$ is **parametrizable**:

$$(\star) \Leftrightarrow \left(\begin{array}{c} x(t) \\ y(t) \\ u(t) \end{array}\right) = P\left(\frac{d}{dt}, \underline{\delta}\right) \xi(t).$$

2. The **parameter** ξ satisfies an equation of the form

$$\pi(\underline{\delta})\, \xi(t) = S\left(\frac{d}{dt}, \underline{\delta}\right) \left(\begin{array}{c} x(t) \\ y(t) \\ u(t) \end{array}\right),$$

$\Leftrightarrow$ there exists a matrix P such that:

$$S\left(\frac{d}{dt}, \underline{\delta}\right) P\left(\frac{d}{dt}, \underline{\delta}\right) = \pi\, I.$$

- **Example**: The following system is δ_h-**free**

$$\dot{y}(t) = u(t-h) \Leftrightarrow \left\{ \begin{array}{l} y(t) = \xi(t-h), \\ u(t) = \dot{\xi}(t), \end{array} \right.$$

because we have $\delta_h\, \xi(t) = y(t)$.

Motion planning

- Let us consider the flexible rod (Mounier and co):

$$\ddot{y}(t) + \ddot{y}(t-2) + \dot{y}(t) - \dot{y}(t-2) = v(t-1).$$

- The system $(\star)$ is **parametrizable**:

$$\begin{cases} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases}$$

- The system **is not flat** but δ**-free** because:

$$\delta\,\xi(t) = y(t).$$

- If y_r is a **desired trajectory**, then we define

$$\xi_r(t) \triangleq y_r(t+1),$$

and we obtain the **open-loop control law**:

$$\begin{aligned} v_r(t) &= \ddot{\xi}_r(t) + \ddot{\xi}_r(t-2) + \dot{\xi}_r(t) - \dot{\xi}_r(t-2) \\ &= \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) - \dot{y}_r(t-1). \end{aligned}$$

- We need to **stabilize the system around the desired trajectory**:

$\Rightarrow$ **closed-loop control law** (difficult problem).

Poles placement

- Let us consider the system:

$$D\left(\frac{d}{dt},\underline{\delta}\right)\,y(t)=N\left(\frac{d}{dt},\underline{\delta}\right)\,u(t)\quad(1).$$

- Let us consider the following feedback law:

$$A\left(\frac{d}{dt},\underline{\delta}\right)\,u(t)=B\left(\frac{d}{dt},\underline{\delta}\right)\,y(t)\quad(2)$$

- If (1) is **parametrizable**, then:

$$(1)\Leftrightarrow\left(\begin{array}{c} y(t)\\ u(t)\end{array}\right)=\left(\begin{array}{c}\widetilde{N}\left(\frac{d}{dt},\underline{\delta}\right)\\ \widetilde{D}\left(\frac{d}{dt},\underline{\delta}\right)\end{array}\right)\xi(t).$$

- The **closed-loop dynamic** is given by:

$$(B\,\widetilde{N}-A\,\widetilde{D})\,\xi(t)=0.$$

- Let us consider a closed-loop dynamic S. Then, there exists a feedback law (2) satisfying

$$(B\,:\,-A)\left(\begin{array}{c}\widetilde{N}\\ \widetilde{D}\end{array}\right)=S\quad(3)$$

iff $S_i\in k[\frac{d}{dt},\underline{\delta}]^{m+p}\left(\begin{array}{c}\widetilde{N}\\ \widetilde{D}\end{array}\right)$, where $S=\left(\begin{array}{c}S_1\\ \vdots\\ S_m\end{array}\right)$.

- If (1) is a **flat system**, then (3) is always feasible:

$$(B\,:\,-A)=S\,(-\widetilde{Y}\,:\,\widetilde{X})+Q\,(D\,:\,-N),\ \ \forall\,Q,$$

$$\text{with}\quad \xi(t)=-\widetilde{Y}\,y(t)+\widetilde{X}\,u(t).$$

Optimal control

- **Problem**: Let us minimize the cost function

$$\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\, dt$$

where $\dot{x}(t) + x(t) - u(t) = 0$, $x(0) = x_0$.

- $\dot{x}(t) + x(t) - u(t) = 0$ is **parametrized** by:

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases} \qquad (1)$$

- By substitution of (1) in the cost, we are led to the following **variational problem without constraints**:

$$\min \ \tfrac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\, dt,$$

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases}$$

$$\Rightarrow \begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{cases}$$

which, by integrations, gives the **controller**:

$$u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\, e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\ e^{-\sqrt{2}(t-T)}}\, x(t).$$

Methodology

1. A **linear system** Σ is defined by a **matrix with entries** R **in a ring** D, i.e., $R\,z = 0\,(\Sigma)$.

1. Using the matrix R, **we define a** D**-module** M.

2. We develop **a dictionary between the properties of the system** Σ **and the module** M.

3. We use **module theory** in order to classify the properties of the module M.

4. We use **homological algebra** in order to check the properties of the module M.

5. Using effective algebra, we develop **effective algorithms** which check the properties of the module M, and thus, of the system Σ.

6. **Implementation** in *OreModules* (Maple):

`http://wwwb.math.rwth-aachen.de/OreModules/`

Ore algebras

• **Definition**: The non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in ∂ is called **skew** if

$$\boxed{\partial\, a = \sigma(a)\, \partial + \delta(a), \quad a \in A,}$$

where $\sigma : A \longrightarrow A$ satisfies $\forall\, a, b \in A$:

$$\begin{cases} \sigma(1) = 1, \\ \sigma(a + b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b), \end{cases}$$

and $\delta : A \longrightarrow A$ is such that $\forall\, a, b \in A$:

$$\begin{cases} \delta(a + b) = \delta(a) + \delta(b), \\ \delta(a\, b) = \sigma(a)\, \delta(b) + \delta(a)\, b. \end{cases}$$

• **Definition**: (Chyzak-Salvy): The skew ring

$$D = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$$

is called an **Ore algebra** if :

$$\boxed{\begin{cases} \sigma_i\, \delta_j = \delta_j\, \sigma_i, \quad 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad j < i. \end{cases}}$$

$\Rightarrow D$ is a **non-commutative ring**.

Examples of Ore algebras

- **Ordinary differential operators**:

$$D = A[\tfrac{d}{dt}; 1, \tfrac{d}{dt}], \quad A = k[t], k(t),$$

$$P = \textstyle\sum_{i=0}^{m} a_i(t) \tfrac{d^i}{dt^i} \in D, \quad \tfrac{d}{dt}\, a(t) = \dot{a}(t).$$

- **Time-delay (time-advance) operators**:

$$D = A[\delta_h; \sigma_h, 0], \quad A = k[t], k(t),$$

$$P = \textstyle\sum_{i=0}^{m} a_i(t) \delta_h^i \in D, \quad \sigma_h a(t) = a(t-h).$$

- **Shift operators**:

$$D = A[\delta_1; \sigma, 0], \quad A = k[n], k(n),$$

$$P = \textstyle\sum_{i=0}^{m} a_i(n) \delta_1^i \in D, \quad \delta_1 a(n) = a(n+1).$$

- **Differential time-delay operators**:

$$D = A[\tfrac{d}{dt}; 1, \tfrac{d}{dt}][\delta_h; \sigma_h, 0], \quad A = k[t], k(t),$$

$$P = \textstyle\sum_{0 \leq i+j \leq m}^{n} a_{ij}(t) \tfrac{d^i}{dt^i}\, \delta_h^j \in D.$$

- **Partial differential operators**:

$$D = A[d_1; 1, \partial_1] \ldots [d_n; 1, \partial_n], \; A = k[x_1, \ldots, x_n],$$

$$P = \textstyle\sum_{0 \leq |\mu| \leq m} a_\mu(x)\, d^\mu, \; d^\mu = d_1^{\mu_1} ... d_n^{\mu_n}, \; \partial_i = \tfrac{\partial}{\partial x_i}.$$

Gröbner bases

• **<u>Theorem</u>**: (Kredel): Let D be an Ore algebra s.t.

$$\sigma(x_j) = a_{ij}\, x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij},$$

$0 \neq a_{ij},\, b_{ij} \in k,\; c_{ij} \in k[x_1, \ldots, x_n],\; \deg(c_{ij}) \leq 1,$ then, **Gröbner bases** w.r.t. any term order can be computed algorithmically.

• **<u>Implementations</u>**:

$\Rightarrow$ **Maple Ore_algebra** (Chyzak, ALGO, INRIA).

`http://algo.inria.fr/libraries/`

$\Rightarrow$ **Plural** (University of Kaiserslautern).

`http://www.singular.uni-kl.de/plural/`

$\Rightarrow$ **Mathematica NCAlgebra** (Helton and co.).

`http://www.math.ucsd.edu/ ncalg/`

Systems-Modules

- Let a system be defined by the **equations**

$$R\,\eta = 0, \quad R \in D^{q\times p},$$

where η is the **system variables** (inputs, outputs, states, latent variables...) and D is an **Ore algebra**.

Like in **algebraic geometry**, we associate with the system the **left** D-module:

$$\boxed{M = D^{1\times p}/(D^{1\times q}\,R)}$$

- **Example**: The wind tunnel model (Manitius 84):

$$\left\{\begin{array}{l} \dot{x}_1(t) = -a\,x_1(t) + k\,a\,x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\,x_2(t) - 2\,\zeta\,\omega\,x_3(t) + \omega^2\,u(t). \end{array}\right. \qquad (\star)$$

- The system $(\star)$ is equivalent to:

$$\underbrace{\begin{pmatrix} \frac{d}{dt} + a & -k\,a\,\delta_h & 0 & 0 \\ 0 & \frac{d}{dt} & -1 & 0 \\ 0 & \omega^2 & \frac{d}{dt} + 2\,\zeta\,\omega & -\omega^2 \end{pmatrix}}_{R \in D^{3\times 4}} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{pmatrix} = 0$$

$$D = \mathbb{R}(a, k, \omega, \zeta)\left[\frac{d}{dt}, \delta_h\right], \quad \boxed{M = D^{1\times 4}/(D^{1\times 3}\,R)}.$$

Classification of modules

• **Definition**:

$a)$ M is **free** if $\exists\, r \in \mathbb{Z}_+ :\ M \cong D^r$.

$b)$ M is **projective** if $\exists\, r \in Z_+$ and a D-module P:

$$M \oplus P \cong D^r.$$

$c)$ M is **reflexive** if ϵ is an isomorphism:

$$\begin{array}{rrcl} \epsilon: & M & \longrightarrow & M^{\star\star}, \\ & m & \longmapsto & \epsilon(m), \qquad \epsilon(m)(f) = f(m). \end{array}$$

$d)$ M is **torsion-free** if:

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

$m \in t(M)$ is called a **torsion element** of M.

• **Theorem**:

1. **free** $\Rightarrow$ **projective** $\Rightarrow$ .. $\Rightarrow$ **reflexive** $\Rightarrow$ **torsion-free**.

2. If D **is a principal domain** (e.g. $K[\frac{d}{dt}]$), then:

torsion-free = free.

3. If $D = k[x_1, \ldots, x_n]$, where k is a field:

projective = free (Th. Quillen-Suslin).

Modules-systems-homological algebra

• $R \in D^{q\times p}, \quad M = D^{1\times p}/(D^{1\times q}\, R), \quad \widetilde{N} = D^{1\times q}/(D^{1\times p}\, \theta(R)).$

Systems	**Modules**	**Homological algebra**	$d(\widetilde{N})$
$\exists$ autonomous elements	with torsion	$\mathrm{ext}^1_D(\widetilde{N}, D) \neq 0$	$n-1$
Controllability Parametrizable $\pi\left(\delta_{\sigma(1)}, \ldots, \delta_{\sigma(n-1)}\right)$-free	torsion-free	$\mathrm{ext}^1_D(\widetilde{N}, D) = 0$	$n-2$
The parametrization is parametrizable $\pi\left(\delta_{\sigma(1)}, \ldots, \delta_{\sigma(n-2)}\right)$-free	reflexive	$\mathrm{ext}^i_D(\widetilde{N}, D) = 0,$ $i = 1, 2$	$n-3$
...	...	...	...
Bézout Identities Chain of n parametrizations Internal Stabilization	projective	$\mathrm{ext}^i_D(\widetilde{N}, D) = 0,$ $1 \leq i \leq n$	-1
Flatness	free	$\emptyset$	

Contributions due to Oberst/Fliess-Mounier/Pommaret-Q.

Involution

• **Definition**: An **involution** of an Ore algebra D is a k-linear map $\theta : D \to D$ satisfying:

1. $\theta(a_1\, a_2) = \theta(a_2)\, \theta(a_1), \quad a_1, \, a_2 \in D,$

2. $\theta^2 = id_D$.

• If $R \in D^{q \times p}$, then we have:

$$\boxed{\theta(R) \triangleq (\theta(R_{ij}))^T \in D^{p \times q}}.$$

• **Example**: 1. If $D = k[x_1, \ldots, x_n]$, then $\theta = id_D$.

2. If $D = A\left[\frac{d}{dt}, \, \delta_h, \, \delta_{-h}\right]$, then an involution of D is defined by:

$$t \mapsto t, \quad \frac{d}{dt} \mapsto -\frac{d}{dt}, \quad \delta_h \mapsto \delta_{-h}, \quad \delta_{-h} \mapsto \delta_h.$$

Let $R = [t\, \frac{d}{dt} \, : \, -t^2\, \delta_h] \in D^{1 \times 2}$, then we have:

$$\theta(R) = \begin{pmatrix} -\frac{d}{dt}\, t \\ -\delta_{-h}\, t^2 \end{pmatrix} = \begin{pmatrix} -t\, \frac{d}{dt} + 1 \\ -(t+h)^2\, \delta_{-h} \end{pmatrix}.$$

• **Left** D-module $N \stackrel{\theta}{\Longleftrightarrow}$ **Right** D-module $\widetilde{N}$:

$$\forall\, P \in D, \; \forall n \in N : \; P \circ n = n\, \theta(P).$$

Extension functor

- **Parametrizability**:

$$P\left(\frac{d}{dt},\underline{\delta}\right)\,\xi(t)=\eta(t) \overset{?}{\Longleftrightarrow} R\left(\frac{d}{dt},\underline{\delta}\right)\,\eta(t)=0$$

- **Hints**: We follow steps 1, 2, 3 and 4:

$$\begin{array}{lccl}
4. \quad \theta(Q)\left(\frac{d}{dt},\underline{\delta}\right)\,\xi(t)=\eta(t) & \Longrightarrow & R\left(\frac{d}{dt},\underline{\delta}\right)\,\eta(t)=0 & 1. \\
\Uparrow & & \Downarrow & \\
\text{involution } \theta & & \text{involution } \theta & \\
\Uparrow & & \Downarrow & \\
3. \quad 0=Q\left(\frac{d}{dt},\underline{\delta}\right)\,\mu(t) & \overset{\text{G.B.}}{\Longleftrightarrow} & \mu(t)=\theta(R)\left(\frac{d}{dt},\underline{\delta}\right)\,\lambda(t) & 2.
\end{array}$$

$$\begin{aligned} Q\circ\theta(R)=0 \Rightarrow \theta(Q\circ\theta(R)) &= \theta^2(R)\circ\theta(Q) \\ &= R\circ\theta(Q)=0. \end{aligned}$$

- **The last step 5**:

$$\theta(Q)\left(\frac{d}{dt},\underline{\delta}\right)\,\xi(t)=\eta(t) \overset{\text{G.B.}}{\Longleftrightarrow} R'\left(\frac{d}{dt},\underline{\delta}\right)\,\eta(t)=0$$

$$\boxed{\text{ext}^1_D(\widetilde{N},D)=(D^{1\times q'}\,R')/(D^{1\times q}\,R)} \qquad \text{(G.B.)}$$

where $\widetilde{N}=D^{1\times q}/(D^{1\times p}\,\theta(R)) \Longleftrightarrow \theta(R)\,\lambda=0.$

The wind tunnel model

- **Is the wind tunnel model parametrizable?**

$$\begin{cases} \dot{x}_1(t) + a\, x_1(t) - k\, a\, x_2(t-h) = 0, \\ \dot{x}_2(t) - x_3(t) = 0, \\ \dot{x}_3(t) + \omega^2\, x_2(t) + 2\,\zeta\,\omega\, x_3(t) - \omega^2\, u(t) = 0. \end{cases} \quad (1)$$

$$\begin{cases} \left(\frac{d}{dt} + a\right)\, \lambda_1 = \mu_1, \\ -k\, a\, \delta_h\, \lambda_1 + \frac{d}{dt}\, \lambda_2 + \omega^2\, \lambda_3 = \mu_2, \\ -\lambda_2 + \left(\frac{d}{dt} + 2\,\zeta\,\omega\right)\, \lambda_3 = \mu_3, \\ -\omega^2\, \lambda_3 = \mu_4. \end{cases} \quad (2)$$

$$\omega^2\, k\, a\, \delta_h\, \mu_1 + \left(\omega^2\, \tfrac{d}{dt} - \omega^2\, a\right)\, \mu_2 + \left(\omega^2\, \tfrac{d^2}{dt^2} + \omega^2\, a\, \tfrac{d}{dt}\right)\, \mu_3$$
$$+ \left(\tfrac{d^3}{dt^3} + 2\,\zeta\,\omega\, \tfrac{d^2}{dt^2} + a\, \tfrac{d^2}{dt^2} + \omega^2\, \tfrac{d}{dt} + 2\, a\,\zeta\,\omega\, \tfrac{d}{dt} + a\,\omega^2\right)\, \mu_4 = 0. \quad (3)$$

$$\begin{cases} x_1(t) = & -\omega^2\, k\, a\, \xi(t-h), \\ x_2(t) = & -\omega^2\, \dot{\xi}(t) + a\,\omega^2\, \xi(t), \\ x_3(t) = & \omega^2\, \ddot{\xi}(t) - \omega^2\, a\, \dot{\xi}(t), \\ u(t) = & -\xi(t)^{(3)} + (2\,\zeta\,\omega + a)\, \ddot{\xi}(t) \\ & -(\omega^2 + 2\, a\,\omega\,\zeta)\, \dot{\xi}(t) + a\,\omega\, \xi(t). \end{cases} \quad (4)$$

The compatibility conditions of (4) are exactly generated by (1) $\Rightarrow$ **the system is parametrized by (4).**

The wind tunnel model

• **Is the wind tunnel model flat?**

$$\begin{cases} x_1(t) = & -\omega^2\, k\, a\, \xi(t-h), \\ x_2(t) = & -\omega^2\, \dot{\xi}(t) + a\, \omega^2\, \xi(t), \\ x_3(t) = & \omega^2\, \ddot{\xi}(t) - \omega^2\, a\, \dot{\xi}(t), \\ u(t) = & -\xi(t)^{(3)} + (2\,\zeta\,\omega + a)\, \ddot{\xi}(t) \\ & -(\omega^2 + 2\, a\, \omega\, \zeta)\, \dot{\xi}(t) + a\, \omega\, \xi(t). \end{cases} \tag{4}$$

• The system **is not flat** but δ**-free** because:

$$\xi(t) = -\frac{1}{\omega^2\, k\, a}\, \delta^{-1}\, x_1(t) = -\frac{1}{\omega^2\, k\, a}\, x_1(t+h).$$

• If $y(t) = x_1(t)$ is the output of the system, then we can solve the **tracking problem**:

$$\xi_r(t) = -\frac{1}{\omega^2\, k\, a}\, y_r(t+h)$$

$$\begin{aligned} \Rightarrow u_r(t) = \quad & -\tfrac{1}{\omega^2\, k\, a}\, (-y_r(t+h)^{(3)} + (2\,\zeta\,\omega + a)\, \ddot{y}_r(t+h) \\ & -(\omega^2 + 2\, a\, \omega\, \zeta)\, \dot{y}_r(t+h) + a\, \omega\, y_r(t+h)). \end{aligned}$$

• We need to find a **controller stabilizing the system around the desired trajectory**.

OreModules

- ***OreModules*** is a tool-box developed in *Maple*.
- ***OreModules*** uses *Mgfun* developed by F. Chyzak

`http://algo.inria.fr/chyzak/mgfun.html.`

- ***OreModules*** is developed by Chyzak-Q.-Robertz.
- ***Oremodules*** can handle linear systems of ODEs, PDEs, differential time-delay systems, multidimensional discrete systems. . .
- ***OreModules*** computes:

1. autonomous elements, non-controllable elements,
2. parametrizations of under-determined systems,
3. left-/right-/generalized inverses,
4. flat outputs of a flat system, π-polynomials,
5. first integrals of motion,
6. Euler-Lagrange equations. . .

- A **second release is available** on the web page:

`http://wwwb.math.rwth-aachen.de/OreModules.`

List of the functions

Main functions
Parametrization MinimalParametrization(s) AutonomousElements LeftInverse(Rat) LocalLeftInverse RightInverse(Rat) GeneralizedInverse(Rat) PiPolynomial FirstIntegral LQEquations
Module theory
TorsionElements Exti(Rat) Extn(Rat) Quotient(Rat) SyzygyModule(Rat) Resolution(Rat) FreeResolution(Rat) OreRank
Some low-level functions
DefineOreAlgebra Involution Factorize Mult ApplyMatrix

Extensions to non-linear systems?

Electrical circuit containing a **LC transmission line** (V. Rasvan, S-I. Niculescu, *Oscillations in lossless propagation models: a Liapunov-Krasovskii approach*, IMA J. Math. Contr. Inform., 19 (2002), 151-172):

$$\left\{\begin{array}{l} R_1\, C_1 \dot{v}_1(t) + \left(1 + R_1 \sqrt{\frac{C}{L}}\right)\, v_1(t) - 2\, R_1 \sqrt{\frac{C}{L}}\, \eta_2(t-h) \\ + R_1\, f_1(v_1(t)) + E(t) = 0, \\ \left(1 + R_1 \sqrt{\frac{C}{L}}\right)\, C_2\, \dot{v}_2(t) + \sqrt{\frac{C}{L}}\, v_2(t) - 2\, \sqrt{\frac{C}{L}}\, \eta_1(t-h) = 0, \\ \eta_1(t) - v_1(t) + \eta_2(t-h) = 0, \\ \eta_2(t) - \frac{1}{(1+R_2\sqrt{C}L)}\, v_2(t) + \frac{(1-R_2\sqrt{C}L)}{(1+R_2\sqrt{C}L)}\, \eta_1(t-h) = 0, \end{array}\right.$$

where $h = \sqrt{L\, C}$.

• Applying **OreModules** to the system without the non-linearity and coming back to the original system, we obtain that the system is δ-flat.

• An **explicit parametrization** of the system is:

$$
\left\{
\begin{array}{l}
v_1(t) = \frac{1}{2}\,\frac{C_2}{\sqrt{\frac{C}{L}}}\,\dot v_2(t+h) + \frac{1}{2}\,C_2\,R_2\,\dot v_2(t+h) + \frac{1}{2}\,v_2(t+h),\\
v_2(t) = v_2(t),\\
\eta_1(t) = \frac{1}{2}\,\left(v_2(t+h) + \frac{\left(C2 + C2\,R2\,\sqrt{\frac{C}{L}}\right)}{\sqrt{\frac{C}{L}}}\,\dot v_2(t)\right),\\
\eta_2(t) = \frac{1}{2}\,\left(v_2(t) + \frac{\left(-C2 + C2\,R2\,\sqrt{\frac{C}{L}}\right)}{\sqrt{\frac{C}{L}}}\,\dot v_2(t)\right),\\
E(t) = -\frac{1}{2}\,C_2\,R_2\,R_1\,C_1\,\ddot v_2(t+h) + \frac{1}{2}\,C_2\,R_1\,\dot v_2(t-h)\\
-\frac{1}{2}\,C_2\,R_2\,\dot v_2(t+h) - \frac{1}{2}\,\frac{C_2\,R_1\,C_1}{\sqrt{\frac{C}{L}}}\,\ddot v_2(t+h) - \frac{1}{2}\,v_2(t+h)\\
+\frac{3}{2}\,\sqrt{\frac{C}{L}}\,R_1\,v_2(t-h) - \frac{1}{2}\,\sqrt{\frac{C}{L}}\,R_1\,v_2(t+h) - \frac{1}{2}\,\frac{C_2}{\sqrt{\frac{C}{L}}}\,\dot v_2(t+h)\\
-\frac{1}{2}\,R_1\,C_1\,\dot v_2(t+h) - \frac{1}{2}\,C_2\,R_2\,\sqrt{\frac{C}{L}}\,R_1\,\dot v_2(t+h)\\
-\frac{1}{2}\,C_2\,R_1\,\dot v_2(t+h) + \frac{1}{2}\,\frac{C_2}{\sqrt{\frac{C}{L}}}\,\dot v_2(t-h) - \frac{1}{2}\,v_2(t-h)\\
+\frac{1}{2}\,\frac{C_2\,R_1\,C_1}{\sqrt{\frac{C}{L}}}\,\ddot v_2(t-h) - \frac{1}{2}\,C_2\,R_2\,\dot v_2(t-h)\\
-\frac{1}{2}\,C_2\,R_2\,\sqrt{\frac{C}{L}}\,R_1\,\dot v_2(t-h) - \frac{1}{2}\,C_2\,R_2\,R_1\,C_1\,\ddot v_2(t-h)\\
-\frac{1}{2}\,R_1\,C_1\,\dot v_2(t-h) + R_1\,f_1\,\Bigg(\frac{1}{2}\,\frac{C_2}{\sqrt{\frac{C}{L}}}\,\dot v_2(t+h)\\
+\frac{1}{2}\,C_2\,R_2\,\dot v_2(t+h) + \frac{1}{2}\,v_2(t+h) - \frac{1}{2}\,\frac{C_2}{\sqrt{\frac{C}{L}}}\,\dot v_2(t-h)\\
+\frac{1}{2}\,C_2\,R_2\,\dot v_2(t+h) + \frac{1}{2}\,v_2(t-h)\ \Bigg).
\end{array}
\right.
$$

• See Mounier-Rudolph (Internat. J. Control 98) for different examples of **non-linear δ-flat systems**.

Bibliography

• J. Assan, **Analyse et synthèse de l'approche géométrique pour les systèmes linéaires sur un anneau**, PhD thesis, Ecole Centrale de Nantes, 25/10/99.

• F. Chyzak, **Fonctions holonomes en calcul formel**, PhD thesis, Ecole Polytechnique, 27/05/98.

• F. Chyzak, B. Salvy, **Non-commutative elimination in Ore algebras proves multivariate identities**, J. Symbolic Computation, 26 (1998), 187-227.

• F. Chyzak, A. Quadrat, D. Robertz, **Effective algorithms for parametrizing linear control systems over Ore algebras**, INRIA report 5181, 29/04/04, submitted.

• G. Conte, A. M. Perdon, **Systems over rings: geometric theory and applications**, Annual Reviews in Control 24 (2000), 113-124.

• D. Cox, J. Little, D. O'Shea, **Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra**, UTM, Springer 1997.

• M. Fliess, H. Mounier, **Controllability and observability of linear delay systems: an algebraic approach**, ESAIM COCV, 3 (1998), 301-314.

• G-M. Greuel, G. Pfister, **A Singular Introduction to Commutative Algebra**, Springer 2002.

• L. Habets, **Algebraic and computational aspects of time-delay systems**, PhD thesis, Eindhoven University of Technology, 1994.

• L. Habets, **Computational aspects of systems over rings – reactability and stabilizability**, CWI Quarterly, 9 (1996), 85-95.

• H. Li, **Non-commutative Gröbner Bases and Filtered-Graded Transfer**, Lecture Notes in Mathematics 1795, Springer, 2002.

• A. Manitius, **Feedback controllers for a wind tunnel model involving a delay: analytical design and numerical simulations**, IEEE Trans. Automa. Contr., 29 (1984), 1058-1068.

• H. Mounier, **Propriétés structurelles des systèmes linéaires à retards: aspects théoriques et pratiques**, PhD thesis, University of Paris XI, 24/10/95.

• H. Mounier, J. Rudolph, **Flatness-based control of nonlinear delay systems: a chemical reactor example**, Internat. J. Control, 71 (1998), 871-890.

• J.-F. Pommaret, **Partial Differential Control Theory**, Kluwer 2001.

• J.-F. Pommaret, A. Quadrat, **Algebraic analysis of linear multidimensional control systems**, IMA J. Math. Contr. Inform., 16 (1999), 275-297.

• A. Quadrat, **Analyse algébrique des systèmes linéaires multidimensionnels**, PhD thesis, Ecole Nationale des Ponts et Chaussées, 23/09/99.

• V. Rasvan, S-I. Niculescu, **Oscillations in lossless propagation models: a Liapunov-Krasovskii approach**, IMA J. Math. Contr. Inform., 19 (2002), 151-172.

• E. Sontag, **Linear systems over commutative rings: a survey**, Ricerche Automat., 7 (1976), 1-34.