Equivalences of Linear Control Systems

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Abstract

We show how homological algebra and algebraic analysis allow to give various notions of equivalence for linear control systems which do not depend on their presentations and therefore preserve their structural properties.

Keywords: System equivalence, homotopic equivalence, projective equivalence, primeness of multidimensional systems, extension and torsion functors.

1 Introduction

Many notions of equivalence have been developed for linear control systems after the work of Rosenbrock [8]. See for example [1] as well as the different references inside. One fondamental idea of equivalence theory is to know which informations on the system are preserved when passing from one form to another (e.g. Kalman form, polynomial forms, transfer matrices...).

It is well known that we can associate an A-module M to any matrix R with entries in an integral domain A. The interest of using M rather than R is that the algebraic properties of M do only depend on the module itself and not on its presentation matrix R. Indeed, a module M can be defined by plenty of equivalent presentations, i.e. by totally different matrices having sometimes quite different sizes. For example, a second order ordinary differential (OD) equation is equivalent to two first order OD equations when A is a polynomial ring in one indeterminate.

For studying systems, we shall propose techniques of homological algebra which only depend on M and not on the choice of a presentation of the system, i.e. on the choice of the resolution of the corresponding A-module M. In particular, one can associate an A-module M to any linear control system and introduce a new A-module N. A major idea of this paper, first noted in [2], is to study M by means of homological properties of N that only depend on M and to achieve by this way a complete solution of the conjecture recently proposed on the various types of primeness [11].

This new approach, using modules, is very close to the behavioural approach of Willems [10] and has never been used for applications, up to our knowledge, as one must notice that a concept like projective equivalence that will be used in this paper, does not admit any classical/operator counterpart.

2 Homotopic equivalence

We shall denote by A an integral domain which is supposed to be either a commutative ring or a *left Ore domain*, i.e. a domain such that:

$$\forall\,(a,b)\in A^2,\ \exists\,(u,v)\in (A\backslash 0)^2:\ u\,a=v\,b.$$

Definition 1. [9] Let M be a finitely generated left A-module. Then,

- M is *free* if $M\cong A^r$ for a certain $r\in\mathbb{N}$,
- M is *projective* if there exist an A-module N and $r\in\mathbb{N}$ such that $M\oplus N\cong A^r$
- M is *reflexive* if the A-morphism

  $$\epsilon: M\rightarrow \hom_A(\hom_A(M,A),A),$$

  defined by $\epsilon(m)(f)=f(m),\ \forall\, f\in\hom_A(M,A)$, is an isomorphism,
- $t(M)=\{m\in M\mid \exists\, 0\neq a\in A,\ a\,m=0\}$ is the *torsion* submodule of M. M is a *torsion-free* A-module if $t(M)=0$ and M is a *torsion* A-module if $t(M)=M$.

Let us recall the following definition of complexes and exact sequences [9].

Definition 2. • A *complex* $P=(P_i,d_i)$ is a sequence of left A-modules P_i and of A-morphisms $d_i: P_i\rightarrow P_{i-1}$ such that:

$$d_i\circ d_{i+1}=0\Leftrightarrow \mathrm{im}\, d_{i+1}\subseteq \ker d_i.$$

- We call the r^{th} *module of homology* of a complex $P = (P_i, d_i)$, the left A-module

$$H_r(P) = \ker\, d_r / \mathrm{im}\, \mathrm{d_{r+1}}.$$

- A complex $P = (P_i, d_i)$ is said to be *exact at* F_r if $\mathrm{im}\, d_{r+1} = \ker d_r \Leftrightarrow H_r(P) = 0$, and $P = (P_i, d_i)$ is *exact* if it is exact at any F_r.

Example 1. If $P = (P_i, d_i)$ is any complex, then we have the following exact sequence $0 \longrightarrow \mathrm{im}\, d_{i+1} \xrightarrow{j_i} \ker d_i \xrightarrow{\pi_i} H_i(P) \longrightarrow 0$ for any i, where j_i denotes the inclusion A-morphism and π_i the A-morphism which maps any element of $\ker d_i$ into its class in $H_i(P)$.

We have the following proposition. See [9] for a proof.

Proposition 1. *Any A-module M has a projective resolution, that is to say, there exists an exact sequence of the form*

$$\ldots \xrightarrow{d_i} P_{i-1} \xrightarrow{d_{i-1}} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0, \quad (1)$$

where P_i is a projective A-module for any $i \geq 0$. If P_i is a free A-module for any $i \geq 0$, then (1) is called a free resolution of M.

Example 2. Let us consider the matrix $R_1 = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ with entries in the ring $D = \mathbb{R}[d_1, d_2]$ of differential operators with real coefficients and

$$\begin{array}{ccccc} D^2 & \xrightarrow{.R_1} & D & \xrightarrow{\pi} & M \longrightarrow 0 \\ (a_1\ a_2) & \longrightarrow & (a_1\ a_2)\, R_1 & & \end{array}$$

the beginning of a free resolution of the D-module M defined by the system of partial differential equations (PDE)

$$\begin{cases} d_1\, y = 0, \\ d_2\, y = 0, \end{cases}$$

where $y = \pi(1)$ is the class in M of $1 \in D$. The kernel of the D-morphism $.R_1$ is defined by the couple $(a_1\ a_2) \in D^2$ satisfying:

$$a_1\, d_1 + a_2\, d_2 = 0. \quad (2)$$

D is a polynomial ring and d_1 (resp. d_2) does not divide d_2 (resp. d_1). Thus, using the Gauss lemma [3, 9], all the solutions of (2) have the form:

$$\begin{cases} a_1 = b\, d_2, \quad b \in D, \\ a_2 = -b\, d_1, \end{cases}$$

Finally, if we note $R_2 = (d_2\ \ -d_1)$, we obtain the following free resolution of M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^2 \xrightarrow{.R_1} D \xrightarrow{\pi} M \longrightarrow 0.$$

Definition 3.

- Let $\ldots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \ldots$ and $\ldots \longrightarrow P'_i \xrightarrow{d'_i} P'_{i-1} \longrightarrow \ldots$ be two complexes of A-modules. We call *morphism of complexes* $f : (P_i, d_i) \rightarrow (P'_i, d'_i)$ a set of A-morphisms $f_i : P_i \rightarrow P'_i$ such that $d'_i \circ f_i = f_{i-1} \circ d_i$ for any i, i.e. such that we have the following commutative diagram:

$$\begin{array}{ccccccc} \ldots \longrightarrow & P'_{i+1} & \xrightarrow{d'_{i+1}} & P'_i & \xrightarrow{d'_i} & P'_{i-1} & \longrightarrow \ldots \\ & \uparrow f_{i+1} & & \uparrow f_i & & \uparrow f_{i-1} & \\ \ldots \longrightarrow & P_{i+1} & \xrightarrow{d_{i+1}} & P_i & \xrightarrow{d_i} & P_{i-1} & \longrightarrow \ldots \end{array}$$

- A morphism of complexes $f : (P_i, d_i) \rightarrow (P'_i, d'_i)$ is *homotopic* to zero if there exist A-morphisms $s_i : P_i \rightarrow P'_{i+1}$ such that $f_i = d'_{i+1} \circ s_i + s_{i-1} \circ d_i$ for any i, i.e. such that we have the following diagram:

$$\begin{array}{ccccccc} \ldots \longrightarrow & P'_{i+1} & \xrightarrow{d'_{i+1}} & P'_i & \xrightarrow{d'_i} & P'_{i-1} & \longrightarrow \ldots \\ \nwarrow s_{i+1} & \uparrow f_{i+1} & \nwarrow s_i & \uparrow f_i & \nwarrow s_{i-1} & \uparrow f_{i-1} & \\ \ldots \longrightarrow & P_{i+1} & \xrightarrow{d_{i+1}} & P_i & \xrightarrow{d_i} & P_{i-1} & \longrightarrow \ldots \end{array}$$

  By extension, we shall say that two morphisms of complexes $f, f' : (P_i, d_i) \rightarrow (P'_i, d'_i)$ are homotopic if $f - f'$ is homotopic to zero.

- A morphism of complexes $f : (P_i, d_i) \rightarrow (P'_i, d'_i)$ is an *homotopism* if there exists $f' : (P'_i, d'_i) \rightarrow (P_i, d_i)$ such that $f \circ f' - \mathrm{id}_{P'}$ and $f' \circ f - \mathrm{id}_P$ are homotopic to zero, and the complexes $P = (P_i, d_i)$ and $P' = (P'_i, d'_i)$ are said to be *homotopy equivalent*.

Proposition 2. *If (P_i, d_i) (resp. (P'_i, d'_i)) is a projective resolution of an A-module M (resp. A-module M'), then any A-morphism $f : M \rightarrow M'$ induces a morphism of complexes $f : (P_i, d_i) \rightarrow (P'_i, d'_i)$ uniquely defined up to an homotopy.*

Proof. We have the following diagram

$$\begin{array}{ccc} P'_0 & \xrightarrow{\pi'} & M' \longrightarrow 0, \\ f_0 \nwarrow & & \uparrow f \circ \pi \\ & & P_0 \end{array}$$

where f_0 exists and satisfies $\pi' \circ f_0 = f \circ \pi$ because P_0 is a projective A-module [9]. Then, we have $\pi' \circ f_0 \circ d_1 = f \circ \pi \circ d_1 = 0 \Rightarrow \mathrm{im}\, (f_0 \circ d_1) \subseteq \ker \pi' = \mathrm{im}\, d'_1$. Thus, we have the following diagram

$$\begin{array}{ccc} P'_1 & \xrightarrow{d'_1} & \mathrm{im}\, d'_1 \longrightarrow 0, \\ f_1 \nwarrow & & \uparrow f_0 \circ d_1 \\ & & P_1 \end{array}$$

where f_1 exists and satisfies $d'_1 \circ f_1 = f_0 \circ d_1$ because P_1 is a projective A-module... Hence, there exists a morphism of complexes f_i satisfying $d'_i \circ f_i = f_{i-1} \circ d_i, \ \forall\ i \geq 0$, and $f_{-1} = f$.

Let us suppose that there exists an other A-morphism $g_i : P_i \rightarrow P'_i$ satisfying $d'_i \circ g_i = g_{i-1} \circ d_i, \ i \geq 0$, with

$g_{-1} = f$. Then, we have $\pi' \circ (f_0 - g_0) = (f - f) \circ \pi = 0 \Rightarrow \operatorname{im}(f_0 - g_0) \subseteq \ker \pi' = \operatorname{im} d_1'$ and we obtain the following diagram

$$\begin{array}{ccl} P_1' & \xrightarrow{d_1'} & \operatorname{im} d_1' \longrightarrow 0, \\ s_0 & \nwarrow & \uparrow f_0 - g_0 \\ & & P_0 \end{array}$$

where s_0 exists and satisfies $f_0 - g_0 = d_1' \circ s_0$ because P_0 is a projective A-module. If we note $s_{-1} = 0 : M \to P_0'$, then we have $f_0 - g_0 = d_1' \circ s_0 + s_{-1} \circ \pi$. Moreover, we have $d_1' \circ (f_1 - g_1 - s_0 \circ d_1) = d_1' \circ f_1 - d_1' \circ g_1 - d_1' \circ s_0 \circ d_1 = f_0 \circ d_1 - g_0 \circ d_1 - d_1' \circ s_0 \circ d_1 = (f_0 - g_0 - d_1' \circ s_0) \circ d_1 = 0$ because $f_0 - g_0 - d_1' \circ s_0 = s_{-1} \circ \pi$ and thus $\operatorname{im}(f_1 - g_1 - s_0 \circ d_1) \subseteq \ker d_1' = \operatorname{im} d_2'$. Thus, we have the following diagram

$$\begin{array}{ccl} P_2' & \xrightarrow{d_2'} & \operatorname{im} d_2' \longrightarrow 0, \\ s_1 & \nwarrow & \uparrow f_1 - g_1 - s_0 \circ d_1 \\ & & P_1 \end{array}$$

where s_1 exists and satisfies $f_1 - g_1 = d_2' \circ s_1 + s_0 \circ d_1$ because P_1 is a projective A-module... Hence, f and g are homotopic. □

Theorem 1. *Let (P_i, d_i) and (P_i', d_i') be two projective resolutions of an A-module M, then there exists an homotopism between (P_i, d_i) and (P_i', d_i').*

Proof. Let $\mathrm{id}_M : M \to M$ be the identity A-morphism, then from proposition 2, there exist $f_i : P_i \to P_i'$ satisfying $f_{i-1} \circ d_i = d_i' \circ f_i$, $i \geq 0$, with $f_{-1} = \mathrm{id}_M$. Similarly, there exist $g_i : P_i' \to P_i$ satisfying $g_{i-1} \circ d_i' = d_i \circ g_i$, $i \geq 0$, with $g_{-1} = \mathrm{id}_M$. Thus, $(g_{i-1} \circ f_{i-1}) \circ d_i = g_{i-1} \circ d_i' \circ f_i = d_i \circ (g_i \circ f_i)$, and $\mathrm{id} : (P_i, d_i) \to (P_i', d_i')$ and $h : (P_i, d_i) \to (P_i, d_i)$, defined by $h_i = g_i \circ f_i$, are homotopic by proposition 2. Then, there exist $s_i : P_i \to P_{i+1}$ such that $\mathrm{id}_{P_i} - g_i \circ f_i = d_{i+1} \circ s_i + s_{i-1} \circ d_i$, $i \geq 0$. Moreover, we have $d_i' \circ (f_i \circ g_i) = (f_{i-1} \circ d_i) \circ g_i = f_{i-1} \circ g_{i-1} \circ d_i'$, which implies that the morphisms of complexes $\mathrm{id}_{P'} : P' \to P'$ and $k : P' \to P'$, defined by $k_i = f_i \circ g_i$, are homotopic and thus there exist $s_i' : P_i' \to P_{i+1}'$ such that $\mathrm{id}_{P_i} - f_i \circ g_i = d_{i+1} \circ s_i' + s_{i-1}' \circ d_i$, $i \geq 0$. Hence, the two projective resolutions (P_i, d_i) and (P_i', d_i') are homotopy equivalent. □

If (P_i, d_i) and (P_i', d_i') are two free resolutions of the left A-module M, then, using canonical basis of $P_i \cong A^{l_i}$, we can represent d_i by a matrix R_i. Thus, if (A^{l_i}, R_i) and $(A^{l_i'}, R_i')$ are two free resolutions of M, then there exist matrices $T_i \in A^{l_i \times l_i'}$, $T_i' \in A^{l_i' \times l_i}$ and $S_i \in A^{l_i \times l_{i+1}'}$, $S_i' \in A^{l_i' \times l_{i+1}}$ such that:

$$\left\{ \begin{array}{l} T_i\, R_i' = R_i\, T_{i-1}, \\ R_i'\, T_{i-1}' = T_i'\, R_i, \\ T_i\, T_i' = I_{l_i} + S_i\, R_{i+1} + S_{i-1}\, R_i, \\ T_i'\, T_i = I_{l_i'} + S_i'\, R_{i+1}' + S_{i-1}'\, R_i', \end{array} \right. \tag{3}$$

where I_{l_i} is the $l_i \times l_i$ identity matrix. See [7] for more details and examples.

Example 3. Let us consider the system $\ddot{y} - 2\,\dot{y} - \dot{u} + u = 0$, $A = \mathbb{R}[\frac{d}{dt}]$, and the A-module M defined by the free resolution $0 \longrightarrow A \xrightarrow{.R} A^2 \xrightarrow{\pi} M \longrightarrow 0$ with $R = (\frac{d^2}{dt^2} - 2\,\frac{d}{dt} \quad -\frac{d}{dt} + 1)$, $\pi(f_1) = y$, $\pi(f_2) = u$, where $\{f_1, f_2\}$ is the canonical basis of A^2. Moreover, let us consider a second system

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 + v, \\ \dot{x}_2 = 2\,x_2 + v, \end{array} \right.$$

and the A-module M' defined by

$$0 \longrightarrow A^2 \xrightarrow{.R'} A^3 \xrightarrow{\pi'} M \longrightarrow 0,$$

with

$$R' = \left(\begin{array}{ccc} \frac{d}{dt} & -1 & -1 \\ 0 & \frac{d}{dt} - 2 & -1 \end{array} \right),$$

$\pi'(e_1) = x_1$, $\pi'(e_2) = x_2$, $\pi'(e_3) = v$, where $\{e_1, e_2, e_3\}$ is the canonical basis of A^3.

Let us define the A-morphisms $f : M \to M'$ and $g : M' \to M$ by:

$$\left\{ \begin{array}{l} f(y) = x_1, \\ f(u) = v, \end{array} \right. \qquad \left\{ \begin{array}{l} g(x_1) = y, \\ g(x_2) = \dot{y} - u, \\ g(v) = u. \end{array} \right.$$

We easily verify that $g \circ f = \mathrm{id}_M$, $f \circ g = \mathrm{id}_{M'} \Rightarrow M \cong M'$. Let us show that the two different free resolutions of M are homotopy equivalent. We have the commutative exact diagram

$$\begin{array}{ccl} A^2 & \xrightarrow{\pi} & M \longrightarrow 0, \\ g_0 & \nwarrow & \uparrow g \circ \pi' \\ & & A^3 \end{array}$$

where $g \circ \pi' : A^3 \to M$ is defined in the canonical basis by:

$$\left\{ \begin{array}{l} (g \circ \pi')(e_1) = y = \pi(f_1), \\ (g \circ \pi')(e_2) = \dot{y} - u = \pi(\dot{f}_1 - f_2), \\ (g \circ \pi')(e_3) = u = \pi(f_2). \end{array} \right.$$

Therefore, we can define the morphism $g_0 : A^3 \to A^2$ by:

$$\left\{ \begin{array}{l} g_0(e_1) = f_1, \\ g_0(e_2) = \dot{f}_1 - f_2, \\ g_0(e_3) = f_2, \end{array} \right. \Leftrightarrow g_0((a_1\ a_2\ a_3)) = (a_1\ a_2\ a_3)\, V_0,$$

where

$$V_0 = \left(\begin{array}{cc} 1 & 0 \\ \frac{d}{dt} & -1 \\ 0 & 1 \end{array} \right)$$

and $a_i \in A$. Then, we have the following commutative diagram

$$\begin{array}{ccl} A & \xrightarrow{.R} & \operatorname{im}(.R) \longrightarrow 0, \\ f_1 & \nwarrow & \uparrow (.R') \circ (.V_0) = .(R'\, V_0) \\ & & A^2 \end{array}$$

with

$$R'\,V_0 = \begin{pmatrix} 0 & 0 \\ \frac{d^2}{dt^2} - 2\,\frac{d}{dt} & -\frac{d}{dt} + 1 \end{pmatrix},$$

and we easily verify that $f_1((a_1\ a_2)) = (a_1\ a_2)\,V_1$ where $V_1 = (0\ \ 1)'$ and $a_i \in A$. We let the reader check by himself that, doing similarly with f, we obtain the following commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & A & \xrightarrow{.R} & A^2 & \xrightarrow{\pi} & M \longrightarrow 0 \\ & .V_1 \uparrow\downarrow .U_1 & & .V_0 \uparrow\downarrow .U_0 & & g \uparrow\downarrow f \\ 0 \longrightarrow & A^2 & \xrightarrow{.R'} & A^3 & \xrightarrow{\pi'} & M' \longrightarrow 0, \end{array}$$

where $U_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_1 = (\frac{d}{dt} - 2\ \ 1),$

$$R\,U_0 = U_1\,R', \qquad V_1\,R = R'\,V_0.$$

Therefore, we have the following commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & A^2 & \xrightarrow{.R'} & A^3 & \xrightarrow{\pi'} & M' \longrightarrow 0 \\ & \uparrow .(V_1\,U_1) & & \uparrow .(V_0\,U_0) & & \uparrow g \circ f = \mathrm{id}_{M'} \\ 0 \longrightarrow & A^2 & \xrightarrow{.R'} & A^3 & \xrightarrow{\pi'} & M' \longrightarrow 0, \end{array}$$

which implies from proposition 2 that the morphisms of complexes $f \circ g$ and $\mathrm{id}_{M'}$ are homotopic. Thus, from theorem 1, there exists a 3×2 matrix S such that we have the following commutative diagram

$$\begin{array}{ccc} 0 \longrightarrow & A^2 \xrightarrow{.R'} & R' A^2 \longrightarrow 0, \\ & .S' \nwarrow & \uparrow .(I_3 - V_0 U_0) = .\begin{pmatrix} 0 & 0 & 0 \\ -\frac{d}{dt} & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ & & A^3 \end{array}$$

i.e. $I_3 - V_0\,U_0 = S'\,R'$, and a trivial computation gives:

$$S' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, we finally find that $I_2 - V_1\,U_1 = R'\,S'$. Doing similarly for $g \circ f$ and id_M, we let the reader check by himself that we obtain $S = 0$, $U_0\,V_0 = I_2$ and $U_1\,V_1 = 1$.

3 Projective equivalence

Definition 4. Two A-modules M and M' are said to be *projective equivalent* if there exist two projective A-modules P and P' such that $N \oplus P \cong N' \oplus P'$.

We have the following *Schanuel's lemma* [9].

Lemma 1. *If* $0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$ *and* $0 \longrightarrow L' \longrightarrow P' \longrightarrow M \longrightarrow 0$ *are two exact sequences with* P *and* P' *two projective* A*-modules, then* $L \oplus P' \cong L' \oplus P$.

The main result of the paper is to prove the following delicate theorem which does not seem to appear in the literature and is essential for applications [5, 6].

Theorem 2. *[7] If* M *is a left* A*-module defined by the projective resolution* $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$ *and* N *is the right* A*-module defined by* $0 \longleftarrow N \longleftarrow P_0^\star \xleftarrow{d_1^\star} P_1^\star$*, where* $P_i^\star = \hom_A(P_i, A)$ *and* $d_i^\star(f) = f \circ d_i$*, then* N *is defined up to a projective equivalence.*

To prove this theorem, we need the following lemma which is proved in [3].

Lemma 2. *Let* $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$ *and* $P_1' \xrightarrow{d_1'} P_0' \xrightarrow{\pi'} M' \longrightarrow 0$ *be projective resolutions of two* A*-modules* M *and* M' *and* $\phi : M \to M'$ *an isomorphism. Then, there exist an isomorphism* $\alpha : P_0 \oplus P_0' \to P_0 \oplus P_0'$ *and an isomorphism* $\beta : P_1 \oplus P_1' \oplus P_0 \oplus P_0' \to P_1 \oplus P_1' \oplus P_0 \oplus P_0'$ *such that we have the following commutative diagram:*

$$\begin{array}{ccc} P_1 \oplus P_0' \oplus P_0 \oplus P_1' & \xrightarrow{(d_1 \oplus \mathrm{id}_{P'_0}, 0)} & P_0 \oplus P_0' \\ \downarrow \beta & & \downarrow \alpha \\ P_1 \oplus P_0' \oplus P_0 \oplus P_1' & \xrightarrow{(0, \mathrm{id}_{P_0} \oplus d'_1)} & P_0 \oplus P_0'. \end{array}$$

Moreover, we have:

$$\mathrm{coker}(d_1 \oplus \mathrm{id}_{P_0'}, 0)^\star \cong \mathrm{coker}(0, \mathrm{id}_{P_0} \oplus d_1')^\star.$$

Proof. Now, we can prove theorem 2. If $P = (P_i, d_i)$ and $P' = (P_i', d_i')$ are two projective resolutions of an A-module M, then we have the commutative exact diagram given by the figure 1.

Let Q' be the kernel of the morphism $\ker(d_1 \oplus \mathrm{id}_{P_0'}, 0) \to \ker d_1$ induced by $P_1 \oplus P_1' \oplus P_0 \oplus P_1' \to P_1$. Then, a chase in the diagram of the figure 1 gives the following exact sequence [9]:

$$0 \longrightarrow Q' \longrightarrow P_1' \oplus P_0 \oplus P_0' \xrightarrow{(0, \mathrm{id}_{P'_0})} P_0' \longrightarrow 0.$$

Thus, $Q' \cong \ker(0, \mathrm{id}_{P_0'}) = P_1' \oplus P_0$ is a projective A-module. Applying the functor $\hom_A(\cdot, A)$ [9] to the diagram defined in the figure 1, we obtain the exact commutative diagram given by the figure 2 where $N'' = \mathrm{coker}\,(d_1 \oplus \mathrm{id}_{P_0'}, 0)^\star$, $N = \mathrm{coker}\, d_1$ and the two central vertical and the upper horizontal sequences are exact because they are dual of exact sequences composed only with projective modules [9]. A chase in the diagram of the figure 2 gives the exact sequence $0 \longrightarrow N \longrightarrow N'' \longrightarrow Q'^\star \longrightarrow 0$ which splits because $Q'^\star$ is a projective A-module [9]. Therefore, we have $N'' \cong N \oplus Q'^\star$.

Doing similarly with the resolution $P_1' \xrightarrow{d_1'} P_0' \longrightarrow M \longrightarrow 0$ and substituting the medium horizontal sequence of the figure 1 by the exact sequence given by the

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & P'_0 \oplus P_0 \oplus P'_1 & & P'_0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker(d_1 \oplus \mathrm{id}_{P'_0}, 0) & \longrightarrow & P_1 \oplus P'_0 \oplus P_0 \oplus P'_1 & \xrightarrow{(\mathrm{d_1 \oplus id_{P'_0}}, 0)} & P_0 \oplus P'_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \ker d_1 & \longrightarrow & P_1 & \xrightarrow{\mathrm{d}_1} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

Figure 1: commutative exact diagram

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
0 & \longleftarrow & Q'^{\star} & \longleftarrow & (P'_0 \oplus P_0 \oplus P'_1)^{\star} & \longleftarrow & P'^{\star}_0 & \longleftarrow & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & N'' & \longleftarrow & (P_1 \oplus P'_0 \oplus P_0 \oplus P'_1)^{\star} & \xleftarrow{(\mathrm{d_1 \oplus id_{P'_0}}, 0)^{\star}} & (P_0 \oplus P'_0)^{\star} & \longleftarrow & M^{\star} & \longleftarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \| & & \\
0 & \longleftarrow & N & \longleftarrow & P^{\star}_1 & \xleftarrow{\mathrm{d}^{\star}_1} & P^{\star}_0 & \longleftarrow & M^{\star} & \longleftarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 2: dual of the previous commutative exact diagram

$$
0 \longrightarrow \ker(0, \mathrm{id}_{P_0} \oplus d'_1) \longrightarrow P_1 \oplus P'_0 \oplus P_0 \oplus P'_1 \xrightarrow{(0, \mathrm{id_{P_0} \oplus d'_1})} P_0 \oplus P'_0 \longrightarrow M \longrightarrow 0
$$

Figure 3: horizontal sequence

figure 3, we obtain $N''' \cong N' \oplus Q^\star$, where $N' = \mathrm{coker} d_1'^\star$ and $N''' = \mathrm{coker}(0, \mathrm{id}_{P_0} \oplus d_1')^\star$.

But, from lemma 2, we know that $N'' \cong N'''$ and thus $N \oplus Q'^\star \cong N' \oplus Q^\star$, that is to say:

$$N \oplus P_0^\star \oplus P_1'^\star \cong N' \oplus P_0'^\star \oplus P_1^\star. \qquad (4)$$

□

Theorem 3. *[3] If a left A-module M is defined by two projective resolutions $0 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$ and $0 \longrightarrow P_1' \xrightarrow{d_1'} P_0' \xrightarrow{\pi'} M \longrightarrow 0$, then we have:*

$$N = \mathrm{coker} d_1^\star \cong N' = \mathrm{coker} d_1'^\star.$$

Example 4. Let us consider the following equivalent systems $\ddot{y} - \dot{u} = 0$ and:

$$\left\{ \begin{array}{l} \dot{x}_1 = 0, \\ \dot{x}_2 = x_1 + v. \end{array} \right.$$

We easily check that we have the following commutative exact diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & A & \xrightarrow{.R} & A^2 & \xrightarrow{\pi} & M \longrightarrow 0 \\ & .V_1 \uparrow\downarrow .U_1 & & .V_0 \uparrow\downarrow .U_0 & & g \uparrow\downarrow f \\ 0 \longrightarrow & A^2 & \xrightarrow{.R'} & A^3 & \xrightarrow{\pi'} & M' \longrightarrow 0, \end{array} \qquad (5)$$

with $\quad R = (\frac{d^2}{dt^2} \quad -\frac{d}{dt}), \quad R' = \left(\begin{array}{ccc} \frac{d}{dt} & 0 & 0 \\ -1 & \frac{d}{dt} & -1 \end{array} \right),$

$$\left\{ \begin{array}{l} f(y) = x_2, \\ f(u) = v, \end{array} \right. \quad \left\{ \begin{array}{l} g(x_1) = \dot{y} - u, \\ g(x_2) = y, \\ g(v) = u, \end{array} \right. \quad V_1 = \left(\begin{array}{c} 1 \\ 0 \end{array} \right),$$

$$V_0 = \left(\begin{array}{cc} \frac{d}{dt} & -1 \\ 1 & 0 \\ 0 & 1 \end{array} \right), \; U_1 = \left(\begin{array}{c} 1 \\ \frac{d}{dt} \end{array} \right)', \; U_0 = \left(\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

Moreover, we easily check that $g \circ f = \mathrm{id}_M$ and $f \circ g = \mathrm{id}_{M'}$ and thus $M \cong M'$.

Hence, we have the following commutative exact diagram:

$$\begin{array}{ccccc} 0 \longleftarrow & N & \xleftarrow{p} & A & \xleftarrow{R.} & A^2 \\ & & & V_1. \downarrow\uparrow U_1. & & \\ 0 \longleftarrow & N' & \xleftarrow{p'} & A^2 & \xleftarrow{R'.} & A^3. \end{array} \qquad (6)$$

The A-module N is defined by the equation

$$z\,(\frac{d^2}{dt^2} - \frac{d}{dt}) = 0,$$

where $z = p(1)$, whereas the A-module N' is defined by

$$\left\{ \begin{array}{l} z_1 \frac{d}{dt} - z_2 = 0, \\ z_2 \frac{d}{dt} = 0, \\ z_2 = 0, \end{array} \right.$$

where, $z_1 = p'(f_1), z_2 = p'(f_2)$ and $\{f_1, f_2\}$ is the canonical basis of A^2. The matrices V_1 and U_1 induce the morphisms $h : N \to N'$ and $k : N' \to N$ respectively defined by $h(n) = p'(V_1\, a)$ with $p(a) = n$ and $k(n') = p(U_1\, l)$ with $p'(l) = n'$. Then, we obtain $h(z) = p'(V_1) = z_1$ and

$$\left\{ \begin{array}{l} k(z_1) = p(U_1\, f_1) = z, \\ k(z_2) = p(U_1\, f_2) = z\, \frac{d}{dt}. \end{array} \right.$$

We check that we have $k \circ h = \mathrm{id}_N$ and $h \circ k = \mathrm{id}_{N'}$, i.e. $N \cong N'$.

Hence, even if the A-modules N and N' are defined by totally different numbers of unknowns and equations, we have $N \cong N'$. See [7] for more details, examples of multidimensional control systems and applications to the theory of linear elasticity. See also [6] for applications in control theory.

We let the reader check by himself that the A-modules M and M' defined in example 3 satisfy $N = A/R\,A^2 = 0$, $N' = A^2/A^3\,R' = 0$ and thus $N = N'$.

4 Applications of equivalences

Proposition 3. *[9] If (P_i, d_i) and (P_i', d_i') are two homotopy equivalent complexes, then we have:*

$$H_i(P) \cong H_i(P'), \; \forall\, i.$$

If $\ldots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\pi} M \longrightarrow 0$ is a free resolution of the left A-module M and S a left A-module, then the abelian groups of cohomology of the complex

$$\ldots \xleftarrow{d_2^\star} \hom_A(F_1, S) \xleftarrow{d_1^\star} \hom_A(F_0, S) \longleftarrow 0, \qquad (7)$$

where $d_i^\star(f) = f \circ d_i, \; \forall\, f \in \hom_A(F_{i-1}, S)$, do not depend on the free resolution of M and are called $\mathrm{ext}_A^i(M, S)$ (see [9] for more details). Hence, we have:

$$\left\{ \begin{array}{l} \mathrm{ext}_A^0(M, S) = \hom_A(M, S), \\ \mathrm{ext}_A^i(M, S) = \ker d_{i+1}^\star / \mathrm{im}\, d_i^\star, \; \forall\, i \geq 1. \end{array} \right.$$

The fact that the abelian groups of cohomology of the complex (7) do not depend on the projective resolution of M comes from the fact that two different projective resolutions of M are homotopy equivalent by theorem 1. Then, the dual sequences (7) of the two projective resolutions are homotopy equivalent, a fact which implies, by proposition 3, that the abelian groups of cohomology of the two corresponding complexes (7) are isomorphic.

Example 5. Let us consider again example 4. We let the reader check by himself that $\mathrm{ext}_A^1(M, A) = N$ and $\mathrm{ext}_A^1(M', A) = N'$. Hence, using theorem 3, we have $N \cong N' \Rightarrow \mathrm{ext}_A^1(M, A) \cong \mathrm{ext}_A^1(M', A)$.

Let K be a differential field containing $\mathbb{Q}$ and $D = K[d_1, ..., d_n]$ the ring of scalar differential linear operators with coefficients in K [5]. Let M be the left D-module defined by $D^l \xrightarrow{.R} D^m \xrightarrow{\pi} M \longrightarrow 0$, and N the right D-module by:

$$0 \longleftarrow N \longleftarrow D^l \stackrel{R.}{\longleftarrow} D^m \longleftarrow \hom_D(M, D) \longleftarrow 0.$$

We have seen that theorem 1 and proposition 3 imply that the right D-modules $\mathrm{ext}^i_D(N, D),\ i \geq 1$, do not depend on the resolution of N. Moreover, by theorem 2, $\mathrm{ext}^i_D(N, D),\ i \geq 1$, do only depend on M [7] because N is defined up to a projective equivalence and $\mathrm{ext}^i_D(P, D) = 0,\ \forall\, i \geq 1$, for any projective module P (see [9] for more details).

We are now able to give applications of these two notions of equivalence. For that, let us recall a few well known definitions of primeness [4, 11].

Definition 5. Let R be a $l \times m\,(1 \leq l \leq m)$ full rank matrix with entries in $D = \mathbb{C}[d_1, ..., d_n] \cong \mathbb{C}[\chi_1, ..., \chi_n]$. Then, we say that:

- R is *minor left-prime* if there is no common factor in the $l \times l$ minors of R,
- R is *weakly zero left-prime* if all the $l \times l$ minors of R vanish all together on a finite set of points of $\mathbb{C}^n$,
- R is *zero left-prime* if all the $l \times l$ minors of R never vanish all together.

We have the following inclusions [4, 11]:

zero left-prime	$\subseteq$ weakly zero left-prime
weakly zero left-prime	$\subseteq$ minor left-prime.

In 1998, Wood, Rogers and Owens have conjectured in [11] that these three above definitions and inclusions were in fact elements of a chain of n successive definitions. We have the following results where the homotopic and projective equivalences are essential to prove that the algebraic properties of M do not depend on the presentation (i.e. on the matrix R) and where $d(M) = \dim(D/\mathrm{ann}(M))$ is the *krull dimension* of $D/\mathrm{ann}(M)$ [3], $\mathrm{ann}(M) = \{a \in A \mid a\,m = 0,\ \forall\, m \in M\}$ while $\mathrm{ext}^1_D(N, D) \cong \ker \epsilon$ and $\mathrm{ext}^2_D(N, D) \cong \mathrm{coker}\,\epsilon$ [2].

Module M	$\mathbf{ext}^i_D(N, D)$ [2, 5]	***d(N)*** [2, 5]	**Primeness** [4, 5, 11]
	$\mathrm{ext}^0_D(N, D) \neq 0$	n	
with torsion	$\mathrm{ext}^1_D(N, D) \cong t(M) \neq 0$	$n-1$	$\emptyset$
torsion-free	$\mathrm{ext}^1_D(N, D) = 0$	$n-2$	minor left prime
reflexive	$\mathrm{ext}^i_D(N, D) = 0,\ 1 \leq i \leq 2$	$n-3$	
.	.	.	.
.	.	.	.
	$\mathrm{ext}^i_D(N, D) = 0,\ 1 \leq i \leq n-1$	0	weakly zero left prime
projective	$\mathrm{ext}^i_D(N, D) = 0,\ 1 \leq i \leq n$	-1	zero left prime

In conclusion, we hope that these new techniques will open new perspectives for application of algebraic analysis to linear control theory.

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