# COHERENT $H_\infty(D)$-MODULES IN CONTROL THEORY

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Abstract: If $D$ is $\mathbb{C}_+$ or $\mathbb{D}$ , then we prove that $H_\infty(D)$ is a Hermite regular coherent ring of weak global dimension 2 (the coherence was already proved in [4]). We show:

- there is a one-to-one equivalence between systems defined by matrices with entries in $H_\infty(D)$ and coherent $H_\infty(D)$-modules,
- coherent $H_\infty(D)$-modules are stable by elementary algebraic operations, and thus, we can characterize the algebraic properties of coherent $H_\infty(D)$-modules using *algebraic analysis*,
- we generalize certain results known on $H_\infty(\mathbb{C}_+)$ to Hermite coherent domains with weak global dimension less or equal to 2. 

Keywords: $H_\infty(D)$, coherent rings and modules, Hermite rings, weak global dimension, primeness, extended Bézout identities, algebraic analysis.

## 1. INTRODUCTION

Internal stabilization of certain distributed parameters control systems uses Banach algebras as $L_\infty(\mathbb{R})$, $H_\infty(\mathbb{C}_+)$ or the Callier-Desoer algebras $\mathcal{A}(\beta)$ and $\hat{\mathcal{A}}(\beta)$ [3]. However, it is known that all noetherian Banach algebras are finite-dimensional [10]. Hence, $L_\infty(\mathbb{R})$, $H_\infty(\mathbb{C}_+)$ or the Callier-Desoer algebras $\mathcal{A}(\beta)$ and $\hat{\mathcal{A}}(\beta)$ are not noetherian rings, i.e. any ideal $I$ of these algebras cannot be written in general under the form $I = \sum_{i=1}^n A\, a_i$ for some $a_i \in A$ and $n \in \mathbb{N}$. A direct consequence of this remark is that we cannot study algebraic properties of these Banach algebras by means of concepts and techniques developed for noetherian rings, i.e. by means of the main part of classical algebra.

The concepts of *coherent rings* and *modules* first appeared in 1964 in some exercises of Bourbaki [1]. Coherent rings include noetherian rings, Bézout domains, semi-hereditary rings...[9]. We shall show that the class of coherent modules is stable by sums, intersections, quotients, tensor products, homomorphisms, localizations, and thus, constitutes a good class to work with.

It was proved in [4] that if $\mathbb{D}$ is the unity disc and $\mu$ is a positive measure, then $H_\infty(\mathbb{D})$ and $L_\infty(\mu)$ are coherent rings. The result on $H_\infty(\mathbb{D})$ was extended in [8] to any finitely connected domain $D$ of $\mathbb{C}$. Thus, $H_\infty(\mathbb{C}_+)$ is also a coherent domain.

We prove that $H_\infty(\mathbb{C}_+)$ is a regular ring of weak global dimension 2 and a Hermite ring [12] (see also [13]). The paper shows that there is a one-to-one correspondence between coherent $H_\infty(\mathbb{C}_+)$-modules and systems defined by matrices with entries in $H_\infty(\mathbb{C}_+)$. This correspondence is used to characterize *intrinsically* the algebraic properties of these systems. We show that the results of algebraic analysis obtained in [6,7] for linear multidimensional systems are still valid for regular coherent rings and, in particular, for $H_\infty(\mathbb{C}_+)$. Hence, we try to develop in this paper a theory of coherent $H_\infty(\mathbb{C}_+)$-modules for the study of certain classes of distributed systems as the theory of coherent $D$-modules is for the study of systems of linear partial differential equations [2].

## 2. COHERENT RINGS AND MODULES

In the course of the paper, $A$ denotes a commutative domain with a unity. We refer the reader [1,9] to for basic definitions of module theory.

*Definition 1.*
- An $A$-module $M$ is *finitely generated* if there exists an exact sequence of the form $F_0 \xrightarrow{d_0} M \longrightarrow 0$, where $F_0$ is a finite free $A$-modules, i.e. $F_0 \cong A^{r_0}$, $r_0 \in \mathbb{Z}_+$.
- An $A$-module $M$ is *finitely presented* (f.p.) if there exists an exact sequence of the form $F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0$, where $F_0$ and $F_1$ are finite free $A$-modules.

*Remark 1.* From Definition 1, we deduce that $M$ is a finitely generated $A$-module iff there exist $r_0 \in \mathbb{Z}_+$ and a finite family $\{y_1, \ldots, y_{r_0}\}$ of elements of $M$ such that:

$$m = \sum_{i=1}^{r_0} a_i\, y_i,\ a_i \in A,\ \forall\, m \in M.$$

If $\{e_1, \ldots, e_{r_0}\}$ (resp. $\{f_1, \ldots, f_{r_1}\}$) denotes the canonical basis of $F_0 \cong A^{r_0}$ (resp. $F_1 \cong A^{r_1}$), then $M$ is finitely presented iff $(y_j)_{1 \leq j \leq r_0}$ satisfies:

$$(d_0 \circ d_1)(f_i) = d_0 \left( \sum_{j=1}^{r_0} R_{ij}\, e_j \right) = \sum_{j=1}^{r_0} R_{ij}\, y_j = 0.$$

Thus, a finitely presented $A$-module is defined by a system with a finite number of unknowns ($r_0$) and equations ($r_1$).

*Definition 2.* An $A$-module $M$ is *coherent* if $M$ is a finitely generated $A$-module and if any finitely generated submodule of $M$ is finitely presented. A ring $A$ is *coherent* if it is coherent as an $A$-module.

*Proposition 1.* [1,2] If $A$ is a coherent ring, then an $A$-module is coherent iff it is finitely presented.

*Proposition 2.* (1) [1,2] Let $M_i$, $1 \leq i \leq 4$, and $M$ be $A$-modules such that we have the exact sequence:

$$M_1 \longrightarrow M_2 \longrightarrow M \longrightarrow M_3 \longrightarrow M_4.$$

If, $M_i$ are coherent $A$-modules for $1 \leq i \leq 4$, then $M$ is a coherent $A$-module.

(2) [1,2] Let $M, N, M' \subset M, M'' \subset M$ be coherent $A$-modules, $I$ a coherent ideal and any $A$-morphism $\phi : M \longrightarrow N$, then:

(a) $\ker \phi$, $\mathrm{im}\, \phi$, $\mathrm{coker}\, \phi$ and $\mathrm{coim}\, \phi$ are coherent $A$-modules.

(b) $M \oplus N, M' + M'', M' \cap M'', M/M'$ are coherent $A$-modules.

(c) $M \otimes_A N$ and $\hom_A(M, N)$ are coherent $A$-modules.

(d) If $S$ is a multiplicative set, then $S^{-1} A$ is coherent $A$-module.

(e) $IM$ is coherent $A$-module.

(f) $\mathrm{ann}(M)$ is a coherent ideal of $A$.

*Definition 3.*
- A *projective* (resp. *free, flat*) *resolution* of an $A$-module $M$ is an exact sequence of the form

$$\ldots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0, \quad (1)$$

where $P_i$ is a projective (resp. free, flat) $A$-module.
- We call *projective* (resp. *flat*) dimension $\mathrm{pd}_A(M)$ (resp. $\mathrm{w.dim}_A(M)$) of $M$ the minimum number $n \in \mathbb{N} \cup \{+\infty\}$ such that there exists a projective (resp. flat) resolution of $M$ of length $n$, i.e.:

$$0 \longrightarrow P_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \longrightarrow M \longrightarrow 0.$$

- The *weak global dimension* of a ring $A$ is:

$$\mathrm{w.gl.dim}(A) = \sup \{\mathrm{w.dim}_A(M) \mid \forall\ A - \text{module } M\}.$$

*Corollary 1.* Let $A$ be a coherent ring and $M$ a finitely presented $A$-module, then there exists a finite free resolution of $M$ ($F_i \cong A^{r_i}$, $r_i \in \mathbb{Z}_+$):

$$\ldots \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0. \quad (2)$$

*Proof.* By 1 of Proposition 2, we prove by induction that $A^{r_i}$, $r_i \in \mathbb{Z}$, is a coherent $A$-module. The kernel of an homomorphism between two coherent $A$-modules is a coherent $A$-module and, by Proposition 1, it is a finitely presented $A$-module...

*Remark 2.* Using the canonical basis of $F_i \cong A^{r_i}$, the exact sequence (2) can be written as

$$\ldots \xrightarrow{.R_2} A^{r_1} \xrightarrow{.R_1} A^{r_0} \longrightarrow M \longrightarrow 0, \quad (3)$$

where $R_i$ is a $r_i \times r_{i-1}$ matrix with entries in $A$ and $.R_i : A^{r_i} \to A^{r_{i-1}}$ is defined by letting operate a row vector of length $r_i$ on the left of $R_i$ to obtain a row vector of length $r_{i-1}$. By Remark 1, $M$ is defined by $R_1\, y = 0$, where $y_i$ is the class of $e_i$ in $M$ and $\{e_1, \ldots, e_{r_0}\}$ is the canonical basis of $A^{r_0}$.

*Definition 4.* Let $M$ and $N$ be two $A$-modules and a projective resolution (1) of $M$, then:

- The defects of exactness of

$$\ldots \xleftarrow{d_2^\star} \hom_A(P_1, N) \xleftarrow{d_1^\star} \hom_A(P_0, N) \longleftarrow 0, \quad (4)$$

where $d_i^\star$ is defined by $d_i^\star(f) = f \circ d_i$, $\forall\, f \in \hom_A(P_{i-1}, N)$, only depend on $M$ and $N$ and not on (1) [1,9]. They are called $\mathrm{ext}_A^i(M, N)$. Therefore, we have:

$$\begin{cases} \mathrm{ext}_A^0(M, N) = \ker d_1^\star = \hom_A(M, N), \\ \mathrm{ext}_A^i(M, N) = \ker d_{i+1}^\star / \mathrm{im}\, d_i^\star,\ i \geq 1. \end{cases}$$

- The defects of exactness of

$$\ldots \xrightarrow{\mathrm{id}_N\otimes d_2} N\otimes_A P_1 \xrightarrow{\mathrm{id}_N\otimes d_1} N\otimes_A P_0 \longrightarrow 0, \quad (5)$$

  where $\mathrm{id}_N \otimes d_i$ is defined by $(\mathrm{id}_N \otimes d_i)(n \otimes m) = n \otimes d_i(m)$, $\forall n \in N, \forall m \in P_i$, only depend on $M$ and $N$ and not on (1) [1,9]. They are called $\mathrm{tor}_i^A(M,N)$ and we have:

$$\begin{cases} \mathrm{tor}_0^A(M,N) = \mathrm{coker}(\mathrm{id}_N \otimes d_1) = N \otimes_A M, \\ \mathrm{tor}_i^A(M,N) = \ker(\mathrm{id}_N \otimes d_i)/\mathrm{im}(\mathrm{id}_N \otimes d_{i+1}). \end{cases}$$

*Remark 3.* If $A$ is a coherent ring, $M$ a coherent $A$-module, then $M$ has a finite free resolution of the form (3) and (4) is defined by

$$\ldots \xleftarrow{R_3.} N^{r_2} \xleftarrow{R_2.} N^{r_1} \xleftarrow{R_1.} N^{r_0} \longleftarrow 0,$$

where $R_i. : N^{r_{i-1}} \to N^{r_i}$ is defined by letting operate a column vector of length $r_{i-1}$ with entries in $N$ on the right of $R_i$ to obtain a column vector of length $r_i$ with entries in $N$. Therefore, we have:

$$\mathrm{ext}_A^i(M,N) = \ker_N(R_{i+1}.)/\,\mathrm{im}_N(R_i.),\ \forall i \geq 1.$$

Similarly, (5) becomes the complex

$$\ldots \xrightarrow{.R_2} N^{r_1} \xrightarrow{.R_1} N^{r_0} \longrightarrow 0,$$

where $.R_i : N^{r_i} \to N^{r_{i-1}}$ is defined by letting operate a row vector of length $r_i$ with entries in $N$ on the left of $R_i$ to obtain a row vector of length $r_{i-1}$ with entries in $N$. Therefore, we have:

$$\mathrm{tor}_i^A(M,N) = \ker_N(.R_i)/\mathrm{im}_N(.R_{i+1}),\ \forall i \geq 1.$$

*Corollary 2.* If $A$ is a coherent domain, $M$ and $N$ two coherent $A$-modules, then $\mathrm{ext}_A^i(M,N)$ and $\mathrm{tor}_i^A(M,N)$ are coherent $A$-modules for $i \geq 0$, and $\mathrm{ext}_A^i(M,A)$ is a torsion $A$-module for $i \geq 1$.

Let us note $M^\star = \hom_A(M,A)$.

*Definition 5.* Let $M$ be an $A$-module defined by a finite presentation:

$$F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0.$$

We call the *transposed module* of $M$, the $A$-module $T(M) = \mathrm{coker}\, d_1^\star$ defined by:

$$0 \longleftarrow T(M) \longleftarrow F_1^\star \xleftarrow{d_1^\star} F_0^\star.$$

*Theorem 1.* Let $A$ be a coherent integral domain and $K = Q(A)$ its field of fractions. If $M$ is a finitely presented $A$-module such as its transposed $A$-module $N = T(M)$ has a finite projective resolution of length $n$, then we have:

- $t(M) \cong \mathrm{tor}_1^A(K/A, M) \cong \mathrm{ext}_A^1(N,A)$,
- $M$ is torsion-free iff $\mathrm{ext}_A^1(N,A) = 0$,
- $M$ is reflexive iff $\mathrm{ext}_A^i(N,A) = 0,\ i = 1,2$,
- $M$ is projective iff $\mathrm{ext}_A^i(N,A) = 0, 1 \leq i \leq n$.

The proof can be obtained as in [6,7] in changing finitely generated modules (resp. noetherian rings) by finitely presented (resp. coherent) ones.

*Proposition 3.* Let $M$ be an $A$-module defined by the following finite free presentation

$$0 \longrightarrow F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0,$$

then $M$ is projective iff $T(M) \cong \mathrm{ext}_A^1(M,A) = 0$.

## 3. $H_\infty(D)$ IS A REGULAR COHERENT RING & W.GL.DIM$(H_\infty(D)) = 2$

*Theorem 2.* [8,4] If $D$ is a finitely connected domain of $\mathbb{C}$ and $\mu$ a positive measure, then $H_\infty(D)$ and $L_\infty(\mu)$ are coherent rings. In particular, $H_\infty(\mathbb{C}_+)$, $H_\infty(\mathbb{D})$, where $\mathbb{D}$ is the unit disc, $L_\infty(\mathbb{R}_+)$ and $L_\infty(\mathbb{R})$ are coherent rings.

We shall only consider the cases $D = \mathbb{C}_+$ and $\mathbb{D}$.

The proof of the coherence of $H_\infty(D)$ is based on the following theorem, which is a weak$-\star$ version of the Beurling-Lax theorem [5]. The condition on $m$ is given by point 2 of the *lemma on the local rank* page 44 of [5] (see also the remark page 45).

*Theorem 3.* Let $R$ be a $q \times p$-matrix with entries in $H_\infty(D)$ and the following $H_\infty(D)$-morphism:

$$\begin{array}{rrcl} R: & H_\infty(D)^p & \longrightarrow & H_\infty(D)^q \\ & z = (z_1,\ldots,z_p)^t & \longrightarrow & R\,z. \end{array}$$

Then, there exists a $p \times m$-matrix $R_{-1}$, with entries in $H_\infty(D)$, such that:

$$\ker R = R_{-1}\, H_\infty(D)^m, \quad (6)$$

$$\overline{R_{-1}^t(-\overline{s})}\, R_{-1}(s) = I_m, \quad (7)$$

$$m = p - \mathrm{rank}\, R, \quad (8)$$

where $\mathrm{rank}\, R$ is the maximal number of $H_\infty(D)$-linearly independent rows or columns of $R$.

*Corollary 3.* If $M$ is a finitely presented $H_\infty(D)$-module, then $\mathrm{pd}_{H_\infty(D)}(M) \leq 2$.

*Proof.* Let $H_\infty(D)^q \xrightarrow{.R} H_\infty(D)^p \longrightarrow M \longrightarrow 0$ be a finite presentation of $M$. By Theorem 3, up to a transposition, there exists a $r \times q$-matrix $R_1$ with entries in $H_\infty(D)$ such that we have the exact sequence:

$$H_\infty(D)^r \xrightarrow{.R_1} H_\infty(D)^q \xrightarrow{.R} H_\infty(D)^p \longrightarrow M \longrightarrow 0. \quad (9)$$

From the exactness of (9), we obtain:

$$\mathrm{rank}\,(\ker .R_1) + \mathrm{rank}\, M = r + p - q.$$

From the exact sequence

$$0 \longrightarrow \mathrm{im}.R \longrightarrow H_\infty(D)^p \longrightarrow M \longrightarrow 0,$$

we deduce $\mathrm{rank}\, M = p - \mathrm{rank}\, R$. Finally, from (8), we have $r = q - \mathrm{rank}\, R$. Therefore, $\mathrm{rank}\,(\ker .R_1) =$

0, i.e. $\ker .R_1$ is a torsion $H_\infty(D)$-module. But, $\ker .R_1$ is a sub-module of the free $H_\infty(D)$-module $H_\infty(D)^q$, which is only possible if $\ker .R_1 = 0$, because a free module is torsion-free. Hence, any finitely presented $H_\infty(D)$-module $M$ has a finite free resolution of length less or equal to 2.

*Example 1.* The ideal $I = \left(\frac{1}{s+1},\ e^{-s}\right)$ of $H_\infty(\mathbb{C}_+)$ has the following finite free resolution

$$0 \longrightarrow H_\infty(\mathbb{C}_+) \xrightarrow{R_{-1}.} H_\infty(\mathbb{C}_+)^2 \xrightarrow{R.} I \longrightarrow 0,$$

with $R = \left(\frac{1}{s+1}\ \ e^{-s}\right)$, $R_{-1} = \left(\frac{-1}{s+\sqrt{2}}\ \ \frac{s+1}{s+\sqrt{2}}\, e^{-s}\right)^t$. Hence, the $H_\infty(\mathbb{C}_+)$-module $N = H_\infty(\mathbb{C}_+)/I$ has a finite free resolution of length 2. Finally, $N$ is defined by the following two equations

$$e^{-s}\, z = 0, \quad \frac{1}{(s+1)}\, z = 0,$$

where $z$ is the class of 1 in $N$. We have

$$\inf_{\mathrm{Re} s > 0} \left( \mid e^{-s} \mid + \frac{1}{\mid s+1 \mid} \right) = 0,$$

and, by the Corona theorem [5], $1 \notin I \Leftrightarrow N \neq 0$, i.e. $I$ is not a free (projective) $H_\infty(\mathbb{C}_+)$-module.

*Corollary 4.* $\mathrm{w.gl.dim}(H_\infty(D)) = 2$.

*Proof.* Using Corollary 3 and the fact that every finitely presented flat module is projective [1,9], then any finitely presented $H_\infty(D)$-module $M$ has a finite flat resolution of length less or equal to 2:

$$\mathrm{w.dim}(M) = \mathrm{pd}(M) \leq 2.$$

$\mathrm{w.gl.dim}(H_\infty(D))$ is attained by taking the supremum of the weak dimension of finitely presented modules [1,9], and, using Example 1, we obtain:

$$\mathrm{w.gl.dim}(H_\infty(D)) = \sup\{\mathrm{pd}(M) \mid M \text{ f.p.}\} = 2.$$

*Remark 4.* Corollary 4 shows that, for any finitely presented $H_\infty(D)$-module $M$, we have:

$$\mathrm{ext}^i_{H_\infty(D)}(T(M), H_\infty(D)) = 0,\ \forall\, i \geq 3.$$

Hence, using Theorem 1, any finitely presented $H_\infty(D)$-module $M$ satisfies only one of the following cases: $t(M) \neq 0$, $M$ is torsion-free but not projective or $M$ is projective. These three cases are the *intrinsic formulations* of the well-known concepts for a matrix $R$ to be (or not to be ?) weakly-prime or to be strongly-prime [11].

*Example 2.* Let us consider the $H_\infty(\mathbb{C}_+)$-module $M$ defined by the following free presentation

$$0 \longrightarrow H_\infty(\mathbb{C}_+) \xrightarrow{.R} H_\infty(\mathbb{C}_+)^2 \longrightarrow M \longrightarrow 0,$$

where $R$ is defined in Example 1. We have the following exact finite free resolution of $N = T(M)$:

$$0 \longleftarrow N \longleftarrow H_\infty(\mathbb{C}_+) \xleftarrow{R.} H^2_\infty(\mathbb{C}_+) \xleftarrow{R_{-1}.} H_\infty(\mathbb{C}_+) \longleftarrow 0. \quad (10)$$

Dualizing (10), we obtain the sequence:

$$H_\infty(\mathbb{C}_+) \xrightarrow{.R} H^2_\infty(\mathbb{C}_+) \xrightarrow{.R_{-1}} H_\infty(\mathbb{C}_+) \longrightarrow 0.$$

We easily check that:

$$\begin{aligned} \mathrm{ext}^1(N, H_\infty(\mathbb{C}_+)) &= \ker(.R_{-1})/H_\infty(\mathbb{C}_+)\, R = 0, \\ \mathrm{ext}^2(N, H_\infty(\mathbb{C}_+)) &= H_\infty(\mathbb{C}_+)/H^2_\infty(\mathbb{C}_+)\, R_{-1}, \\ &= H_\infty(\mathbb{C}_+)/J. \end{aligned}$$

where $J = \left(\frac{-1}{s+\sqrt{2}},\ \frac{s+1}{s+\sqrt{2}}\, e^{-s}\right)$ is an ideal of $H_\infty(\mathbb{C}_+)$. We easily check that $J = I$, where $I$ is defined in Example 1. Thus:

$$1 \notin J = I \Rightarrow \mathrm{ext}^2_{H_\infty}(N, H_\infty(\mathbb{C}_+)) \neq 0.$$

Hence, $M$ is torsion-free but not projective.

*Example 3.* Similarly, we easily check that the $H_\infty(\mathbb{C}_+)$-module $M = H_\infty(\mathbb{C}_+)^2/H_\infty(\mathbb{C}_+)\, R$, where $R = \left(\frac{s-1}{s+1}\ \ -\frac{e^{-s}}{s+1}\right)$, is projective. We can also use the Corona theorem [5] to show that $T(M) = 0$, and, by Proposition 3, we obtain that $M$ is a projective $H_\infty(\mathbb{C}_+)$-module.

*Definition 6.* A ring $A$ is *regular* if any finitely generated ideal has a finite projective dimension.

*Corollary 5.* $H_\infty(D)$ is a regular ring.

*Corollary 6.* Any f.p. projective $H_\infty(D)$-module is free, i.e. $H_\infty(D)$ is a Hermite ring.

*Proof.* Let $M$ be a projective finitely presented $H_\infty(D)$-module $F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$, and $N = T(M)$. Then, $N$ is a coherent $A$-module and, by Theorem 3, $N$ has a free resolution of length 2:

$$0 \longleftarrow N \longleftarrow F_1^\star \xleftarrow{d_1^\star} F_0^\star \xleftarrow{d_2^\star} F_{-1}^\star \longleftarrow 0.$$

$M$ is a projective $H_\infty(D)$-module, then by Theorem 1, we have

$$\mathrm{ext}^1_{H_\infty(D)}(N, H_\infty(D)) = \mathrm{ext}^2_{H_\infty(D)}(N, H_\infty(D)) = 0,$$

which means that we have the exact sequence $F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_2} F_{-1} \longrightarrow 0$, i.e. $M \cong F_{-1}$ is free.

Corollary 6 was only proved in [12] for full row rank $q \times p$ matrices with entries in $H_\infty(\mathbb{D})$.

*Example 4.* The $H_\infty(\mathbb{C}_+)$-module $M$ defined in Example 3 is projective, and thus, free by Corollary 6. $M$ is defined by the equation

$$\left(\frac{s-1}{s+1}\right) y - \left(\frac{e^{-s}}{s+1}\right) u = 0,$$

where $y$ (resp. $u$) is the class of $e_1 = (1 \quad 0)$ (resp. $e_2 = (0 \quad 1)$) in $M$. We can check that $z = 2\, e\, y + \left(1 + 2\, \left(\frac{1-e^{-(s-1)}}{s-1}\right)\right) u$ is a basis of $M$:

$$y = \left(\frac{e^{-s}}{s+1}\right) z, \quad u = \left(\frac{s-1}{s+1}\right) z.$$

Let us call $H_\infty(D)$-*system*, any system defined by a matrix $R$ with entries in $H_\infty(D)$, i.e. a system of the form $R\,z = 0$, where $z$ is a set of variables.

*Corollary 7.* $H_\infty(D)$-systems and coherent $H_\infty(D)$-modules are in a one-to-one correspondence.

## 4. COHERENT RINGS OF WEAK GLOBAL DIMENSION 2

*Theorem 4.* If $A$ is a Hermite coherent integral domain with $\text{w.gl.dim}\, A \leq 2$, then, for any $A$-module $M$ defined by $F_1 \xrightarrow{d_1} F_0 \longrightarrow M \longrightarrow 0$, there exist a free $A$-module $F_1''$ and two homomorphisms $d_1'' : F_1'' \to F_0$ and $d_1' : F_1 \to F_1''$ such that $d_1 = d_1'' \circ d_1'$ and we have the exact sequences:

$$0 \longrightarrow F_1'' \xrightarrow{d_1''} F_0 \longrightarrow M/t(M) \longrightarrow 0, \quad (11)$$

$$0 \longrightarrow \ker d_1 \longrightarrow F_1 \xrightarrow{d_1'} F_1'' \longrightarrow t(M) \longrightarrow 0. \quad (12)$$

*Proof.* We have the commutative exact diagram

$$\begin{array}{ccccccc}
 & 0 & & & & 0 & \\
 & \downarrow & & & & \downarrow & \\
 & \ker d_1' & & 0 & & t(M) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\pi} & M & \longrightarrow 0 \\
 & \downarrow d_1' & & \| & & \downarrow p & \\
0 \longrightarrow & \ker \phi & \longrightarrow & F_0 & \xrightarrow{\phi} & M/t(M) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \operatorname{coker} d_1' & & 0 & & 0 & \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}$$

where $\phi = p \circ \pi$ and $d_1' : F_1 \to \ker \phi$ is induced by the identity homorphism from $F_0$ to $F_0$. An easy chase in the diagram shows that $\ker d_1' \cong \ker d_1$ and $\operatorname{coker} d_1' \cong t(M)$. Now, let us prove that $\ker \phi$ is a free $A$-module. $M/t(M)$ is a coherent $A$-module over a coherent ring, thus, it has a finite free presentation $P_1 \longrightarrow F_0 \xrightarrow{\phi} M/t(M) \longrightarrow 0$. Let us take a finite free resolution of $T(M/t(M))$:

$$0 \longleftarrow T(M/t(M)) \longleftarrow P_1^\star \longleftarrow F_0^\star \longleftarrow F_{-1}^\star.$$

$M/t(M)$ is a torsion-free $A$-module, and thus, $\text{ext}_A^1(T(M/t(M)), A) = 0$, i.e. we have the exact sequence:

$$0 \longrightarrow M/t(M) \longrightarrow F_{-1} \longrightarrow F_{-1}/(M/t(M)) \longrightarrow 0.$$

Hence, we have the following exact sequence:

$$0 \longrightarrow \ker \phi \longrightarrow F_0 \xrightarrow{\phi} F_{-1} \longrightarrow F_1/(M/t(M)) \longrightarrow 0.$$

An argument of homological algebra shows that

$$\text{pd}_A(F_1/(M/t(M))) \leq 2 \Rightarrow \text{pd}_A(\ker \phi) = 0,$$

i.e. $\ker \phi$ is a finitely generated projective (free) $A$-module ($A$ is a Hermite ring). Thus, $\ker \phi \cong F_1''$ which gives (11) and (12).

*Corollary 8.* If $A$ is a Hermite coherent domain with weak global dimension $\text{w.gl.dim}(A) \leq 2$ and $K = Q(A)$, then every transfer matrix $T \in K^{q \times (p-q)}$ can be written as $T = D^{-1}\,N$ where the matrix $(D \quad -N) \in A^{q \times p}$ is weakly left-prime, i.e. $K^q\,(D \quad -N) \cap A^p = A^p\,(D \quad -N)$.

*Proof.* By Theorem 4, for any full rank $q \times p$-matrix $R$ $(0 < q \leq p)$, there exist a full rank $q \times q$-matrix $R'$ and a full rank $q \times p$-matrix $R''$ such that $R = R'\,R''$ and $M/t(M) = A^p/A^q\,R''$. $M/t(M)$ is a torsion-free $A$-module, and thus, $R''$ is weakly left-prime [7]. If $T$ is a matrix with entries in $K = Q(A)$, then we can use the previous result with $R = (dI_q \quad -H)$, where $d$ is the product of all the denominators of the entries of $T$. Hence, we have $(dI_q \quad -H) = R'\,(D \quad -N)$, where $R'' = (D \quad -N)$ is left weak-prime and $\det(R') \neq 0$. Therefore:

$$\begin{cases} dI_q = R'\,D \quad (\Rightarrow \det(D) \neq 0), \\ H = R'\,N, \end{cases}$$

$$\Rightarrow T = (dI_q)^{-1}\,H = (R'\,D)^{-1}(R'\,N) = D^{-1}\,N.$$

*Theorem 5.* Let $A$ be a Hermite coherent integral domain with $\text{w.gl.dim}\, A \leq 2$, $R$ a full rank $q \times p$-matrix $(0 < q \leq p)$ with entries in $A$ and $M = A^p/A^q\,R$. Then, there exist $\pi \in A$, $R_{-1} \in A^{p \times (p-q)}$, $S \in A^{p \times q}$ and $S_{-1} \in A^{(p-q) \times q}$ such that we have the *extended Bézout identities*:

$$(1)\ \ (\,S \ \ R_{-1}\,) \begin{pmatrix} R \\ S_{-1} \end{pmatrix} = \pi\, I_p,$$

$$(2)\ \ \begin{pmatrix} R \\ S_{-1} \end{pmatrix} (\,S \ \ R_{-1}\,) = \pi \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}.$$

*Proof.* The matrix $R$ is full rank, and thus, $M$ is defined by the following finite free presentation:

$$0 \longrightarrow A^q \xrightarrow{.R} A^p \longrightarrow M \longrightarrow 0.$$

The $A$-module $N = T(M)$ is defined by the following exact sequence:

$$0 \longleftarrow N \longleftarrow A^q \xleftarrow{R.} A^p \longleftarrow \ker R. \longleftarrow 0.$$

$A$ is a coherent ring with $\text{w.gl.dim}\, A \leq 2$, and thus, any finitely presented $A$-module has a projective dimension less or equal to 2. In particular, $\text{pd}_A(N) \leq 2 \Rightarrow \text{pd}_A(\ker R.) = 0$, i.e. $\ker R.$ is a projective $A$-module. But, $A$ is a Hermite ring, and thus, $\ker R.$ is a free $A$-module. Moreover, $\text{rank}(\ker R.) = p - q$, because $N$ is a torsion $A$-module. Therefore, $\ker R. \cong A^{p-q}$. Hence, we have the following exact sequence:

$$0 \longleftarrow N \longleftarrow A^q \xleftarrow{R.} A^p \xleftarrow{R_{-1}.} A^{p-q} \longleftarrow 0. \quad (13)$$

Dualizing (13), we obtain the following complex:

$$0 \longrightarrow A^q \xrightarrow{.R} A^p \xrightarrow{.R_{-1}} A^{p-q} \longrightarrow 0. \quad (14)$$

The two possible defects of exactness of (14) are:

$$\begin{cases} \text{ext}^1_A(N, A) = \ker .R_{-1}/A^q\, R, \\ \text{ext}^2_A(N, A) = A^{p-q}/A^p\, R_{-1}. \end{cases}$$

We have shown in Corollary 2 that $\text{ext}^i_A(N, A)$ is a torsion coherent $A$-module for $i \geq 1$. Therefore, using (f) of Proposition 2, $\text{ann}(\text{ext}^i_A(N, A))$ is a coherent ideal of $A$ for $i \geq 1$. In particular, $\text{ann}(\text{ext}^i_A(N, A))$ is a finitely generated ideal. Let $\pi^i \in \text{ann}(\text{ext}^i_A(N, A))$ for $i = 1, 2$ and:

$$\pi = \prod_{\{i=1,2 \,|\, \pi^i \neq 0\}} \pi^i.$$

Then, we have $\pi\, \text{ext}^i_A(N, A) = 0$ for $i = 1, 2$. Let us denote by $S$ the multiplicative set formed with $\pi$, i.e. $S = \{1, \pi, \pi^2, \ldots, \pi^k, \ldots\}$ and:

$$S^{-1}\, A = \left\{ \frac{a}{s} \;\mid\; a \in A,\; s \in S \right\}.$$

We denote $S^{-1}\, A$ by $A_\pi$. $\text{ext}^i_A(N, A)$ is a coherent $A$-module over a coherent ring $A$, and thus, $\text{ext}^i_A(N, A)$ is a finitely presented $A$-module. $A_\pi$ is a flat $A$-module, and thus, we have [1,9]:

$$\text{ext}^i_{A_\pi}(A_\pi \otimes_A N, A_\pi) \cong A_\pi \otimes_A \text{ext}^i_A(N, A) = 0. \quad (15)$$

Moreover, we can easily prove that:

$$T(A_\pi \otimes_A M) \cong A_\pi \otimes_A T(M) \qquad (16)$$

Finally, using (15) and (16), we see that $A_\pi \otimes_A M$ is a projective $A_\pi$-module. Hence, the tensor product $A_\pi \otimes_A \cdot$ of (14), we obtain the sequence:

$$0 \longrightarrow A^q_\pi \xrightarrow{.R} A^p_\pi \xrightarrow{.R_{-1}} A^{p-q}_\pi \longrightarrow 0, \qquad (17)$$

which is exact because of (15). Therefore, (17) splits [1,9], and, in particular, there exist $S' \in A^{p\times q}_\pi$ and $S'_{-1} \in A^{(p-q)\times q}_\pi$ such that we have:

- $\begin{pmatrix} S' & R_{-1} \end{pmatrix} \begin{pmatrix} R \\ S'_{-1} \end{pmatrix} = I_p,$
- $\begin{pmatrix} R \\ S'_{-1} \end{pmatrix} \begin{pmatrix} S' & R_{-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}.$

Chasing the denominators of $S'$ and $S'_{-1}$, we obtain the extended Bézout identities 1 and 2.

## 5. CONCLUSION

Following ideas of Zames, a class of SISO plants needs to have a structure of an algebra if we want to put two systems in series, in parallel or in feedback. Any mathematical models of plants are only approximations of real systems. This remark shows that the algebra of SISO plants has to be endowed with a norm in order to take into account the errors in the modelization. It seems to be fair to ask this normed algebra to be complete to define a good topology and to deal with a concept of closeness for two systems. Therefore, we require that this algebra is a Banach algebra. But, non-trivial Banach algebras are not noetherian rings, and thus, a main part of classical algebra cannot be used to study the properties of these systems.

We hope to have convinced the readers that the only alternative to study *finitely generated algebraic objects* is to ask the Banach algebra $A$, modelling the class of plants, to be a *coherent domain*. Indeed, the fact that any MIMO plant is defined by a means of a finite number of unknowns and equations, i.e. by means of a matrix with entries in $A$, implies that MIMO plants are in correspondence with coherent $A$-modules. Coherent modules are stable by the main standard algebraic manipulations, and thus, we can characterize the structural properties of MIMO systems by means of modules properties using homological algebra.

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