Linear control theory:
An effective algebraic analysis approach

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Introduction

Unicursal curves

• **Definition**: A curve $f(x,y)=0$ admits a **rational parametrization** if there exist rational functions α and β such that:

$$f(\alpha(t),\beta(t))=0,$$

$$\text{i.e.} \qquad f(x,y)=0 \Leftrightarrow \begin{cases} x=\alpha(t), \\ y=\beta(t). \end{cases}$$

Such a curve $f(x,y)=0$ is called **unicursal**.

• **Example**: The circle $x^2+y^2-1=0$ admits the rational parametrization:

$$x=\frac{1-t^2}{1+t^2}, \quad y=\frac{2\,t}{1+t^2}, \quad t=\tan(\theta/2).$$

• **Example**: The cuspidal cubic $y^2-x^3=0$ admits the rational parametrization:

$$x=t^2, \quad y=t^3.$$

• **Example**: The nodal cubic $y^3+x^3-x\,y=0$ admits the rational parametrization:

$$x=\frac{t}{1+t^3}, \quad y=\frac{t^2}{1+t^3}.$$

• **Example**: The curve $x^n+y^n-1=0$ is **not unicursal** for $n\geq 3$ (otherwise the Fermat-Wiles would be wrong!).

Diophantine equations

• **Definition**: A **diophantine equation** is a polynomial equation with integral coefficients and unknowns.

• **Example**: Find the integral solutions of:

$$x^2 + y^2 = z^2.$$

• If the curve $f(x, y) = 0$ is **unicursal**, we have to find t such that $(x = \alpha(t),\ y = \beta(t)) \in \mathbb{Z}^2$.

• **Example**: Find the integral solutions of

$$x^2+y^2 = z^2 \Leftrightarrow \left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2 = 1,\ x/z,\ y/z \in \mathbb{Q}.$$

$$\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 = 1 \Leftrightarrow \begin{cases} \frac{x}{z} = \frac{1-t^2}{1+t^2}, \\ \frac{y}{z} = \frac{2\,t}{1+t^2}, \end{cases} \forall\, t \in \mathbb{R}.$$

$$((x/z),\ (x/y)) \in \mathbb{Q}^2 \Leftrightarrow t = y/(1+x) \in \mathbb{Q}.$$

Let $t = a/b$ ($a,\ b \in \mathbb{Z}$), then the integral solutions of $x^2 + y^2 = z^2$ are:

$$\begin{cases} x = a^2 - b^2, \\ y = 2\,a\,b, \qquad \forall\, a,\ b \in \mathbb{Z}. \\ z = a^2 + b^2. \end{cases}$$

• **Problem:** Parametrize the solutions of $f(x, y) = 0$ which satisfy some constraints.

Integral computation

• Integration of rational function $g(x, y)$ on a curve $f(x, y) = 0$:

$$\begin{cases} \int g(x,\, y)\, dx, \\ f(x, y) = 0. \end{cases}$$

• **Example**: The integration of $y/(1 + x)$ on the circle $x^2 + y^2 = 1$ is equal to:

$$I = \int_0^1 \frac{y\, dx}{(1 + x)} = \int_0^1 \frac{\sqrt{1 - x^2}\, dx}{(1 + x)}.$$

• If the curve $f(x, y) = 0$ admits a **rational parametrization** $x = \alpha(t)$ and $y = \beta(t)$, then:

$$\begin{cases} \int g(x,\, y)\, dx, \\ f(x, y) = 0, \end{cases} \Leftrightarrow \begin{cases} \int g(\alpha(t), \beta(t))\, \dot{\alpha}(t)\, dt, \\ x = \alpha(t),\ y = \beta(t). \end{cases} \qquad (\star)$$

$\Rightarrow g(\alpha(t), \beta(t))\, \dot{\alpha}(t) \in \mathbb{R}(t) \Rightarrow$ integration of $(\star)$.

• **Example**: Using the parametrization of the circle:

$$\frac{y\, dx}{(1 + x)} = -\frac{4\, t^2\, dt}{(1 + t^2)^2} \Rightarrow I = -1 + \frac{\pi}{2}.$$

• **Example**: The computation of the length of an **elliptic arc** $y^2 = x\,(x - 1)\,(x - \lambda)$

$\Rightarrow$ **elliptic (abelian) integrals**.

First problem

• **An underdetermined linear system** is defined by

$$\begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & \vdots & \vdots \\ a_{q1} & \dots & a_{qp} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = 0,$$

where $a_{ij} \in D$ (integral domain, field), and:

$$\boxed{p > \mathrm{rank}_D(a_{ij}).}$$

• Can we **parametrize all the solutions** of the **linear system of partial differential equations** (PDE):

$$\partial_1^2 \, y_1(x) + \partial_2^2 \, y_2(x) - 2 \, \partial_1 \, \partial_2 \, y_3(x) = 0? \; (\star)$$

$$x = (x_1, \, x_2), \; \partial_i = \frac{\partial}{\partial \, x_i}.$$

Yes, we have:

$$(\star) \Leftrightarrow \begin{cases} y_1(x) = \partial_2 \, \phi(x), \\ y_2(x) = \partial_1 \, \psi(x), \\ y_3(x) = \frac{1}{2} \, (\partial_1 \, \phi(x) + \partial_2 \, \psi(x)), \end{cases} \quad \forall \, \phi, \, \psi \in C^{\infty}(\Omega).$$

Problem I:

1. Recognize if an undertermined linear system of PDE can be parametrized by means of free functions.

2. If yes, compute such parametrizations.

Underdetermined systems

- **Mathematical physics**:

$$\begin{cases} \frac{\partial \vec{B}}{\partial t} + \nabla \wedge \vec{E} = \vec{0}, \\ \nabla \,.\, \vec{B} = 0, \\ \nabla \wedge \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{\jmath}, \\ \nabla \,.\, \vec{D} = \rho, \end{cases}$$

Maxwell equations.

$$\omega^{rs}\,(\partial_{ij}\,\Omega_{rs} + \partial_{rs}\,\Omega_{ij} - \partial_{ri}\,\Omega_{sj} - \partial_{sj}\,\Omega_{ri})$$
$$-\omega_{ij}\,(\omega^{rs}\,\omega^{uv}\,\partial_{rs}\,\Omega_{uv} - \omega^{ru}\,\omega^{sv}\,\partial_{rs}\,\Omega_{uv}) = 0,$$

$\omega = \mathrm{diag}(1,1,1,-1)$ **Linearized Einstein equations**.

$\partial_1^2\,\epsilon_{22} + \partial_2^2\,\epsilon_{11} - 2\,\partial_1\,\partial_2\,\epsilon_{12} = 0,$ **Linear elasticity**.

- **Differential geometry**: “We deal in this book with a class of partial differential equations which arise in differential geometry rather than in physics. Our equations are, for the most part, *undertermined* and their solutions are rather dense in spaces of functions”, Gromov “Partial Differential Relations”.

$$\sum_{l=1}^{m}\left(\frac{\partial z_l}{\partial x_i}\,\frac{\partial y_l}{\partial x_j} + \frac{\partial z_l}{\partial x_j}\,\frac{\partial y_l}{\partial x_i}\right) = g_{ij},\, 1 \leq i < j \leq n,$$
$$m > \frac{n\,(n+1)}{2},$$

Isometric embedding problem.

- **Nonholonomic mechanics**:

$$\begin{cases} \frac{dx}{du} = \sin u\,\frac{dz}{du}, \\ \frac{dy}{du} = \cos u\,\frac{dz}{du}, \end{cases}$$

Integrating wheel.

Examples

• **Example**:

$$\left\{\begin{array}{l} \frac{\partial \vec{B}}{\partial t} + \nabla \wedge \vec{E} = \vec{0}, \\ \nabla \, . \, \vec{B} = 0, \end{array}\right. \quad \Leftrightarrow \left\{\begin{array}{l} \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \nabla \wedge \vec{A}. \end{array}\right.$$

• **Example**:

$$d_1 \, \zeta_1 + d_2 \, \zeta_2 + x_2 \, \zeta_1 = 0$$

$$\Leftrightarrow \left\{\begin{array}{l} \zeta_1 = d_2^2 \, \eta_2 + d_1 \, d_2 \, \eta_1 + x_2 \, d_2 \, \eta_1 + 2 \, \eta_1, \\ \zeta_2 = -d_1 \, d_2 \, \eta_2 - d_1^2 \, \eta_1 - 2 \, x_2 \, d_1 \, \eta_1 - x_2 \, d_2 \eta_2 - x_2^2 \, \eta_1 + \eta_2, \end{array}\right.$$

$$\left\{\begin{array}{l} d_2^2 \, \eta_2 + d_1 \, d_2 \, \eta_1 + x_2 \, d_2 \, \eta_1 + 2 \, \eta_1 = 0, \\ -d_1 \, d_2 \, \eta_2 - d_1^2 \, \eta_1 - 2 \, x_2 \, d_1 \, \eta_1 - x_2 \, d_2 \eta_2 - x_2^2 \, \eta_1 + \eta_2 = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \eta_1 = -d_2 \, \xi, \\ \eta_2 = d_1 \, \xi + x_2 \, \xi. \end{array}\right.$$

• **Example**:

$$\left\{\begin{array}{l} 0 = \eta_1, \\ d_1 \, \xi = \eta_2, \\ \xi = \eta_3, \end{array}\right. \quad \Rightarrow \quad \underbrace{\left\{\begin{array}{l} d_2 \, \eta_1 - d_1 \, \eta_1 + 2 \, \eta_1 + 2 \, \eta_2 - 2 \, d_1 \, \eta_3 = 0, \\ d_2 \, \eta_1 + d_2 \, \eta_2 - d_1 \, d_2 \, \eta_3 = 0. \end{array}\right.}_{R \, \eta = 0}$$

$\left\{\begin{array}{l} \eta_1 = x_2, \\ \eta_2 = -(x_2 + \frac{1}{2}), \\ \eta_3 = 0, \end{array}\right.$ is a **solution of** $R \, \eta = 0$ **but**
is not of the form $(0 : d_1 \, \xi : \xi)^T$ for a certain ξ.

Multidimensional control systems

• **1-D systems**:

⋆ Kalman system:

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t), \\ y(t) = C(t)\,x(t) + D(t)\,u(t). \end{cases}$$

⋆ Polynomial system:

$$P\left(t, \frac{d}{dt}\right)\, y(t) - Q\left(t, \frac{d}{dt}\right)\, u(t) = 0.$$

• **n-D systems**:

⋆ Time-delay system ($\delta_{t_i} f(t) = f(t - t_i)$):

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n} A_i\, x(t - t_i) + \sum_{i=1}^{n} B_i\, u(t - t_i), \\ y(t) = \sum_{i=1}^{n} C_i\, x(t - t_i) + \sum_{i=1}^{n} D_i\, u(t - t_i). \end{cases}$$

$$P\left(\delta_{t_1}, \ldots, \delta_{t_n}, \frac{d}{dt}\right)\, y(t) - Q\left(\delta_{t_1}, \ldots, \delta_{t_n}, \frac{d}{dt}\right)\, u(t) = 0.$$

⋆ n-D filters ($z_1\, u(k_1, \ldots, k_n) = u(k_1 + 1, \ldots, k_n)$):

$$P\,(z_1, \ldots, z_n)\, y_{(k_1, \ldots, k_n)} - Q\,(z_1, \ldots, z_n)\, u_{(k_1, \ldots, k_n)} = 0.$$

Examples

• **Example**: Let us consider the system

$$\ddot{y}(t) + \alpha(t)\,\dot{y}(t) + y(t) = \dot{u}(t) + \alpha(t)\,u(t),$$

where α is a **function of time**.

A **parametrization** of this system is defined by:

$$\begin{cases} y(t) = \dot{\xi}(t) + \alpha(t)\,\xi(t), \\ u(t) = \ddot{\xi}(t) + \alpha(t)\,\dot{\xi}(t) + (1 + \dot{\alpha}(t))\,\xi(t). \end{cases}$$

The **parameter** ξ is an element of the system:

$$\xi(t) = -\dot{y}(t) + u(t).$$

• **Example**: We have the following **parametrization**:

$$\begin{cases} \dot{x}_1(t) - x_1(t-1) + 2\,x_1(t) + 2\,x_2(t) - 2\,u(t-1) = 0, \\ \dot{x}_1(t) + \dot{x}_2(t) - u(t-1) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1(t) = 2\,\dot{\xi}(t-1) - 2\,\xi(t-1), \\ x_2(t) = -\xi(t-2) - \dot{\xi}(t-1) + 2\,\xi(t-1), \\ u(t) = \ddot{\xi}(t) - \dot{\xi}(t-1). \end{cases}$$

The **parameter** ξ satisfies the equation:

$$\delta\,(1-\delta)\,\xi(t) = \frac{1}{2}\,x_1(t) + x_2(t), \quad \delta\,\xi(t) = \xi(t-1).$$

Differential time-delay systems

- **Example**: Study of a **wave equation**:

$$\begin{cases} \frac{\partial^2 z}{\partial t^2}(t,x) - \frac{\partial^2 z}{\partial x^2}(t,x) = 0, \quad (\star) \\ \frac{\partial z}{\partial x}(t,0) = 0, \\ \frac{\partial z}{\partial x}(t,1) = u(t), \\ y(t) = \frac{\partial z}{\partial t}(1,t). \end{cases}$$

- The **solution** of $(\star)$ has the form (D'Alembert):

$$z(t,x) = \phi(t+x) + \psi(t-x), \quad \forall\, \phi,\, \psi \in C(\mathbb{R}),$$

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial x}(t,0) = \dot{\phi}(t) - \dot{\psi}(t) = 0, \\ \frac{\partial z}{\partial x}(t,1) = \dot{\phi}(t+1) - \dot{\psi}(t-1) = u(t), \\ \frac{\partial z}{\partial t}(t,1) = \dot{\phi}(t+1) + \dot{\psi}(t-1) = y(t). \end{cases}$$

$$\Rightarrow \begin{cases} \dot{\phi}(t+1) - \dot{\phi}(t-1) = u(t), \\ \dot{\phi}(t+1) + \dot{\phi}(t-1) = y(t). \end{cases}$$

$$\Rightarrow y(t) - y(t-2) = u(t) + u(t-2). \; (\star\star)$$

- The **parametrization** of the system $(\star\star)$ is given by:

$$\begin{cases} y(t) = \xi(t) + \xi(t-2), \\ u(t) = \xi(t) - \xi(t-2). \end{cases}$$

Differential time-delay systems

- Study of **a flexible rod** (H. Mounier):

$$(1)\quad \begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases}$$

$$\Leftarrow \begin{cases} y_1(t) = \xi(t) + \xi(t-2), \\ y_2(t) = 2\,\xi(t-1), \\ u(t) = \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases} \qquad (2)$$

(1) **is not parametrizable** because the solution

$$y_1(t) = -c/2,\ y_2(t) = -c,\ u(t) = 0,\ 0 \neq c \in \mathbb{R},$$

is not parametrized by (2) because:

$$\xi(t-1) = y_2(t)/2 = -c/2 \Rightarrow y_1(t) = -c \neq -c/2.$$

- In fact, we must **integrate the element**

$$\begin{cases} \theta(t) = 2\,y_1(t-1) - y_2(t) - y_2(t-2), \\ \dot{\theta}(t) = 0, \end{cases}$$

which shows that (1) is **not controllable**.

- We have the following **parametrization** of (1):

$$\begin{cases} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} y_1(t) = \xi(t) + \xi(t-2) - \frac{c}{2}, \\ y_2(t) = 2\,\xi(t-1) - c, \\ u(t) = \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases}$$

Flat linear systems

- **Definition**: (Fliess-Mounier 94) A linear system

$$R\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) \begin{pmatrix} x(t) \\ y(t) \\ u(t) \end{pmatrix} = 0 \ \ (\star)$$

is **flat** ($\frac{d}{dt}y(t) = \dot{y}(t)$, $\delta_h y(t) = y(t-h)$) if:

1. The system $(\star)$ is **parametrizable**:

$$(\star) \Leftrightarrow \begin{pmatrix} x(t) \\ y(t) \\ u(t) \end{pmatrix} = P\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) \xi(t).$$

2. The **parameter** ξ is an element of the system i.e.

$$\xi(t) = S\left(\frac{d}{dt}, \delta_{h_1} \ldots, \delta_{h_n}\right) \begin{pmatrix} x(t) \\ y(t) \\ u(t) \end{pmatrix},$$

$\Leftrightarrow$ the matrix P admits a **left inverse**:

$$S\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) P\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) = I.$$

- **Example**: The following system is **flat**

$$\dot{y}(t-h)+y(t) = u(t-h) \Leftrightarrow \begin{cases} y(t) = \xi(t-h), \\ u(t) = \dot{\xi}(t-h) + \xi(t), \end{cases}$$

because $\xi(t) = -\dot{y}(t) + u(t)$ (called **flat output**).

π-free linear systems

• **<u>Definition</u>**: (Fliess-Mounier 94) The system

$$R\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) \begin{pmatrix} x(t) \\ y(t) \\ u(t) \end{pmatrix} = 0 \;\; (\star)$$

is $\pi(\delta_{h_1} \ldots, \delta_{h_n})$-**free** if:

1. The system $(\star)$ is **parametrizable**:

$$(\star) \Leftrightarrow \begin{pmatrix} x(t) \\ y(t) \\ u(t) \end{pmatrix} = P\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) \xi(t).$$

2. The **parameter** ξ satisfies an equation of the form

$$\pi(\delta_{h_1} \ldots, \delta_{h_n})\xi(t) = S\left(\frac{d}{dt}, \delta_{h_1} \ldots, \delta_{h_n}\right) \begin{pmatrix} x(t) \\ y(t) \\ u(t) \end{pmatrix},$$

$\Leftrightarrow$ there exists a matrix P such that:

$$S\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) P\left(\frac{d}{dt}, \delta_{h_1}, \ldots, \delta_{h_n}\right) = \pi\, I.$$

• **<u>Example</u>**: The following system is δ-**free**

$$\dot{y}(t) = u(t-h) \Leftrightarrow \begin{cases} y(t) = \xi(t-h), \\ u(t) = \dot{\xi}(t), \end{cases}$$

because we have $\delta_h\, \xi(t) = y(t)$.

Motion planning problem

• Let us consider the following neutral system

$$\ddot{y}(t) + \ddot{y}(t-2) + \dot{y}(t) - \dot{y}(t-2) = v(t-1)$$

which corresponds to a **flexible rod**.

• The system $(\star)$ is **parametrizable**:

$$\begin{cases} y(t) = \xi(t-1), \\ v(t) = \ddot{\xi}(t) + \ddot{\xi}(t-2) + \dot{\xi}(t) - \dot{\xi}(t-2). \end{cases}$$

• The system **is not flat** but δ**-free** because:

$$\delta\, \xi(t) = y(t).$$

• If y_r is a **desired trajectory**

$$\Rightarrow \delta\, \xi_r(t) = y_r(t) \Rightarrow \xi_r(t) = y_r(t+1),$$

thus, we obtain the following **open-loop control law**:

$$\begin{aligned} v_r(t) &= \ddot{\xi}_r(t) + \ddot{\xi}_r(t-2) + \dot{\xi}_r(t) - \dot{\xi}_r(t-2) \\ &= \ddot{y}_r(t+1) + \ddot{y}_r(t-1) + \dot{y}_r(t+1) \\ &\quad -\dot{y}_r(t-1). \end{aligned}$$

• We need to **stabilize the system around the desired trajectory**:

$\Rightarrow$ **closed-loop control law** (difficult problem).

Autonomous elements

• **Integration:** Let us consider $y_1(x),\ y_2(x),\ y_3(x)$, functions of $x = (x_1,\ x_2,\ x_3)$ satisfying:

$$\left\{ \begin{array}{l} \partial_3\, y_2(x) - \partial_2\, y_3(x) = 0, \\ \partial_3\, y_1(x) - \partial_1\, y_3(x) = 0. \end{array} \right.$$

The **scalar element** $z(x) = \partial_1\, y_2(x) - \partial_2\, y_1(x)$ satisfies the **scalar equation**:

$$\partial_3\, z(x) = 0 \Rightarrow \partial_1\, y_2(x) - \partial_2\, y_1(x) = \phi(x_1, x_2).$$

• **Controllability:** Let us consider the system:

$$\left\{ \begin{array}{l} \dot{x}_1(t) = x_2(t) - u(t), \quad x_1(0) = x_1^0, \\ \dot{x}_2(t) = x_1(t) + u(t), \quad x_2(0) = x_2^0. \end{array} \right.$$

The **scalar element**

$$z(t) = x_1(t) + x_2(t)$$

satisfies the **scalar equation**:

$$\dot{z}(t) - z(t) = 0.$$

$\Rightarrow$ The system is **not controllable:**

$$\exists\,(y_1,\ y_2) \in \mathbb{R}^2,\ \forall\, T \in \mathbb{R}_+,\ \forall\, u \in C^\infty[0,\ +\infty[:$$
$$(x_1(T),\ x_2(T)) \neq (y_1,\ y_2).$$

For example, take $(y_1,\ y_2) \in \mathbb{R}^2$ such that $\forall\, T \geq 0$:

$$y_1 + y_2 \neq e^T\,(x_1^0 + x_2^0) = x_1(T) + x_2(T).$$

Second problem

• **Decoupling:** Linearized Euler equations for an incompressible fluid:

$$\begin{cases} \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = 0, \\ \partial_t v_1 + \partial_1 p = 0, \\ \partial_t v_2 + \partial_2 p = 0, \\ \partial_t v_3 + \partial_3 p = 0, \end{cases}$$

($\vec{v} = (v_1 : v_2 : v_3)^T$ speed, p pressure)

$$\Rightarrow \begin{cases} \Delta p = (\partial_1^2 + \partial_2^2 + \partial_3^2)\, p = 0, \\ \partial_t \Delta v_1 = 0, \\ \partial_t \Delta v_2 = 0, \\ \partial_t \Delta v_3 = 0. \end{cases}$$

Every dependent variable of a **determined system** (degree of generality $\leq n-1$) satisfies a **scalar equation**.

• **Example:** We **cannot decouple** y and u from the following system $\dot{y}(t) - u(t) = 0$.

Problem II:

1. Recognize if there exist scalar elements of the system (combination of the dependent variables and their derivatives) satisfying scalar equations.

2. If yes, compute a basis of such elements.

Linear systems over Ore algebras

Methodology

1. A **linear system** Σ is defined by a **matrix with entries** R **in a ring** D (i.e. $\Sigma: \; R\,y = 0$).

1. Using the matrix R, **we define a** D**-module** M.

2. We develop **a dictionary between the properties of the system** Σ **and the module** M.

3. We use **module theory** in order to classify the properties of the module M.

4. We use **homological algebra** in order to check the properties of the module M.

5. Using effective algebra, we develop some **effective algorithms** which check the properties of the module M, and thus, of the system Σ.

6. **Implementation** in Maple, Cocoa, Singular...

An Introduction to Differential Modules (D-modules)

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Rings of differential operators

- **Definition**: A **differential field** K is a field with n derivations $\partial_1, \ldots, \partial_n$ which satisfies ($\mathbb{Q} \subseteq K$):

  – $\partial_i \, \partial_j = \partial_j \, \partial_i$,

  – $\partial_i(a + b) = \partial_i \, a + \partial_i \, b$,

  – $\partial_i(a \, b) = (\partial_i \, a) \, b + a \, \partial_i \, b$,

  – $\partial_i(a/b) = ((\partial_i a) \, b - a \, (\partial_i \, b))/b^2$.

- **Example**: We have the following examples:

$$(\mathbb{R}(t), d/dt), \quad (\mathbb{R}(x_1, \ldots, x_n), \{\partial_1, \ldots, \partial_n\}).$$

- Let $D = K[d_1, \ldots, d_n]$ be the **ring of differential operators** with coefficients in K:

$$P = \sum_{0 \leq |\mu| < \infty} a_\mu \, d_\mu \in D,$$

$$\text{with} \quad \begin{cases} a_\mu \in K, \quad \mu = (\mu_1 : \ldots : \mu_n) \in \mathbb{Z}_+^n, \\ d_\mu = d_1^{\mu_1} \ldots d_n^{\mu_n}, \\ d_i(a \, d_j) = a \, d_i \, d_j + (\partial_i a) \, d_j, \; a \in K. \end{cases}$$

- **Remark**: We can also use a **differential ring** A instead of a differential field K

(e.g. $K = k[x_1, \ldots, x_n] \Rightarrow D = A_n$ Weyl algebra).

Ore algebras

• **Definition**: The non-commutative polynomial ring $D = A[\partial; \sigma, \delta]$ in ∂ is called **skew** if

$$\partial\, a = \sigma(a)\, \partial + \delta(a), \quad a \in A,$$

where $\sigma : A \to A$ satisfies $\forall\, a, b \in A$:

$$\left\{ \begin{array}{l} \sigma(1) = 1, \\ \sigma(a + b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b), \end{array} \right.$$

and $\delta : A \to A$ is such that $\forall\, a, b \in A$:

$$\left\{ \begin{array}{l} \delta(a + b) = \delta(a) + \delta(b), \\ \delta(a\, b) = \sigma(a)\, \delta(b) + \delta(a)\, b. \end{array} \right.$$

• **Definition**: (Chyzak-Salvy): The skew ring

$$D = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$$

is called an **Ore algebra** if :

$$\left\{ \begin{array}{l} \sigma_i\, \delta_j = \delta_j\, \sigma_i, \quad 1 \leq i, j \leq m, \\ \sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0, \quad j < i. \end{array} \right.$$

• **Theorem**: (Kredel): Let D be an Ore algebra s.t.

$$\sigma(x_j) = a_{ij}\, x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij},$$

$$0 \neq a_{ij},\, b_{ij} \in k,\ c_{ij} \in k[x_1, \ldots, x_n],\ \deg(c_{ij}) \leq 1,$$

then, for every term order on $x_1, \ldots, x_n, \partial_1, \ldots, \partial_m$, there exist some **non-commutative Gröbner bases**.

Examples of Ore algebras

- **Ordinary differential operators**:

$$D = A[\tfrac{d}{dt}; 1, \tfrac{d}{dt}], \quad A = k[t], k(t),$$

$$P = \sum_{i=0}^{m} a_i(t) \frac{d^i}{dt^i} \in D, \quad \frac{d}{dt} a(t) = \dot{a}(t).$$

- **Time-delay (time-advance) operators**:

$$D = A[\delta_h; \sigma_h, 0], \quad A = k[t], k(t),$$

$$P = \sum_{i=0}^{m} a_i(t) \delta_h^i \in D, \quad \sigma_h a(t) = a(t-h).$$

- **Shift operators**:

$$D = A[\delta_1; \sigma, 0], \quad A = k[n], k(n),$$

$$P = \sum_{i=0}^{m} a_i(n) \delta_1^i \in D, \quad \delta_1 a(n) = a(n+1).$$

- **differential time-delay operators**:

$$D = A[\tfrac{d}{dt}; 1, \tfrac{d}{dt}][\delta_h; \sigma_h, 0], \quad A = k[t], k(t),$$

$$P = \sum_{0 \leq i+j \leq m}^{n} a_{ij}(t) \frac{d^i}{dt^i} \delta_h^j \in D.$$

- **Partial differential operators**:

$$D = A[d_1; 1, \partial_1] \ldots [d_n; 1, \partial_n], \quad A = k[x_1, \ldots, x_n],$$

$$P = \sum_{0 \leq |\mu| \leq m} a_\mu(x)\, d^\mu, \; d^\mu = d_1^{\mu_1} \ldots d_n^{\mu_n}, \; \partial_i = \frac{\partial}{\partial x_i}.$$

Properties

- **Proposition**: If A has the **left Ore property**, namely $\forall\ (a_1,\ a_2) \in A^2$, $\exists\ (0,\ 0) \neq (b_1,\ b_2) \in A^2$ s.t.

$$b_1\, a_1 = b_2\, a_2,$$

then $A[\partial; \sigma, \delta]$ has the **left Ore property**.

- **Proposition**: If A is an **integral domain** ($a\, b = 0$, $a \neq 0 \Rightarrow b = 0$), the skew polynomial ring $A[\partial; \sigma, \delta]$ is an **integral domain**.

- **Proposition**: If A is a **left noetherian ring** and σ is an automorphism, then the skew polynomial ring $A[\partial; \sigma, \delta]$ is a **left noetherian ring**.

Systems and modules

- Let $D = K[d_1, \ldots, d_n]$ and $R \in D^{q \times p}$.

The vectors of D^p and D^q are row vectors.

Let $.R$ be the D-morphism defined by:

$$\begin{array}{ccl} D^q & \xrightarrow{.R} & D^p \\ P & \longrightarrow & P\,R = (P_1 : P_2 : \ldots : P_q) \begin{pmatrix} R_{11} & \ldots & R_{1p} \\ \ldots & \ldots & \ldots \\ R_{q1} & \ldots & R_{qp} \end{pmatrix} \end{array}$$

- **In algebraic analysis, we use the left D-module:**

$$M = \text{coker}.R = D^p/\text{im}.R = D^p/D^q\,R.$$

- Let $\pi : D^p \rightarrow M = D^p/D^q\,R$ be the D-morphism:

$$\pi(P) = \pi(Q) \Leftrightarrow \exists\,\lambda \in D^q : P - Q = \lambda\,R \in D^p.$$

- Let $\{e_1, \ldots, e_p\}$ and $\{f_1, \ldots, f_q\}$ be the canonical bases of D^p and D^q, $\quad y_j = \pi(e_j), j = 1, \ldots, p$.

$$f_i\,R = (R_{i1}\ \ldots\ R_{ip}) = \sum_{j=1}^{p} R_{ij}\,e_j \in D^q R$$

$$\Rightarrow \pi(f_i\,R) = \sum_{j=1}^{p} R_{ij}\,\pi(e_j) = \sum_{j=1}^{p} R_{ij}\,y_j = 0.$$

M is defined by the D-linear combinations of the equations $R\,y = 0$

Example

- Let $D = k[d_1, d_2, d_3]$ and R be the **curl matrix**:

$$R = \left(\begin{array}{ccc} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{array} \right).$$

- Let us consider the D-morphism $.R$:

$$\begin{array}{ccc} D^3 & \xrightarrow{.R} & D^3 \\ (P_1 : \; P_2 : \; P_3) & \longrightarrow & (P_2\, d_3 - P_3\, d_2 : \\ & & P_1\, d_1 + P_3\, d_1 : \; P_1\, d_2 - P_2\, d_1). \end{array}$$

- Let us define $\;y_i = \pi(e_i), \; i = 1, 2, 3.$

Then, $M = D^3/D^3\, R$ **is defined by the equations**

$$\left\{ \begin{array}{l} \pi(f_1\, R) = \pi((0 : \; -d_3 : \; d_2)) = -d_3\, y_2 + d_2\, y_3 = 0, \\ \pi(f_2\, R) = \pi((d_3 : \; 0 : \; -d_1)) = d_3\, y_1 - d_1\, y_3 = 0, \\ \pi(f_3\, R) = \pi((-d_2 : \; d_1 : \; 0)) = -d_2\, y_1 + d_1\, y_2 = 0, \end{array} \right.$$

and their D-linear combinations.

Example

- Let us consider the wind tunnel model (Manitius 84):

$$\left\{\begin{array}{l} \dot{x}_1(t) = -a\, x_1(t) + k\, a\, x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\, x_2(t) - 2\,\zeta\,\omega\, x_3(t) + \omega^2\, u(t). \end{array}\right. \qquad (\star)$$

- Let us consider the following **Ore algebra**:

$$D = \mathbb{R}\left[\frac{d}{dt}; 1, \frac{d}{dt}\right][\delta_h; \sigma_h, 0] \cong \mathbb{R}\left[\frac{d}{dt}, \delta_h\right].$$

- The system $(\star)$ is equivalent to:

$$\underbrace{\left(\begin{array}{cccc} \frac{d}{dt}+a & -k\,a\,\delta_h & 0 & 0 \\ 0 & \frac{d}{dt} & -1 & 0 \\ 0 & \omega^2 & \frac{d}{dt}+2\,\zeta\,\omega & -\omega^2 \end{array}\right)}_{R \in D^{3\times 4}} \left(\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \\ u(t) \end{array}\right) = 0.$$

- The D-module $M = D^4/D^3\,R$ is **defined by the D-linear combinations of the equations** of $(\star)$.

Classification of modules

• Definition: Let M be a **finitely generated** D-module.

$a)$ M is **free** if $\exists\, r \in \mathbb{Z}_+ :\ M \cong D^r$.

$b)$ M is **projective** if $\exists\, r \in Z_+$ and a D-module P:

$$M \oplus P \cong D^r.$$

$c)$ M is **reflexive** if ϵ is an isomorphism:

$$\begin{array}{llll} \epsilon : & M & \longrightarrow & M^{\star\star} = \hom_D(\hom_D(M, D), D), \\ & m & \longrightarrow & \epsilon(m), \qquad \epsilon(m)(f) = f(m). \end{array}$$

$d)$ M is **torsion-free** if:

$$t(M) = \{m \in M \,|\, \exists\, 0 \neq P \in D : P\, m = 0\} = 0.$$

$m \in t(M)$ is called a **torsion element** of M.

$e)$ M is **torsion** if $M = t(M)$.

• Theorem:

1. **free** $\Rightarrow$ **projective** $\Rightarrow .. \Rightarrow$ **reflexive** $\Rightarrow$ **torsion-free**.

2. If D **is a principal domain** (e.g. $K[\frac{d}{dt}]$), then:

torsion-free = free.

3. If $D = k[x_1, \ldots, x_n]$, where k is a field:

projective = free (Th. Quillen-Suslin).

Free resolution of a D-module

• **Definition**: A sequence of D-morphisms

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is **a complex** at M if:

$$\mathrm{im}\, f \subseteq \ker g.$$

The **defect of exactness at** M is defined by:

$$H(M) = \ker g / \mathrm{im}\, f.$$

A complex is **exact at** M if:

$$H(M) = 0 \Leftrightarrow \mathrm{im}\, f = \ker g.$$

• **Example**: The exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M$$

means that f **is injective** and the exact sequence

$$M \xrightarrow{g} M'' \longrightarrow 0$$

means that g **is surjective**.

• **Definition**: A **free resolution** of a left D-module M **is an exact sequence of the form**:

$$\ldots \xrightarrow{.R_3} D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0,$$

where $R_i \in D^{l_i \times l_{i-1}}$ and:

$$\begin{array}{rcl} .R_i : D^{l_i} & \longrightarrow & D^{l_{i-1}} \\ (P_1 : \ldots : P_{l_i}) & \longrightarrow & (P_1 : \ldots : P_{l_i})\, R_i. \end{array}$$

Example

• Let $R_1 = (d_1\, d_2 :\ d_1^2)^T$ and the $D = k[d_1, d_2]$-module $M = D/D^2\, R_1$ defined by the equations:

$$\begin{cases} d_1\, d_2\, y = 0, \\ d_1^2\, y = 0. \end{cases}$$

• We have the following **exact sequence**:

$$0 \longrightarrow \ker .R_1 \longrightarrow D^2 \xrightarrow{.R_1} D \longrightarrow M \longrightarrow 0.$$

• We have the following **equality** (D is a **GCDD**):

$$\begin{aligned} \ker .R_1 &= \{(P_1 :\ P_2) \in D^2 \mid P_1\, d_1\, d_2 = -P_2\, d_1^2\} \\ &= \{(P_1 :\ P_2) \in D^2 \mid P_1\, d_2 = -P_2\, d_1\} \\ &= \{(P\, d_1 :\ -P\, d_2) \mid P \in D\} \\ &= D\, R_2, \quad \text{where } R_2 = (d_1 :\ -d_2), \end{aligned}$$

$\Rightarrow$ **we have the following free resolution of** M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^2 \xrightarrow{.R_1} D \longrightarrow M \longrightarrow 0.$$

Syzygy D-modules

• Let us consider the D-morphism ($R_1 \in D^{l_1 \times l_0}$):

$$0 \longrightarrow \ker .R_1 \longrightarrow D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0.$$

$\ker .R_1 = \{P = (P_1 : \ldots : P_{l_1}) \in D^{l_1} \mid P\, R_1 = 0\}$ is a D**-submodule** of D^{l_1}.

• D is a **left noetherian ring** $\Rightarrow \ker .R_1$ is a **finitely generated** D-module, i.e. $\exists$ **a finite family** of vectors $\{v_1, \ldots, v_{l_2}\}$ of D^{l_1} which **generates** $\ker .R_1$:

$$\Rightarrow \forall\, v \in \ker .R_1,\ \exists\, Q_1, \ldots, Q_{l_2} \in D :$$

$$v = \sum_{i=1}^{l_2} Q_i\, v_i = (Q_1 : \ldots : Q_{l_2})\, (v_1 : \ldots : v_{l_2})^T,$$

$$\Rightarrow \ker .R_1 = \operatorname{im} .R_2,\ R_2 = (v_1 : \ldots : v_{l_2})^T \in D^{l_2 \times l_1},$$

$$\Rightarrow D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0 \ \ \textbf{exact}.$$

• **Proposition**: Any finitely generated D-module M has a **finite free resolution**.

• **Algorithm**: Find **a basis of the compatibility conditions** of $R_i\, y = u$ (**elimination of** y using a Gröbner basis, formal theory of PDE ...):

$$\forall\, P \in \ker .R_i,\ P\,(R_i\, y) = P\, u = 0 \Rightarrow R_{i+1}\, u = 0.$$

Example

- Consider $D = \mathbb{R}[\frac{d}{dt}; 1, \frac{d}{dt}][\delta_h; \sigma_h, 0]$ and:

$$Q = \begin{pmatrix} \frac{d}{dt} + a & 0 & 0 \\ -k\,a\,\delta_h & \frac{d}{dt} & \omega^2 \\ 0 & -1 & \frac{d}{dt} + 2\,\zeta\,\omega \\ 0 & 0 & -\omega^2 \end{pmatrix} \in D^{4\times 3}.$$

- Consider the D-linear map:

$$\begin{array}{ccc} D^4 & \xrightarrow{.R} & D^3 \\ \lambda = (\lambda_1 : \lambda_2 : \lambda 3 : \lambda_4) & \longmapsto & \lambda\, Q. \end{array}$$

- The D-module $\ker .Q = \{P \in D^4 \,|\, P\,Q = 0\}$ is the **1st syzygy module** of $N = D^3/D^4\,Q$.

- Let us define:

$$\Sigma = \left\{ \left(\frac{d}{dt} + a\right)\, \lambda_1 - \mu_1, -k\,a\,\delta_h\,\lambda_1 + \frac{d}{dt}\,\lambda_2 + \omega^2\,\lambda_3 - \mu_2, \right.$$
$$\left. -\lambda_2 + (\frac{d}{dt} + 2\,\zeta\,\omega)\,\lambda_3 - \mu_3, -\omega^2\,\lambda_3 - \mu_4 \right\},$$

- The intersection of a **Gröbner base** for an elimination order of Σ with $\sum_{i=1}^{3} D\,\mu_i$ is:

$$\left\{ \omega^2\,k\,a\,\delta_h\,\mu_1 + (\omega^2\,\frac{d}{dt} - \omega^2\,a)\,\mu_2 + (\omega^2\,\frac{d^2}{dt^2} + \omega^2\,a\,\frac{d}{dt})\,\mu_3 \right.$$
$$\left. +(\frac{d^3}{dt^3} + 2\,\zeta\,\omega\,\frac{d^2}{dt^2} + a\,\frac{d^2}{dt^2} + \omega^2\,\frac{d}{dt} + 2\,a\,\zeta\,\omega\,\frac{d}{dt} + a\,\omega^2)\,\mu_4 \right\}$$

$$\Rightarrow Q_1 = (\ \omega^2\,k\,a\,\delta_h :\ \omega^2\,\frac{d}{dt} + \omega^2\,a :\ \omega^2\,\frac{d^2}{dt^2} + \omega^2\,a\,\frac{d}{dt} :$$
$$\frac{d^3}{dt^3} + 2\,\zeta\,\omega\,\frac{d}{dt}^2 + a\,\frac{d^2}{dt^2} + \omega^2\,\frac{d}{dt} + 2\,a\,\zeta\,\omega\,\frac{d}{dt} + a\,\omega^2),$$

$\Rightarrow 0 \longrightarrow D \xrightarrow{.Q1} D^4 \xrightarrow{.Q} D^3 \xrightarrow{\pi} N \longrightarrow 0$ **exact**.

Extension functor

- **Definition**: Let M be a left D-module and

$$\ldots \xrightarrow{.R_3} D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0 \quad (1)$$

a free resolution of M.

- We call **the truncated complex** associated with (1) the following complex:

$$\ldots \xrightarrow{.R_4} D^{l_3} \xrightarrow{.R_3} D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow 0 \quad (2) \; .$$

- Let S be a left D-module. The **dual sequence** of (2) is the complex defined by

$$\ldots \xleftarrow{R_4.} S^{l_3} \xleftarrow{R_3.} S^{l_2} \xleftarrow{R_2.} S^{l_1} \xleftarrow{R_1.} S^{l_0} \longleftarrow 0 \quad (3)$$

$$\begin{array}{lrcl} \textbf{with}: & R_i. : S^{l_{i-1}} & \longrightarrow & S^{l_i} \\ & (P_1 : \ \ldots : \ P_{l_{i-1}})^T & \longrightarrow & R_i \, (P_1 : \ \ldots : \ P_{l_{i-1}})^T. \end{array}$$

The complex (3) is **generally not exact** at S^{l_i}.

We denote **the defects of exactness** of (3) by:

$$\left\{ \begin{array}{l} \mathrm{ext}^0_D(M,S) = \ker_S R_1. = \hom_D(M,S), \\ \\ \mathrm{ext}^i_D(M,S) = \ker_S R_{i+1}. / \operatorname{im}_S R_i., \;\; i \geq 1. \end{array} \right.$$

- **Theorem**: The abelian group $\mathrm{ext}^i_D(M,S)$ **only depends on** M and S and **not on resolution** (1).

Example

• Let $R_1 = (d_1\, d_2 : \; d_1^2)^T$ and the $D = k[d_1, d_2]$-module $M = D/D^2\, R_1$ defined by the equations:

$$d_1\, d_2\, y = 0, \;\; d_1^2\, y = 0.$$

• We have the following **free resolution** of M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^2 \xrightarrow{.R_1} D \longrightarrow M \longrightarrow 0 \;\; (\star).$$

• Let $S = \mathcal{D}'(\mathbb{R}^2)$ the space of distributions on $\mathbb{R}^2$ ($S = \mathcal{E}(\mathbb{R}^2) \ldots$), then the **dual sequence** of ($\star$) is:

$$0 \longleftarrow S \xleftarrow{R_2.} S^2 \xleftarrow{R_1.} S \longleftarrow 0.$$

$$\Rightarrow \begin{cases} \mathrm{ext}^0_D(M, S) = \ker_S R_1. = \hom_D(M, S), \\ \mathrm{ext}^1_D(M, S) = \ker_S R_2./R_1\, S, \\ \mathrm{ext}^2_D(M, S) = S/R_2\, S^2. \end{cases}$$

• $\hom_D(M, S)$ is the **solution** in S of $R_1\, y = 0$.

• A **necessary condition** so that there exists a solution in S to the **inhomogeneous system**

$$R_1\, y = u,$$

with $u \in S^2$ **fixed**, is $u \in R_1\, S \Leftrightarrow R_2\, u = 0$.

• $\mathrm{ext}^1_D(M, S)$ gives the **functional obstructions**:

$$\boxed{\exists\, y \in S : R_1\, y = u \Leftrightarrow \pi(u) = 0 \text{ in } \mathrm{ext}^1_D(M, S).}$$

Properties of $\mathrm{Ext}^i_D(M,S)$

• **Proposition**: If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an **exact sequence** of **left** D-modules and S is a **left** D-module, then we have the **exact sequence**:

$$\begin{array}{llll} 0 \longrightarrow & \hom_D(M'',S) \longrightarrow & \hom_D(M,S) \longrightarrow & \hom_D(M'',S) \\ \longrightarrow & \mathrm{ext}^1_D(M'',S) \longrightarrow & \mathrm{ext}^1_D(M,S) \longrightarrow & \mathrm{ext}^1_D(M',S) \\ \longrightarrow & \mathrm{ext}^2_D(M'',S) \longrightarrow & \mathrm{ext}^2_D(M,S) \longrightarrow & \ldots \end{array}$$

• **Proposition**: If M is a **projective** D**-module**, then, for every D-module S, we have:

$$\boxed{\mathrm{ext}^i_D(M,S) = 0, \; i \geq 1.}$$

• **Definition**: A D-module S is called **injective** if, for every finitely generated ideal $I = (P_1, \ldots, P_m)$ of D, there exists $y \in S$ which satisfies

$$\left\{ \begin{array}{lll} P_1\, y & = & u_1, \\ \vdots & & \vdots \\ P_m\, y & = & u_m, \end{array} \right.$$

where $u_1, \ldots, u_m \in S$ satisfy the relations of I, i.e.:

$$\sum_{i=1}^{m} Q_i\, P_i = 0 \Rightarrow \sum_{i=1}^{m} Q_i\, u_i = 0.$$

• **Proposition**: If S is an **injective** D**-module**, then, for every D-module M, we have:

$$\boxed{\mathrm{ext}^i_D(M,S) = 0, \; i \geq 1.}$$

Results of B. Malgrange (59-63)

• **Theorem** (Malgrange): If Ω is an open convex subset of $\mathbb{R}^n$, then $\mathcal{D}'(\Omega)$, $\mathcal{E}(\Omega)$ and $\mathcal{S}'(\Omega)$ are three **injective** $D = \mathbb{C}[d_1, \ldots, d_n]$**-modules**.

• If S is an **injective** D**-module**, then the **solvability** in $y \in S^{l_0}$ of an underdetermined system $R\,y = u$, $u \in S^{l_1}$ fixed, is **just an algebraic problem**:

"**Computing the second syzygy** D**-module of** M (i.e. the **compatibility conditions**)":

$$D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0$$

• **Example**: Let us reconsider $R_1 = (d_1\,d_2 :\ d_1^2)^T$ and the $D = \mathbb{C}[d_1, d_2]$-module $M = D/D^2\,R_1$:

$$d_1\,d_2\,y = 0, \quad d_1^2\,y = 0.$$

• We have the free resolution of M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^2 \xrightarrow{.R_1} D \longrightarrow M \longrightarrow 0.$$

• If $S = \mathcal{D}'(\Omega),\ \mathcal{E}(\Omega)$ or $\mathcal{S}'(\Omega)$, then we have the following **exact sequence**:

$$0 \longleftarrow S \xleftarrow{R_2.} S^2 \xleftarrow{R_1.} S \longleftarrow \hom_D(M, S) \longleftarrow 0.$$

$$\Rightarrow \exists\, y \in S : \begin{cases} d_1\,d_2\,y = u_1, \\ d_1^2\,y = u_2, \end{cases} \Leftrightarrow d_1\,u_1 = d_2\,u_2.$$

$$\Rightarrow \forall\, z \in S,\ \exists\, u_1\,u_2 \in S :\ z = d_1\,u_1 - d_2\,u_2.$$

Flat D-modules

• **Proposition**: If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an **exact sequence** of **left** D-modules and S a **right** D-module, then we have the **exact sequence**:

$$\begin{array}{ccccccc} \ldots & \longrightarrow & \text{tor}_2^D(S, M) & \longrightarrow & \text{tor}_2^D(S, M'') & \longrightarrow & \\ \text{tor}_1^D(S, M') & \longrightarrow & \text{tor}_1^D(S, M) & \longrightarrow & \text{tor}_1^D(S, M'') & \longrightarrow & \\ S \otimes_D M' & \longrightarrow & S \otimes_D M & \longrightarrow & S \otimes_D M'' & \longrightarrow & 0. \end{array}$$

• **Definition**: A D-module S is called **flat** if, for **every relation** of the form

$$\sum_{i=1}^{m} P_i \, y_i = 0, \;\; P_i \in D, \;\; y_i \in S,$$

there exist $z_1, \ldots, z_p \in S$ and $(Q_{ij}) \in D^{m \times p}$ s.t.:

1. $y_i = \sum_{j=1}^{p} Q_{ij} \, z_j, \quad 1 \leq i \leq m,$

2. $\sum_{i=1}^{m} P_i \, Q_{ij} = 0, \quad 1 \leq i \leq p.$

• **Proposition**: If S is a **right flat** D**-module**, then, for every **left** D**-module** M, we have:

$$\boxed{\text{tor}_i^D(S, M) = 0, \; i \geq 1.}$$

• **Proposition**: If M is a **left flat** D**-module**, then, for every **right** S**-module** M, we have:

$$\boxed{\text{tor}_i^D(S, M) = 0, \; i \geq 1.}$$

Results of B. Malgrange (59-63)

• **Theorem** (Malgrange): If Ω is an open convex subset of $\mathbb{R}^n$, then $\mathcal{D}(\Omega)$, $\mathcal{E}'(\Omega)$ and $\mathcal{S}(\Omega)$ are three **flat** $D = \mathbb{C}[d_1, \ldots, d_n]$**-modules**.

• If S is a left **flat** D**-module**, then the determination of **a parametrization** of an underdetermined system of $R_1\, u = 0,\ u \in S^{l_0}$, is just an **algebraic problem**:

"**Computing the second syzygy** D**-module of the right** $N = D^{l_1}/R_1\, D^{l_0}$ (D^{l_i} column vectors)":

$$0 \longleftarrow N \longleftarrow D^{l_1} \xleftarrow{R_1 .} D^{l_0} \xleftarrow{R_0 .} D^{l_{-1}}$$

Indeed, the tensor product $\cdot \otimes_D S$ of the free resolution of N gives the **exact sequence**:

$$0 \longleftarrow N \otimes_D S \longleftarrow S^{l_1} \xleftarrow{R_1 .} D^{l_0} \xleftarrow{R_0 .} S^{l_{-1}}$$

$$\Rightarrow R_1\, u = 0 \Leftrightarrow \exists\, y \in S^{l_0} :\ u = R_0\, y.$$

• **Example**: Let us consider $R_1 = (d_1\, d_2 \,:\, d_1^2)^T$, $R_2 = (d_1 \,:\, -d_2)$. We have the **exact sequence**:

$$0 \longleftarrow M \longleftarrow D \xleftarrow{R_1 .^T} D^2 \xleftarrow{R_2 .^T} D \longleftarrow 0.$$

$\Rightarrow$ If $S = \mathcal{D}(\Omega)$, $\mathcal{E}'(\Omega)$ or $\mathcal{S}(\Omega)$, then:

$$d_1\, d_2\, u_1 + d_1^2\, u_2 = 0 \Leftrightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} d_1 \\ -d_2 \end{pmatrix} y,\ y \in S.$$

$\mathbf{Ext}^i_D(M, D)$

Definition: Let M be a left D-module and

$$\ldots \xrightarrow{.R_3} D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow M \longrightarrow 0 \quad (1)$$

a free resolution of M.

We call **the truncated complex** associated with (1) the following complex:

$$\ldots \xrightarrow{.R_4} D^{l_3} \xrightarrow{.R_3} D^{l_2} \xrightarrow{.R_2} D^{l_1} \xrightarrow{.R_1} D^{l_0} \longrightarrow 0 \quad (2)\ .$$

The **transposed sequence** of (2) is the complex defined by

$$\ldots \xleftarrow{R_4.} D^{l_3} \xleftarrow{R_3.} D^{l_2} \xleftarrow{R_2.} D^{l_1} \xleftarrow{R_1.} D^{l_0} \longleftarrow 0 \quad (3)$$

where:

$$\begin{array}{rcl} R_i. : D^{l_{i-1}} & \longrightarrow & D^{l_i} \\ (P_1 : \ \ldots : \ P_{l_{i-1}})^T & \longrightarrow & R_i\,(P_1 : \ \ldots : \ P_{l_{i-1}})^T. \end{array}$$

Complex (3) is generally not exact at D^{l_i}.

We denote **the defects of exactness** of (3) by:

$$\begin{cases} \mathrm{ext}^0_D(M, D) = \ker_D R_1. = \hom_D(M, D), \\ \mathrm{ext}^i_D(M, D) = \ker_D R_{i+1}. / \mathrm{im}_D\, R_i., \quad i \geq 1. \end{cases}$$

• **Theorem**: The **right D-module** $\mathrm{ext}^i_D(M, D)$ **only depends on** M and **not on the free resolution** (1).

Example

- Let $R_1 = (d_1\, d_2 :\ d_1^2)^T$ and the $D = \mathbb{R}[d_1, d_2]$-module $M = D/D^2\, R_1$ defined by the equations:

$$d_1\, d_2\, y = 0,\ \ d_1^2\, y = 0.$$

- We have the following **free resolution** for M:

$$0 \longrightarrow D \xrightarrow{.R_2} D^2 \xrightarrow{.R_1} D \longrightarrow M \longrightarrow 0,\ \ (\star).$$

- The **transposed of the truncated complex** $(\star)$ is:

$$0 \longleftarrow D \xleftarrow{.R_2^T} D^2 \xleftarrow{.R_1^T} D \longleftarrow 0.$$

- We have the following **defects of exactness**:

$$\left\{ \begin{array}{l} \mathrm{ext}_D^1(M, D) = \ker .R_2^T \,/\, D\, R_1^T, \\ \mathrm{ext}_D^2(M, D) = D/D^2\, R_2^T. \end{array} \right.$$

- $$\begin{aligned} \ker .R_2^T &= \{(P_1 :\ P_2) \in D^2 \mid P_1\, d_1 = P_2\, d_2\} \\ &= \{(P\, d_2 :\ P\, d_1) \mid P \in D\} \\ &= D\,(d_2 :\ d_1). \end{aligned}$$

$\Rightarrow \mathrm{ext}_D^1(M, D) = D\,(d_2 : d_1)/D(d_1\, d_2 : d_1^2) \neq 0$:

$$\left\{ \begin{array}{l} z = \pi((d_2 :\ d_1)) = d_2\, z_1 + d_1\, z_2, \\ d_1\, z = 0, \\ d_1\, d_2\, z_1 + d_1^2\, z_2 = 0. \end{array} \right.$$

- $1 \notin I = (d_1, d_2) \Rightarrow \mathrm{ext}_D^2(M, D) = D/I \neq 0.$

Formal adjoint

• **Definition**: The **formal adjoint** of $R \in D^{q \times p}$ is obtained by **integrating by parts**

$$< z,\, R\,y > = < \widetilde{R}\,z,\, y > + d(\cdot),$$

where $d(\cdot)$ corresponds to the **boundary terms**.

$$\begin{array}{ccc} D^q & \xrightarrow{.R} & D^p \\ D^q & \xleftarrow{.\widetilde{R}} & D^p \end{array}$$

• We can also use the **involution** of $D = K[d_1, \ldots, d_n]$ defined by the following **three rules**:

1. If $a \in K$, then $\widetilde{a} = a$,

2. $\widetilde{d_i} = -d_i$,

3. $\widetilde{P \circ Q} = \widetilde{Q} \circ \widetilde{P}$.

• **Example**: The **adjoint** of $R = (x_2\, d_2 \,:\; d_1)$ is:

$\widetilde{R} = (-d_2(x_2) \,:\; -d_1)^T = (-x_2\, d_2 - 1 \,:\; -d_1)^T.$

$$< z,\, R\,y >= z\,(x_2\, d_2\, y_1 + d_1\, y_2) = (x_2\, z)\, d_2\, y_1 + z\, d_1\, y_2$$

$$= -d_2\,(x_2\, z)\, y_1 - (d_1\, z)\, y_2 + \ldots$$

$$= (y_1 \,:\; y_2)\,(-x_2\, d_2 - 1 \,:\; -d_1)^T\, z + \ldots =< y,\, \widetilde{R}\, z > + d(\cdot).$$

• $\forall\, R \in D^{q \times p}$, **we have:** $\quad \widetilde{\widetilde{R}} = R.$

Involution

• **<u>Definition</u>**: An **involution** of an Ore algebra D is a k-linear map $\theta : D \longrightarrow D$ satisfying:

1. $\theta(a_1\, a_2) = \theta(a_2)\, \theta(a_1), \quad a_1, a_2 \in D,$

2. $\theta^2 = id_D$.

• **<u>Exemple</u>**: 1. If $D = k[x_1, \ldots, x_n]$, then $\theta = id_D$.

2. If $D = k[x_1, \ldots, x_n][d_1; 1, \partial_1] \ldots [d_n; 1, \partial_n]$, then an involution of D is defined by:

$$x_i \longmapsto x_i, \quad d_i \longmapsto -d_i, \quad 1 \leq i \leq n.$$

3. If $D = A[\frac{d}{dt}; 1, \frac{d}{dt}][\delta_h; \sigma_h, 0][\delta_{-h}; \sigma_{-h}, 0]$, then an involution of D is defined by:

$$t \longmapsto t, \quad \frac{d}{dt} \longmapsto -\frac{d}{dt}, \quad \delta_h \longmapsto \delta_{-h}, \quad \delta_{-h} \longmapsto \delta_h.$$

Let $R = [t\, \frac{d}{dt} \; : \; -t^2\, \delta_h] \in D^{1 \times 2}$, then we have:

$$\theta(R) = \begin{pmatrix} -\frac{d}{dt}\, t \\ -\delta_{-h}\, t^2 \end{pmatrix} = \begin{pmatrix} -t\, \frac{d}{dt} - 1 \\ -(t+h)^2\, \delta_{-h} \end{pmatrix}.$$

• **<u>Proposition</u>**: Let D be an Ore algebra with an involution θ. Then, **any right D-module N** corresponds to the **left D-module $\widetilde{N}$** defined by:

$$\forall\, P \in D, \; \forall n \in N : \; P \circ n = n\, \theta(P).$$

Adjoint and transposed D-modules

• **<u>Definition:</u>** Let D be an Ore algebra and θ an **involution** of D. We call **adjoint** of $R \in D^{q\times p}$ the matrix defined by $\widetilde{R} = \theta(R) \in D^{p\times q}$.

• **<u>Definition:</u>** Let $R \in D^{q\times p}$ and $\widetilde{R} \in D^{p\times q}$ the **adjoint matrix** of R and $M = D^p/D^q\,R$ the **left** D-module defined by the exact sequence:

$$\begin{array}{ccccc} D^q & \xrightarrow{.R} & D^p & \longrightarrow M \longrightarrow 0. \\ (P_1 : \ldots : P_q) & \longrightarrow & (P_1 : \ldots : P_q)\,R & \end{array}$$

1. The **right** D-module $N = D^q/R\,D^p$ is the **transposed D-module.** We have the exact sequence:

$$0 \longleftarrow N \longleftarrow \begin{array}{ccc} D^q & \xleftarrow{R.} & D^p. \\ R\,(Q_1 : \ldots : Q_p)^T & \longleftarrow & (Q_1 : \ldots : Q_p)^T \end{array}$$

2. The **left** D-module $\widetilde{N} = D^q/D^p\,\widetilde{R}$ is the **adjoint D-module.** We have the exact sequence:

$$0 \longleftarrow \widetilde{N} \longleftarrow \begin{array}{ccc} D^q & \xleftarrow{.\widetilde{R}} & D^p. \\ (Q_1 : \ldots : Q_p)\,\widetilde{R} & \longleftarrow & (Q_1 : \ldots : Q_p) \end{array}$$

Projective equivalence

• **Proposition:** The D-module N **only depends on** M **up to a projective equivalence**, i.e. if we have

$$M = D^{p_1}/D^{q_1}\, R_1 = D^{p_2}/D^{q_2}\, R_2,$$

then $\exists\ P_1,\ P_2$ two **projective D-modules** such that:

$$P_1 \oplus N_1 = D^{q_1}/R_1\, D^{p_1} \cong P_2 \oplus N_2 = D^{q_2}/R_2\, D^{p_2}$$

$$\Rightarrow \text{ext}^i_D(N_1, S) \cong \text{ext}^i_D(N_2, S),\ i \geq 1.$$

• **Proposition:** $\text{ext}^i_D(N, D) \cong \text{ext}^i_D(\widetilde{N}, D),\ i \geq 1.$

Main results

• **Theorem 1**: If $M = D^{l_0}/D^{l_1}\,R_1,\ \widetilde{N} = D^{l_1}/D^{l_0}\,\widetilde{R_1}$ are the D-modules defined by

$$\begin{array}{ccccccc} & & D^{l_1} & \xrightarrow{.R_1} & D^{l_0} & \longrightarrow M \longrightarrow 0, \\ 0 \longleftarrow \widetilde{N} \longleftarrow & & D^{l_1} & \xleftarrow{.\widetilde{R_1}} & D^{l_0}, & \end{array}$$

where $\widetilde{N}$ is the **adjoint** of M, then:

1. **torsion submodule** $t(M) \cong \text{ext}^1_D(\widetilde{N}, D)$.

2. M is **torsion-free** $\Leftrightarrow \text{ext}^1_D(\widetilde{N}, D) = 0$,

3. M is **reflexive** $\Leftrightarrow \text{ext}^i_D(\widetilde{N}, D) = 0,\ i = 1, 2$,

4. M is **projective** $\Leftrightarrow \text{ext}^i_D(\widetilde{N}, D) = 0, i = 1...\text{gld}(D)$.

• **Theorem 2**: There exists an exact sequence

$$0 \longrightarrow M \longrightarrow D^{l_{-1}} \xrightarrow{.R_{-1}} D^{l_{-2}} \xrightarrow{.R_{-2}} \dots \xrightarrow{.R_{-r+1}} D^{l_{-r}}$$

$$\Longleftrightarrow \quad \text{ext}^i_D(\widetilde{N}, D) = 0, \quad i = 1, \dots, r.$$

• **Theorem 3**: If $D = K[d_1, \dots, d_n], M \neq 0$, then:

$$\begin{aligned} j(M) \ &:= \min_{i \geq 0} \{\, i \mid \text{ext}^i_D(M, D) \neq 0 \} \\ &= n - d(D/\sqrt{\text{ann}(\text{gr}(M))}), \end{aligned}$$

$d(\cdot) =$ **Krull dimension**.

Full row rank matrix & $D = K[d_1, \ldots, d_n]$

• Let $R \in D^{q \times p}$ $(1 \leq q \leq p)$ be a **full row rank matrix**,

$$M = D^p / D^q\, R, \qquad \widetilde{N} = D^q / D^p\, \widetilde{R}.$$

Module M	**ext**$_D^i(\widetilde{N}, D)$	$d(\widetilde{N})$	**Primeness***
	$\text{ext}_D^0(\widetilde{N}, D) \neq 0$	n	$\emptyset$
with torsion	$\text{ext}_D^1(\widetilde{N}, D) \cong t(M)$	$n-1$	$\emptyset$
torsion-free	$\text{ext}_D^1(\widetilde{N}, D) = 0$	$n-2$	minor left prime
reflexive	$\text{ext}_D^i(\widetilde{N}, D) = 0,$ $i = 1, 2$	$n-3$	
...	...	...	...
...	$\text{ext}_D^i(\widetilde{N}, D) = 0,$ $1 \leq i \leq n-1$	0	weakly zero left prime
projective	$\text{ext}_D^i(\widetilde{N}, D) = 0,$ $1 \leq i \leq n$	-1	zero left prime

• $d(\tilde{N})$ **is the degree of generality of the solutions of** $\widetilde{R}\, z = 0$.

$\star$ If $D = \mathbb{C}[d_1, \ldots, d_n]$, then $d(\tilde{N})$ is the dimension of the algebraic variety defined by the $q \times q$ minors of R.

Algorithms

Algorithm: torsion-free D-module

1. **Start with** $R \in D^{q \times p}$.

2. **Compute** its adjoint $\widetilde{R} \in D^{p \times q}$.

3. **Compute** ker $.\widetilde{R} \Rightarrow \exists\, \widetilde{R_{-1}} \in D^{m \times p}$:

$$\ker .\widetilde{R} = D^m\, \widetilde{R_{-1}}.$$

4. **Compute** its adjoint $R_{-1} = \widetilde{\widetilde{R_{-1}}} \in D^{p \times m}$.

5. **Compute** ker $.R_{-1} \Rightarrow \exists\, R' \in D^{q' \times p}$:

$$\ker .R_{-1} = D^{q'}\, R'.$$

$$\Rightarrow \begin{cases} t(M) = \ker .R_{-1}/D^q\, R = D^{q'}\, R'/D^q\, R, \\ M/t(M) = D^p/D^{q'}\, R'. \end{cases}$$

$$5.\quad D^{q'} \xrightarrow{.R'} D^p \xrightarrow{.R_{-1}} D^m$$

$$1.\quad D^q \xrightarrow{.R} D^p \xrightarrow{.R_{-1}} D^m \quad 4.$$

$$2.\quad D^q \xleftarrow{.\widetilde{R}} D^p \xleftarrow{.\widetilde{R_{-1}}} D^m \quad 3.$$

• $t(M) = 0 \Leftrightarrow D^{q'}\, R' = D^q\, R \Rightarrow R_{-1}$ is a **formal parametrization** of R:

$$M = D^p/D^q\, R \cong D^p\, R_{-1}.$$

Algorithm: reflexive D-module

- $R \in D^{q \times p}$, $M = D^p / D^q \, R$.

$$6. \quad D^{p'} \xrightarrow{.R'_{-1}} D^m \xrightarrow{.R_{-2}} D^l$$

$$5. \quad D^{q'} \xrightarrow{.R'} D^p \xrightarrow{.R_{-1}} D^m$$

$$1. \quad D^q \xrightarrow{.R} D^p \xrightarrow{.R_{-1}} D^m \xrightarrow{.R_{-2}} D^l \quad 4.$$

$$2. \quad D^q \xleftarrow{.\widetilde{R}} D^p \xleftarrow{.\widetilde{R_{-1}}} D^m \xleftarrow{.\widetilde{R_{-2}}} D^l \quad 3.$$

- M is a **reflexive** D-module

$$\Leftrightarrow D^{q'} \, R' = D^q \, R \;\; \& \;\; D^{p'} \, R'_{-1} = D^p \, R_{-1}.$$

- If the D-module $M = D^p / D^q \, R$ is **reflexive**, then we have the **exact sequence**

$$0 \longrightarrow M \longrightarrow D^m \xrightarrow{.R_{-2}} D^l$$

and $M^{\star\star} = \ker .R_{-2} = D^p \, R_{-1}$.

- M is a **projective** D-module

$$\Leftrightarrow D^{q'} \, R' = D^q \, R \;\; \& \;\; D^{p'_i} \, R'_{-i} = D^{p_i} \, R_{-i}, \quad 1 \leq i \leq n-1.$$

Algorithm: projective D-module

• **Theorem**: If $R \in D^{q \times p}$ is a **full row rank** matrix $-$ the rows of R are D-linearly independent $-$, then the D-module $M = D^p / D^q \, R$ is **projective** iff:

$$\widetilde{N} = D^q / D^p \, \widetilde{R} = 0 \Leftrightarrow \text{“}\widetilde{R} \, \lambda = 0 \Rightarrow \lambda = 0\text{”} .$$

• **Algorithm**:

1. **Compute the second syzygy** $\ker .R$ of M.

$\Rightarrow$ if $\ker .R \neq 0$, then stop, otherwise, continue.

2. **Check whether or not** $f_i \in D^p \, \widetilde{R}, \; i = 1, \ldots, q$

(check whether or not $\widetilde{R} \, \lambda = 0 \Rightarrow \lambda = 0$).

$\Rightarrow$ if it is not true, then M is **not a projective** D-module, else M is a **projective** D-module.

$$\Rightarrow \forall \, i = 1, \ldots, q : \; \exists \, \widetilde{S_i} \in D^p : \; \widetilde{S_i} \, \widetilde{R} = f_i$$

$$\Rightarrow \widetilde{S} = (\widetilde{S_1}^T : \ldots : \widetilde{S_q}^T)^T \in D^{q \times p} : \; \widetilde{S} \, \widetilde{R} = I_q$$

$$\Rightarrow S = \widetilde{\widetilde{S}} \in D^{p \times q} : \; R \, S = I_q .$$

Examples

Example

- Let us consider the **underdetermined system**:

$$\left\{\begin{array}{l} d_2\,\eta_1 - d_1\,\eta_1 + 2\,\eta_1 + 2\,\eta_2 - 2\,d_1\,\eta_3 = 0,\\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,d_2\,\eta_3 = 0. \end{array}\right.$$

- **Does this system admit a parametrization**?.

1. Let $D = k[d_1, d_2]$ and let us define the matrix

$$R = \left(\begin{array}{ccc} d_2 - d_1 + 2 & 2 & -2\,d_1\\ d_2 & d_2 & -d_1\,d_2 \end{array}\right) \in D^{2\times 3}$$

and the D-module $M = D^3/D^2\,R$.

2. The **formal adjoint** $\widetilde{R}$ of R is defined by:

$$\widetilde{R} = \left(\begin{array}{cc} -d_2 + d_1 + 2 & -d_2\\ 2 & -d_2\\ 2\,d_1 & -d_1\,d_2 \end{array}\right) \in D^{3\times 2}.$$

Then, the D-module $\widetilde{N} = D^2/D^3\,\widetilde{R}$ is defined by the **overdetermined system**:

$$\left\{\begin{array}{l} -d_2\,\lambda_1 + d_1\,\lambda_1 - d_2\,\lambda_2 + 2\,\lambda_1 = 0,\\ -d_2\,\lambda_2 + 2\,\lambda_1 = 0,\\ 2\,d_1\,\lambda_1 - d_1\,d_2\,\lambda_2 = 0. \end{array}\right.$$

Computing the second syzygy of $\widetilde{N}$ corresponds to find a **basis of the compatibility conditions** of:

$$\widetilde{R}\,\lambda = \mu \Rightarrow \widetilde{R}_{-1}\,\mu = 0.$$

Example

3. The computation of the **second syzygy module** of $\widetilde{N}$ gives (**gröbner basis, formal theory of PDE**):

$$\widetilde{R}_{-1} = (0 :\ -d_1 :\ 1) \in D^{1\times 3}.$$

$\Rightarrow$ we have the following **exact sequence**:

$$0 \longleftarrow \widetilde{N} \longleftarrow D^2 \xleftarrow{.\widetilde{R}} D^3 \xleftarrow{.\widetilde{R}_{-1}} D \longleftarrow 0 \ (\star)$$

4. The **dual sequence** of ($\star$) is the **complex**

$$0 \longrightarrow D^2 \xrightarrow{.R} D^3 \xrightarrow{.R_{-1}} D \longrightarrow 0,$$

with $R_{-1} = (0 :\ d_1 :\ 1)^T \in D^{3\times 1}$. **We have**:

$$\text{ext}^1_D(\widetilde{N}, D) = t(M) = \text{ker}.R_{-1}/D^2\, R.$$

5. The D-module $\text{ext}^2_D(\widetilde{N}, D) = D/D^3\, R_{-1}$ is **defined by the equations**:

$$\begin{cases} d_1\, \xi = 0, \\ \xi = 0, \end{cases} \Rightarrow \xi = 0 \Rightarrow \text{ext}^2_D(N, D) = 0.$$

Example

• **Let us compute** $t(M) = \ker .R_{-1}/D^2\,R$.

Finding a family of generators of $\ker .R_{-1}$ corresponds to **find the compatibility conditions** of:

$$R_{-1}\,\xi = \eta \Rightarrow R'\,\eta = 0.$$

We obtain $R' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & d_1 \end{pmatrix} \in D^{2\times 3}$ and:

$$t(M) = D^2\,R'/D^2\,R$$

$$= D^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & d_1 \end{pmatrix} / D^2 \begin{pmatrix} d_2 - d_1 + 2 & 2 & -2\,d_1 \\ d_2 & d_2 & -d_1\,d_2 \end{pmatrix}.$$

• $\theta_1 = \eta_1 \in M$ satisfies the system:

$$\begin{cases} \theta_1 = \eta_1, \\ d_2\,\eta_1 - d_1\,\eta_1 + 2\,\eta_1 + 2\,\eta_2 - 2\,d_1\,\eta_3 = 0, \\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,d_2\,\eta_3 = 0. \end{cases}$$

$$\Rightarrow (d_2^2 - d_1\,d_2)\,\theta_1 = 0 \Rightarrow 0 \neq \theta_1 \in t(M).$$

• $\theta_2 = -\eta_2 + d_1\,\eta_3 \in M$ satisfies the system:

$$\begin{cases} \theta_2 = -\eta_2 + d_1\,\eta_3, \\ d_2\,\eta_1 - d_1\,\eta_1 + 2\,\eta_1 + 2\,\eta_2 - 2\,d_1\,\eta_3 = 0, \\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,d_2\,\eta_3 = 0. \end{cases}$$

$$\Rightarrow (d_2^2 - d_1\,d_2)\,\theta_2 = 0 \Rightarrow 0 \neq \theta_2 \in t(M).$$

Example

- $M = D^3/D^2\, R$ **has torsion elements**

  $\Rightarrow R\,\eta = 0$ **is not formally parametrizable.**

- **By construction**, we know that we have:

$$\left\{\begin{array}{l} 0 = \eta_1, \\ d_1\,\xi = \eta_2, \\ \xi = \eta_3, \end{array}\right. \quad \Rightarrow \quad \underbrace{\left\{\begin{array}{l} d_2\,\eta_1 - d_1\,\eta_1 + 2\,\eta_1 + 2\,\eta_2 - 2\,d_1\,\eta_3 = 0, \\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,d_2\,\eta_3 = 0. \end{array}\right.}_{R\,\eta = 0}$$

- $\left\{\begin{array}{l} \eta_1 = x_2, \\ \eta_2 = -(x_2 + \frac{1}{2}), \\ \eta_3 = 0, \end{array}\right.$ is a **solution of** $R\,\eta = 0$

but is not of the form $(0 : d_1\,\xi : \xi)^T$ for a certain ξ.

A) **Integration of the torsion elements** of M:

$$\left\{\begin{array}{l} \theta_1 = \eta_1, \\ (d_2^2 - d_1\,d_2)\,\theta_1 = 0, \\ \theta_2 = -\eta_2 + d_1\,\eta_3 = (d_2\,\theta_1 - d_1\,\theta_1 + 2\,\theta_1)/2, \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \eta_1 = f(x_1) + g(x_1 + x_2) \\ -\eta_2 + d_1\,\eta_3 = -\frac{1}{2}\,\dot{f}(x_1) + f(x_1) + g(x_1 + x_2), \end{array}\right.$$

where η_1 and η_2 **satisfy**:

$$\left\{\begin{array}{l} d_2\,\eta_1 - d_1\,\eta_1 + 2\,\eta_1 + 2\,\eta_2 - 2\,d_1\,\eta_3 = 0, \\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,d_2\,\eta_3 = 0. \end{array}\right.$$

B) Integration of the **inhomogeneous system**:

$$\left\{\begin{array}{l} \eta_1 = f(x_1) + g(x_1 + x_2) \\ -\eta_2 + d_1\, \eta_3 = -\frac{1}{2}\, \dot{f}(x_1) + f(x_1) + g(x_1 + x_2), \\ d_2\, \eta_1 - d_1\, \eta_1 + 2\, \eta_1 + 2\, \eta_2 - 2\, d_1\, \eta_3 = 0, \\ d_2\, \eta_1 + d_2\, \eta_2 - d_1\, d_2\, \eta_3 = 0. \end{array}\right.$$

i. **Particular solution** ($M/t(M)$ is projective):

$$\left\{\begin{array}{l} \eta_1 = f(x_1) + g(x_1 + x_2), \\ \eta_2 = \frac{1}{2}\, \dot{f}(x_1) - f(x_1) - g(x_1 + x_2), \\ \eta_3 = 0. \end{array}\right.$$

ii. **General solution of the homogeneous system**:

$$\left\{\begin{array}{l} \eta_1 = 0 \\ -\eta_2 + d_1\, \eta_3 = 0, \\ d_2\, \eta_1 - d_1\, \eta_1 + 2\, \eta_1 + 2\, \eta_2 - 2\, d_1\, \eta_3 = 0, \\ d_2\, \eta_1 + d_2\, \eta_2 - d_1\, d_2\, \eta_3 = 0, \end{array}\right. \Leftrightarrow M/t(M)$$

$$\Leftrightarrow \left\{\begin{array}{l} \eta_1 = 0, \\ \eta_2 = d_1\, \xi, \\ \eta_3 = \xi. \end{array}\right.$$

- We have the following **parametrization**:

$$\left\{\begin{array}{l} d_2\, \eta_1 - d_1\, \eta_1 + 2\, \eta_1 + 2\, \eta_2 - 2\, d_1\, \eta_3 = 0, \\ d_2\, \eta_1 + d_2\, \eta_2 - d_1\, d_2\, \eta_3 = 0, \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} \eta_1 = f(x_1) + g(x_1 + x_2), \\ \eta_2 = \frac{1}{2}\, \dot{f}(x_1) - f(x_1) - g(x_1 + x_2) + d_1\, \xi(x), \\ \eta_3 = \xi(x). \end{array}\right.$$

Example

- Let us consider the **underdetermined system**:

$$\begin{cases} d_2\,\eta_1 - d_1\,\eta_1 - 2\,d_1\,\eta_3 + 2\,\eta_1 + 2\,\eta_2 = 0, \\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,\eta_3 = 0. \end{cases}$$

- **Does this system admit a parametrization?**

1. Let $D = k[d_1, d_2]$ and let us define the matrix

$$R = \begin{pmatrix} d_2 - d_1 + 2 & 2 & -2\,d_1 \\ d_2 & d_2 & -d_1 \end{pmatrix} \in D^{2\times 3}$$

and the D-module $M = D^3/D^2\,R$.

2. The **formal adjoint** $\widetilde{R}$ of R is defined by:

$$\widetilde{R} = \begin{pmatrix} -d_2 + d_1 + 2 & -d_2 \\ 2 & -d_2 \\ 2\,d_1 & d_1 \end{pmatrix} \in D^{3\times 2}.$$

Then, the D-module $\widetilde{N} = D^2/D^3\,\widetilde{R}$ is defined by the **overdetermined system**:

$$\begin{cases} -d_2\,\lambda_1 + d_1\,\lambda_1 - d_2\,\lambda_2 + 2\,\lambda_1 = 0, \\ -d_2\,\lambda_2 + 2\,\lambda_1 = 0, \\ 2\,d_1\,\lambda_1 + d_1\,\lambda_2 = 0. \end{cases}$$

Computing the second syzygy of $\widetilde{N}$ corresponds to find a **basis of the compatibility conditions** of:

$$\widetilde{R}\,\lambda = \mu \Rightarrow \widetilde{R}_{-1}\,\mu = 0.$$

Example

3. The computation of the **second syzygy module** of $\widetilde{N}$ gives (**gröbner basis, formal theory of PDE**):

$$\widetilde{R}_{-1} = (2\,d_1\,d_2 + 2\,d_1 \,:\, -d_1\,d_2 - d_1^2 - 2\,d_1 \,:\, d_2^2 - d_1\,d_2).$$

$\Rightarrow$ we have the following **exact sequence**:

$$0 \longleftarrow \widetilde{N} \longleftarrow D^2 \xleftarrow{.\widetilde{R}} D^3 \xleftarrow{.\widetilde{R}_{-1}} D \longleftarrow 0 \quad (\star)$$

4. The **dual sequence** of ($\star$) is the **complex**

$$0 \longrightarrow D^2 \xrightarrow{.R} D^3 \xrightarrow{.R_{-1}} D \longrightarrow 0,$$

with:

$$R_{-1} = (2\,d_1\,d_2 - 2\,d_1 \,:\, -d_1\,d_2 - d_1^2 + 2\,d_1 \,:\, d_2^2 - d_1\,d_2)^T.$$

We have $\begin{cases} t(M) = \mathrm{ext}^1_D(N, D) = \ker .R_{-1}/D^2\,R, \\ \\ \mathrm{ext}^2_D(N, D) = D/D^3\,R_{-1}. \end{cases}$

5. The D-module $\mathrm{ext}^2_D(N, D) = D/D^3\,R_{-1}$ is **defined by the equations**:

$$\begin{cases} 2\,d_1\,d_2\,\xi - 2\,d_1\,\xi = 0, \\ -d_1\,d_2\,\xi - d_1^2\,\xi + 2\,d_1\,\xi = 0, \\ d_2^2\,\xi - d_1\,d_2\,\xi = 0, \end{cases} \Rightarrow \mathrm{ext}^2_D(N, D) \neq 0.$$

Example

• **Let us compute** $\text{ext}^1_D(N,D) = \ker .R_{-1}/D^2\,R$.

Finding a family of generators of $\ker .R_{-1}$ corresponds to **find the compatibility conditions** of:

$$R_{-1}\,\xi = \eta \Rightarrow R'\,\eta = 0.$$

We obtain:

$$R' = \begin{pmatrix} d_2 - d_1 + 2 & 2 & -2\,d_1 \\ d_2 & d_2 & -d_1 \end{pmatrix} = R,$$

$$\Rightarrow t(M) \cong \text{ext}^1_D(N,D) = D^3\,R/D^3\,R = 0.$$

• Finally, we have the **exact sequence**:

$$0 \longrightarrow D^2 \xrightarrow{.R} D^3 \xrightarrow{.R_{-1}} D \longrightarrow \text{ext}^2_D(N,D) \longrightarrow 0.$$

• If $S = \mathcal{D}'(\Omega),\ \mathcal{E}(\Omega)$ or $\mathcal{S}'(\Omega)$, then we have the **exact sequence**:

$$0 \longleftarrow S^2 \xleftarrow{R.} S^3 \xleftarrow{R_{-1}.} S \longleftarrow \text{hom}_D(\text{ext}^2_D(N,D),S) \longleftarrow 0.$$

$$\begin{cases} d_2\,\eta_1 - d_1\,\eta_1 - 2\,d_1\,\eta_3 + 2\,\eta_1 + 2\,\eta_2 = 0, \\ d_2\,\eta_1 + d_2\,\eta_2 - d_1\,\eta_3 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \eta_1 = 2\,d_1\,d_2\,\xi - 2\,d_1\,\xi, \\ \eta_2 = -d_1\,d_2\,\xi - d_1^2\,\xi + 2\,d_1\,\xi, \\ \eta_3 = d_2^2\,\xi - d_1\,d_2\,\xi. \end{cases}$$

Example

- Let us consider the differential time-delay system:

$$\dot{y}(t) - t\,u(t-1) = 0.$$

- We define $D = k(t)[\frac{d}{dt}, \delta, \sigma]$, where:

$$(\delta\, w)(t) = w(t-1), \quad (\sigma\, w)(t) = w(t+1).$$

- Let $R = \left(\frac{d}{dt} \; : \; -t\,\delta\right) \in D^{1\times 2}$ and let us define **the left D-module** $M = D^2/D\,R$.

- We have the **finite free presentation** of M:

$$0 \longrightarrow D \xrightarrow{.R} D^2 \longrightarrow M \longrightarrow 0.$$

- Using the **involution** θ on D, we obtain:

$$\theta(R) = \begin{pmatrix} -\frac{d}{dt} \\ -\sigma\, t \end{pmatrix} = \begin{pmatrix} -\frac{d}{dt} \\ -(t+1)\,\sigma \end{pmatrix}$$

and we define $\widetilde{N} = D/D^2\,\theta(R)$.

- We have the **exact sequence**:

$$0 \longleftarrow \widetilde{N} \longleftarrow D \xleftarrow{.\theta(R)} D^2 \longleftarrow \ker .\theta(R) \longleftarrow 0.$$

Example

- The left D-module $\widetilde{N} = D/D^2\,\theta(R)$ is defined by:

$$\begin{cases} \dot{\lambda} = 0, \\ (t+1)\,\sigma\,\lambda = (t+1)\ \ \lambda(t+1) = 0. \end{cases}$$

- The **compatibility condition** of the system

$$\begin{cases} \dot{\lambda} = \mu_1, \\ (t+1)\,\sigma\,\lambda = \mu_2, \end{cases}$$

is given by $\tau\,\mu_1 - \frac{1}{(t+1)}\,\dot{\mu}_2 + \frac{1}{(t+1)^2}\,\mu_2 = 0.$

- If we define the matrix

$$\theta(R_{-1}) = \left(\tau\ :\ -\frac{1}{(t+1)}\,\frac{d}{dt} + \frac{1}{(t+1)^2}\right) \in D^{1\times 2}$$

then we have the following **exact sequence**:

$$0 \longleftarrow \widetilde{N} \longleftarrow D \overset{.\theta(R)}{\longleftarrow} D^2 \overset{.\theta(R_{-1})}{\longleftarrow} D \longleftarrow 0 \quad (\star).$$

- Dualizing $(\star)$, we obtain the following **complex**

$$0 \longrightarrow D \xrightarrow{.R} D^2 \xrightarrow{.R_{-1}} D \longrightarrow 0.$$

where:

$$R_{-1} = \begin{pmatrix} \delta \\ \frac{1}{(t+1)}\,\frac{d}{dt} \end{pmatrix} \in D^{2\times 1}.$$

Example

• The **left D-module** $\text{ext}^2_D(\widetilde{N}, D) = D/D^2\, R_{-1}$ is defined by the following system:

$$\left\{ \begin{array}{l} \delta\, \xi = 0, \\ \frac{1}{(t+1)}\, \dot{\xi} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \lambda(t-1) = 0, \\ \dot{\xi} = 0, \end{array} \right. \not\Rightarrow \xi = 0$$

$\Rightarrow M$ is **not a reflexive** left D-module.

• The **compatibility condition** of the system

$$\left\{ \begin{array}{l} \delta\, \xi = y, \\ \frac{1}{(t+1)}\, \dot{\xi} = u, \end{array} \right.$$

is defined by **the first system** $\dot{y} - t\, \delta\, u = 0$.

Therefore, we have the **exact sequence**

$$0 \longrightarrow D \xrightarrow{.R} D^2 \xrightarrow{.R_{-1}} D \longrightarrow 0,$$

and the solutions of $\dot{y}(t) - t\, u(t-1) = 0$ are **formally parametrized** by:

$$\left\{ \begin{array}{l} y(t) = \xi(t-1), \\ u(t) = \frac{1}{(t+1)}\, \dot{\xi}(t). \end{array} \right.$$

Example

- Let $R_1 = (d_1 : \ d_2 : \ d_3)$ and $M = D^3 / D\, R_1$.

We have the following exact sequence:

$$0 \longrightarrow D \xrightarrow{.R_1} D^3 \longrightarrow M \longrightarrow 0.$$

- The D-module $N = D / D^3\, R_1^T$ is defined by:

$$d_1\, y = 0, \ \ d_2\, y = 0, \ \ d_3\, y = 0.$$

We have the following **free resolution** for N:

$$0 \longleftarrow N \longleftarrow D \xleftarrow{.\text{grad}} D^3 \xleftarrow{.\text{curl}} D^3 \xleftarrow{.\text{div}} D \longleftarrow 0.$$

with $\text{grad} = R_1^T, \ \text{div} = R_1$ and:

$$\text{curl} = \begin{pmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{pmatrix}.$$

- The **dual of the truncated free resolution** is:

$$0 \longrightarrow D \xrightarrow{.\text{div}} D^3 \xrightarrow{.\text{curl}} D^3 \xrightarrow{.\text{grad}} D \longrightarrow 0,$$

$$\Rightarrow \begin{cases} \text{ext}^i_D(N, D) = 0, \ \ 0 \le i \le 2, \\ \text{ext}^3_D(N, D) = D / D^3\, R_1^T = N \neq 0. \end{cases}$$

$\Rightarrow M$ **is a reflexive but not a free D-module**

$\Rightarrow M^{\star\star} \cong D^3\, \text{grad}$. We have the **exact sequence**:

$$0 \longrightarrow M \cong D^3\, \text{curl} \longrightarrow D^3 \xrightarrow{.\text{grad}} D.$$

- $j(N) = 3 \Leftrightarrow 3 - d(N) = 3 \Leftrightarrow d(N) = 0.$

Example

• If $S = \mathcal{D}'(\Omega)$, $\mathcal{E}(\Omega)$ and $\mathcal{S}'(\Omega)$, then, from the **exact sequence**

$$0 \longrightarrow D \xrightarrow{.\mathrm{div}} D^3 \xrightarrow{.\mathrm{curl}} D^3 \xrightarrow{.\mathrm{grad}} D \longrightarrow N \longrightarrow 0,$$

we obtain the **exact sequence**:

$$0 \longleftarrow S \xleftarrow{\mathrm{div.}} S^3 \xleftarrow{\mathrm{curl.}} S^3 \xleftarrow{\mathrm{grad.}} S \longleftarrow \hom_D(N, S) \longleftarrow 0.$$

• We have the following **consequences**:

- $\forall\, u \in S,\ \exists\, y = (y_1 :\ y_2 :\ y_3)^T \in S^3$:

$$z = d_1\, y_1 + d_2\, y_2 + d_3\, y_3.$$

- $d_1\, y_1 + d_2\, y_2 + d_3\, y_3 = 0$

$$\Leftrightarrow \exists\, (z_1 : z_2 : z_3)^T \in S^3 : \left\{ \begin{array}{l} y_1 = d_2\, z_2 + d_2\, z_3 \\ y_2 = d_3\, z_1 - d_1\, z_3, \\ y_3 = -d_2\, z_1 + d_1\, z_2. \end{array} \right.$$

- $\left\{ \begin{array}{l} d_2\, z_2 + d_2\, z_3 = 0, \\ d_3\, z_1 - d_1\, z_3 = 0, \\ -d_2\, z_1 + d_1\, z_2 = 0. \end{array} \right. \Leftrightarrow \exists\, f \in S : \left\{ \begin{array}{l} z_1 = d_1\, f, \\ z_2 = d_2\, f, \\ z_3 = d_3\, f. \end{array} \right.$

Example

- Let us consider the **underdetermined equation:**

$$d_1\,\zeta_1 + d_2\,\zeta_2 + x_2\,\zeta_1 = 0.$$

- **Does this system admit a parametrization?**

1. Let $D = \mathbb{R}(x_1, x_2)[d_1, d_2]$ and let us define

$$R = (d_1 + x_2 \,:\; d_2) \in D^{1\times 2}$$

and the D-module $M = D^2/D\,R$.

2. The **formal adjoint** $\widetilde{R}$ of R is defined by:

$$\widetilde{R} = \begin{pmatrix} -d_1 + x_2 \\ -d_2 \end{pmatrix} \in D^{2\times 1}.$$

Then, the D-module $\widetilde{N} = D/D^2\,\widetilde{R}$ is defined by the **overdetermined system:**

$$\left\{ \begin{array}{l} -d_1\,\lambda + x_2\,\lambda = 0, \\ -d_2\,\lambda = 0. \end{array} \right.$$

The computation of the second syzygy module of $\widetilde{N}$ leads to $\widetilde{R}\,\lambda = \mu \Rightarrow \lambda = d_2\,\mu_1 - d_1\,\mu_2 + x_2\,\mu_2$

$$\Rightarrow \widetilde{N} = 0 \Rightarrow M \text{ is a } \textbf{projective } D\textbf{-module}.$$

Example

3. The computation of the second syzygy module of $\widetilde{N}$ gives (**Gröbner basis, formal theory of PDE**):

$$\widetilde{R}_{-1} = \left(\begin{array}{cc} d_1\, d_2 - x_2\, d_2 + 1 & -d_1^2 + 2\, x_2\, d_1 - x_2^2 \\ d_2^2 & -d_1\, d_2 + x_2\, d_2 + 2 \end{array} \right).$$

The computation of the third syzygy module of $\widetilde{N}$ gives (**Gröbner basis, formal theory of PDE**):

$$\widetilde{R}_{-2} = (d_1 - x_2 :\ d_2) \in D^{1 \times 2}$$

$\Rightarrow$ we have the following **exact sequence**:

$$0 \longleftarrow \widetilde{N} \longleftarrow D \xleftarrow{.\widetilde{R}} D^2 \xleftarrow{.\widetilde{R}_{-1}} D^2 \xleftarrow{.\widetilde{R}_{-2}} D \longleftarrow 0 \; (\star).$$

4. M is a **projective** D-module $\Rightarrow$ the **dual sequence** of $(\star)$

$$0 \longrightarrow D \xrightarrow{.R} D^2 \xrightarrow{.R_{-1}} D^2 \xrightarrow{.R_{-2}} D \longrightarrow 0$$

is **exact**, with:

$$\left\{ \begin{array}{l} R_{-1} = \left(\begin{array}{cc} d_1\, d_2 + x_2\, d_2 + 2 & d_2^2 \\ -d_1^2 - 2\, x_2\, d_1 - x_2^2 & -d_1\, d_2 - x_2\, d_2 + 1 \end{array} \right), \\ \\ R_{-2} = (-d_2 :\ d_1 + x_2)^T. \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} M \cong D^2\, R_{-1}, \\ 1 \in D^2\, R_{-2} \Rightarrow (d_1 + x_2 :\ d_2)\, R_{-2} = 1. \end{array} \right.$$

Example

• We have the following **split exact sequence**

$$0 \longrightarrow D \underset{\overset{.S}{\longleftarrow}}{\overset{.R}{\longrightarrow}} D^2 \underset{\overset{.S_{-1}}{\longleftarrow}}{\overset{.R_{-1}}{\longrightarrow}} D^2 \underset{\overset{.S_2}{\longleftarrow}}{\overset{.R_{-2}}{\longrightarrow}} D \longrightarrow 0,$$

$$\begin{cases} S = (d_2 : \, -d_1 + x_2)^T, \\ S_{-1} = \begin{pmatrix} d_1\, d_2 - x_2\, d_2 - 2 & d_2^2 \\ d_1^2 - 2\, x_2\, d_1 + x_2^2 & d_1\, d_2 - x_2\, d_2 + 1 \end{pmatrix}, \\ S_{-2} = (d_1 - x_2 : \, -d_2), \end{cases}$$

i.e. we have:

$$R\, S = 1, \;\; R_{-1}\, S_{-1}\, R_{-1} = R_{-1}, \;\; S_{-2}\, R{-}2 = 1.$$

• If S is any D-module, then we have the following **exact sequence**:

$$0 \longleftarrow S \overset{R.}{\longleftarrow} S^2 \overset{R_{-1}.}{\longleftarrow} S^2 \overset{R_{-2}.}{\longleftarrow} S \longleftarrow 0$$

$$d_1\, \zeta_1 + d_2\, \zeta_2 + x_2\, \zeta_1 = 0$$

$$\Leftrightarrow \begin{cases} \zeta_1 = d_2^2\, \eta_2 + d_1\, d_2\, \eta_1 + x_2\, d_2\, \eta_1 + 2\, \eta_1, \\ \zeta_2 = -d_1\, d_2\, \eta_2 - d_1^2\, \eta_1 - 2\, x_2\, d_1\, \eta_1 - x_2\, d_2 \eta_2 - x_2^2\, \eta_1 + \eta_2, \end{cases}$$

$$\begin{cases} d_2^2\, \eta_2 + d_1\, d_2\, \eta_1 + x_2\, d_2\, \eta_1 + 2\, \eta_1 = 0, \\ -d_1\, d_2\, \eta_2 - d_1^2\, \eta_1 - 2\, x_2\, d_1\, \eta_1 - x_2\, d_2 \eta_2 - x_2^2\, \eta_1 + \eta_2 = 0, \end{cases}$$

$$\Leftrightarrow \begin{cases} \eta_1 = -d_2\, \xi, \\ \eta_2 = d_1\, \xi + x_2\, \xi. \end{cases}$$

Applications to mathematical physics

Electromagnetism

- **First set of Maxwell equations**:

$$\left\{ \begin{array}{l} \frac{\partial \vec{B}}{\partial t} + \nabla \wedge \vec{E} = \vec{0}, \\ \nabla \,.\, \vec{B} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} = \nabla \wedge \vec{A}, \end{array} \right.$$

where $(\vec{A}, V)$ is called the **quadri-potential**.

The quadri-potential $(\vec{A}, V)$ determines $(\vec{E}, \vec{B})$ up to a **gauge transformation**:

$$\forall f : \ (\vec{A}, V) \to (\vec{A} + \nabla f, V + \frac{\partial f}{\partial t})$$

because $\left\{ \begin{array}{l} \nabla V + \frac{\partial \vec{A}}{\partial t} = 0, \\ \nabla \wedge \vec{A} = \vec{0}, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} V = \frac{\partial f}{\partial t}, \\ \vec{A} = \nabla f. \end{array} \right.$

- **Second set of Maxwell equations** (duality):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \,.\, \vec{\jmath} = 0 \Leftrightarrow \left\{ \begin{array}{l} \vec{\jmath} = \nabla \wedge \vec{H} - \frac{\partial \vec{D}}{\partial t}, \\ \rho = \nabla \,.\, \vec{D}. \end{array} \right.$$

$$\left\{ \begin{array}{l} -\frac{\partial \vec{D}}{\partial t} + \nabla \wedge \vec{H} = 0, \\ \nabla \,.\, \vec{D} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \vec{H} = \nabla \phi_1 + \frac{\partial \vec{\phi_2}}{\partial t}, \\ \vec{D} = \nabla \wedge \vec{\phi_2}. \end{array} \right.$$

$$\left\{ \begin{array}{l} \nabla \phi_1 + \frac{\partial \vec{\phi_2}}{\partial t} = 0, \\ \nabla \wedge \vec{\phi_2} = 0, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \phi_1 = \frac{\partial \psi}{\partial t}, \\ \vec{\phi_2} = \nabla \psi. \end{array} \right.$$

Linearized Einstein equations

- J. Wheeler's challenge (1970):

Do Einstein equations in vaccum admit a generic potential?

- The **answer is no** (J.F. Pommaret 1995).

- The **linearized Ricci equations in vacuum** are defined by $R \in D^{10\times 10} (D = \mathbb{C}[d_1, d_2, d_3, d_4])$:

$$R = \left(\begin{array}{cccc} d_2^2 + d_3^2 - d_4^2 & d_1^2 & d_1^2 & -d_1^2 \\ d_2^2 & d_1^2 + d_3^2 - d_4^2 & d_2^2 & -d_2^2 \\ d_3^2 & d_3^2 & d_1^2 + d_2^2 - d_4^2 & -d_3^2 \\ d_4^2 & d_4^2 & d_4^2 & d_1^2 + d_2^2 + d_3^3 \\ 0 & 0 & d_1\, d_2 & -d_1\, d_2 \\ d_2\, d_3 & 0 & 0 & -d_2\, d_3 \\ d_3\, d_4 & d_3\, d_4 & 0 & 0 \\ 0 & d_1\, d_3 & 0 & -d_1\, d_3 \\ d_2\, d_4 & 0 & d_2\, d_4 & 0 \\ 0 & d_1\, d_4 & d_1\, d_4 & 0 \end{array} \right.$$

$$\left. \begin{array}{cccccc} -2\, d_1\, d_2 & 0 & 0 & -2\, d_1\, d_3 & 0 & 2\, d_1\, d_4 \\ -2\, d_1\, d_2 & -2\, d_2\, d_3 & 0 & 0 & 2\, d_2\, d_4 & 0 \\ 0 & -2\, d_2\, d_3 & 2\, d_3\, d_4 & -2\, d_1\, d_3 & 0 & 0 \\ 0 & 0 & -2\, d_3\, d_4 & 0 & -2\, d_2\, d_4 & -2\, d_1\, d_4 \\ d_2^3 - d_4^2 & -d_1\, d_3 & 0 & -d_2\, d_3 & d_1\, d_4 & d_2\, d_4 \\ -d_1\, d_3 & d_1^2 - d_4^2 & d_2\, d_4 & -d_1\, d_2 & d_3\, d_4 & 0 \\ 0 & -d_2\, d_4 & d_1^2 + d_2^2 & -d_1\, d_4 & -d_2\, d_3 & -d_1\, d_3 \\ -d_2\, d_3 & -d_1\, d_2 & d_1\, d_4 & d_2^2 - d_4^2 & 0 & d_3\, d_4 \\ -d_1\, d_4 & -d_3\, d_4 & -d_2\, d_3 & 0 & d_1^2 + d_3^2 & -d_1\, d_3 \\ -d_2\, d_4 & 0 & -d_1\, d_3 & -d_3\, d_4 & -d_1\, d_3 & d_2^2 + d_3^2 \end{array} \right)$$

- The torsion submodule of $M = D^{10}/D^{10}\, R$ is defined by **20 torsion elements** and $M/t(M)$ admits a **parametrization with 4 potentials**.

Decoupling problem

• Let us consider the **linearized system of Euler equations for an incompressible fluid**:

$$\left\{\begin{array}{l} \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = 0, \\ \partial_t v_1 + \partial_1 p = 0, \\ \partial_t v_2 + \partial_2 p = 0, \\ \partial_t v_3 + \partial_3 p = 0, \end{array}\right.$$

($\vec{v} = (v_1 : v_2 : v_3)^T$ speed, p pressure).

• By differential eliminations, we can **decouple the variables** v_1, v_2, v_3 and p:

$$\left\{\begin{array}{l} \Delta p = (\partial_1^2 + \partial_2^2 + \partial_3^2)\, p = 0, \\ \partial_t \Delta v_1 = 0, \\ \partial_t \Delta v_2 = 0, \\ \partial_t \Delta v_3 = 0. \end{array}\right.$$

Lie-Poisson structure

- Let us consider the following system (**algebra** E_2):

$$\begin{cases} x_1\, d_3\, y_1 + x_2\, d_3\, y_2 = 0, \\ -x_1\, d_2\, y_1 + x_2\, d_1\, y_1 - y_2 + x_2\, d_3\, y_3 = 0, \quad (\star) \\ -y_1 - x_2\, d_1\, y_2 + x_1\, d_2\, y_2 + x_1\, d_3\, y_3 = 0. \end{cases}$$

- The system $(\star)$ is **not parametrizable** because $(\star)$ has the following **torsion elements** (Seiler 03):

$$\begin{cases} z_1 = x_1\, y_1 + x_2\, y_2, \\ d_3\, z_1 = 0, \\ (-x_1\, d_2 + x_2\, d_1)\, z_1 = 0, \end{cases}$$

$$\begin{cases} z_2 = -x_1^2\, d_2\, y_2 + x_1\, x_2\, d_1\, y_2 - x_2\, y_2 - x_1^2\, d_3\, y_3, \\ d_3\, z_2 = 0, \\ (-x_1\, d_2 + x_2\, d_1)\, z_2 = 0. \end{cases}$$

- The **torsion-free submodule** is defined by

$$\begin{cases} x_1\, y_1 + x_2\, y_2 = 0, \\ -x_1^2\, d_2\, y_2 + x_1\, x_2\, d_1\, y_2 - x_2\, y_2 - x_1^2\, d_3\, y_3 = 0, \end{cases}$$

and is **parametrized by**:

$$\begin{cases} y_1 = x_2\, d_3\, \xi, \\ y_2 = -x_1\, d_3\, \xi, \\ y_3 = -x_2\, d_1\, \xi + x_1\, d_2\, \xi. \end{cases}$$

Applications to linear control theory

Module theory in control theory

1. A **linear control system** is defined by a **matrix with entries in a ring** A:

-$A = \mathbb{R}[\frac{d}{dt}] \subseteq K[\frac{d}{dt}], \quad K = \mathbb{R},\ \mathbb{R}(t),\ C^{\infty}(\Omega)$.

-$A = \mathbb{R}[\frac{d}{dt}, \delta_{t_1}, \ldots, \delta_{t_n}] \cong \mathbb{R}[\chi_1, \ldots, \chi_{n+1}]$.

-$K[\frac{d}{dt}] \ \subseteq \ K[d_1, \ldots, d_n] \ \supseteq \ \mathbb{R}[\chi_1, \ldots, \chi_n]$.

-$A = H_{\infty}(\mathbb{C}_+),\ RH_{\infty},\ \mathcal{A},\ \hat{\mathcal{A}},\ \mathcal{E} \ldots$

2. Study of the **structural properties** of the system

$\Rightarrow$ **linear algebra over the ring** A.

3. **Linear algebra** over a ring $A \Rightarrow$ **module theory**.

4. **Module theory for linear control theory** (Kalman, Oberst, Fliess, Pommaret . . .)

• **Philosophy:**

a) From the equations of the system, **we define an** A**-module**.

b) We develop **a dictionary “properties of systems/properties of modules”**.

c) **Use of module theory** (properties, classifications . . .).

d) **Use of homological algebra** to develop **effective algorithms** (A.Q.).

Dictionary “Modules-Systems”

Modules	Structural properties	Optimal control
Torsion	Poles/zeros classifications	
With torsion	Existence of autonomous elements	
Torsion-free	No autonomous elements, Controllability, Minor left primeness	Variational problem without constraints (Euler-Lagrange equations)
Reflexive	Filter identification	
.	.	.
.	.	.
Projective	Internal stabilization, Computations of the stabilizing controllers, Zero left primeness Bézout identities	Computations of the Lagrange parameters without integrations
Free	Brunovsky canonical form Flatness, Poles placement, Doubly coprime factorization Youla parametrization of the stabilizing controllers,	Optimal controller

$1D$ linear control systems

Controllability of a Kalman system

$$\dot{x} = A\,x + B\,u. \qquad (1)$$

• Let us consider the $D = k[\frac{d}{dt}]$-module defined by:

$$M = D^{n+m}/D^n\,R, \quad R = \left(\frac{d}{dt}\,I_n - A : \; -B\right).$$

We have the following **exact sequence**:

$$0 \longrightarrow D^n \xrightarrow{.R} D^{n+m} \longrightarrow M \longrightarrow 0.$$

• The D-module $N = D^n/D^{n+m}\,\widetilde{R}$ is defined by:

$$\begin{cases} -\dot{\lambda} - A^T\,\lambda = 0, \\ -B^T\,\lambda = 0, \end{cases} \Rightarrow \begin{cases} \dot{\lambda} = -A^T\,\lambda, \\ B^T\,\lambda = 0. \end{cases} \qquad (2)$$

• (1) is **controllable** iff M is **torsion-free**, i.e. $N = 0$.

• The **formal integrability** of (2) gives:

$$B^T\,\dot{\lambda} = 0 \Rightarrow -B^T\,A^T\,\lambda = 0 \Rightarrow B^T\,A^T\,\dot{\lambda} = 0$$

$$\Rightarrow B^T\,(A^2)^T\,\lambda = 0 \Rightarrow \ldots \Rightarrow B^T\,(A^{n-1})^T\,\lambda = 0.$$

• System (1) is **controllable** iff:

$$\text{rk}\,(B : \; A\,B : \; A^2\,B : \ldots : A^{n-1}\,B) = n.$$

Controllability of a Kalman system

$$\dot{x} = A(t)\,x + B(t)\,u. \qquad (3)$$

- Let us consider the $D = K[\frac{d}{dt}]$-module defined by:

$$M = D^{n+m}/D^n\,R, \quad R = \left(\frac{d}{dt}\,I_n - A(t) : \; -B(t)\right).$$

We have the following **exact sequence**:

$$0 \longrightarrow D^n \xrightarrow{.R} D^{n+m} \longrightarrow M \longrightarrow 0.$$

- The D-module $N = D^n/D^{n+m}\,\widetilde{R}$ is defined by:

$$\left\{ \begin{array}{l} -(\dot{\lambda} + A(t)^T\,\lambda) = 0, \\ -B(t)^T\,\lambda = 0, \end{array} \right. \quad \Rightarrow \left\{ \begin{array}{l} \dot{\lambda} = -A(t)^T\,\lambda, \\ B(t)^T\,\lambda = 0. \end{array} \right. \qquad (4)$$

- (3) is **controllable** iff M is **torsion-free**, i.e. $N = 0$.

- The **formal integrability** of (4) gives:

$$B(t)^T\,\dot{\lambda} + \dot{B}(t)^T\,\lambda = 0 \Rightarrow (-B^T\,A^T + \dot{B}^T)\,\lambda = 0$$

$$\Rightarrow (B^T\,(A^2)^T - B^T\,\dot{A}^T - 2\dot{B}^T A^T + \ddot{B}^T)\,\lambda = 0 \ldots$$

- System (3) is **controllable** iff:

$$\text{rk}\,(B \mid AB - \dot{B} \mid A^2 B + \ldots \mid A^{n-1} B + \ldots \mid \ldots) = n.$$

Two pendula mounted on a car

- Let us consider the system:

$$\left\{\begin{array}{l} m_1\,L_1\,\ddot{w}_1+m_2\,L_2\,\ddot{w}_2-w_3+(M+m_1+m_2)\,\ddot{w}_4=0, \\ m_1\,L_1^2\,\ddot{w}_1-m_1\,L_1\,g\,w_1+m_1\,L_1\,\ddot{w}_4=0, \\ m_2\,L_2^2\,\ddot{w}_2-m_2\,L_2\,g\,w_2+m_2\,L_2\,\ddot{w}_4=0, \end{array}\right. \quad (\star).$$

- If $L_1 \neq L_2$, then:

$(\star)$ is **controllable** $\Leftrightarrow$ **parametrizable** $\Leftrightarrow$ **flat**.

- A **parametrization** of $(\star)$ is given by:

$$\left\{\begin{array}{ll} w_1 = & -L_2\,\xi^{(4)}+g\,\ddot{\xi}, \\ w_2 = & -L_1\,\xi^{(4)}+g\,\ddot{\xi}, \\ w_3 = & L_1\,L_2\,M\,\xi^{(6)} \\ & -(L_1\,m_2+L_2\,m_1+g\,(L_1+L_2)\,M)\,\xi^{(4)} \\ & +g^2\,(m_1+m_2+M)\,\xi^{(2)} \\ w_4 = & L_1\,L_2\,\xi^{(4)}-g\,(L_1+L_2)\,\ddot{\xi}+g^2\,\xi. \end{array}\right.$$

- A **flat output** of system $(\star)$ is defined by:

$$\xi=\frac{1}{g^2\,(L_1-L_2)}\,(L_1^2\,w_1-L_2^2\,w_2+(L_1-L_2)\,w_4).$$

- If the output y of the system $(\star)$ is the function ξ

$\Rightarrow$ **tracking problem**.

Genericity of the controllability

• Let us consider the following system

$$\ddot{y} + \alpha(t)\,\dot{y} + \dot{\alpha}(t)\,y = \ddot{u} - \beta\,u, \qquad (5)$$

where α, β are two **parameters** of the system.

• We are searching the **conditions on the parameters** α, β, so that System (5) **is controllable**.

• Let us consider the $D = K[\frac{d}{dt}]$-module defined by:

$$M = D^2/D\,\left(\frac{d^2}{dt^2} + \alpha(t)\,\frac{d}{dt} + \dot{\alpha}(t) : \; -\frac{d^2}{dt^2} + \beta\right).$$

• The D-module $N = D/D^2\widetilde{R}$ is defined by:

$$\begin{cases} \ddot{\lambda} - \alpha(t)\,\dot{\lambda} = 0, \\ -\ddot{\lambda} + \beta\,\lambda = 0. \end{cases}$$

• (5) is **controllable** iff M is **torsion-free**, i.e. $N = 0$

$\Rightarrow$ we need to **study the formal integrability** of:

$$\begin{cases} \ddot{\lambda} - \alpha(t)\,\dot{\lambda} = 0, \\ -\ddot{\lambda} + \beta\,\lambda = 0. \end{cases} \Leftrightarrow \begin{cases} \ddot{\lambda} - \beta\,\lambda = 0, \\ \alpha(t)\dot{\lambda} - \beta\lambda = 0. \end{cases}$$

1. If $\alpha(t) = 0$, then we have:

$$\begin{cases} \ddot{\lambda} - \beta\,\lambda = 0, \\ \beta\,\lambda = 0. \end{cases} \tag{6}$$

   (a) If $\beta = 0$, then $(6) \Leftrightarrow \ddot{\lambda} = 0 \Rightarrow N \neq 0$
   $\Rightarrow$ **system not controllable**.

   (b) If $\beta \neq 0$, then $(6) \Leftrightarrow \lambda = 0 \Rightarrow N = 0$
   $\Rightarrow$ **system controllable**.

2. If $\alpha(t) \neq 0$, then we have:

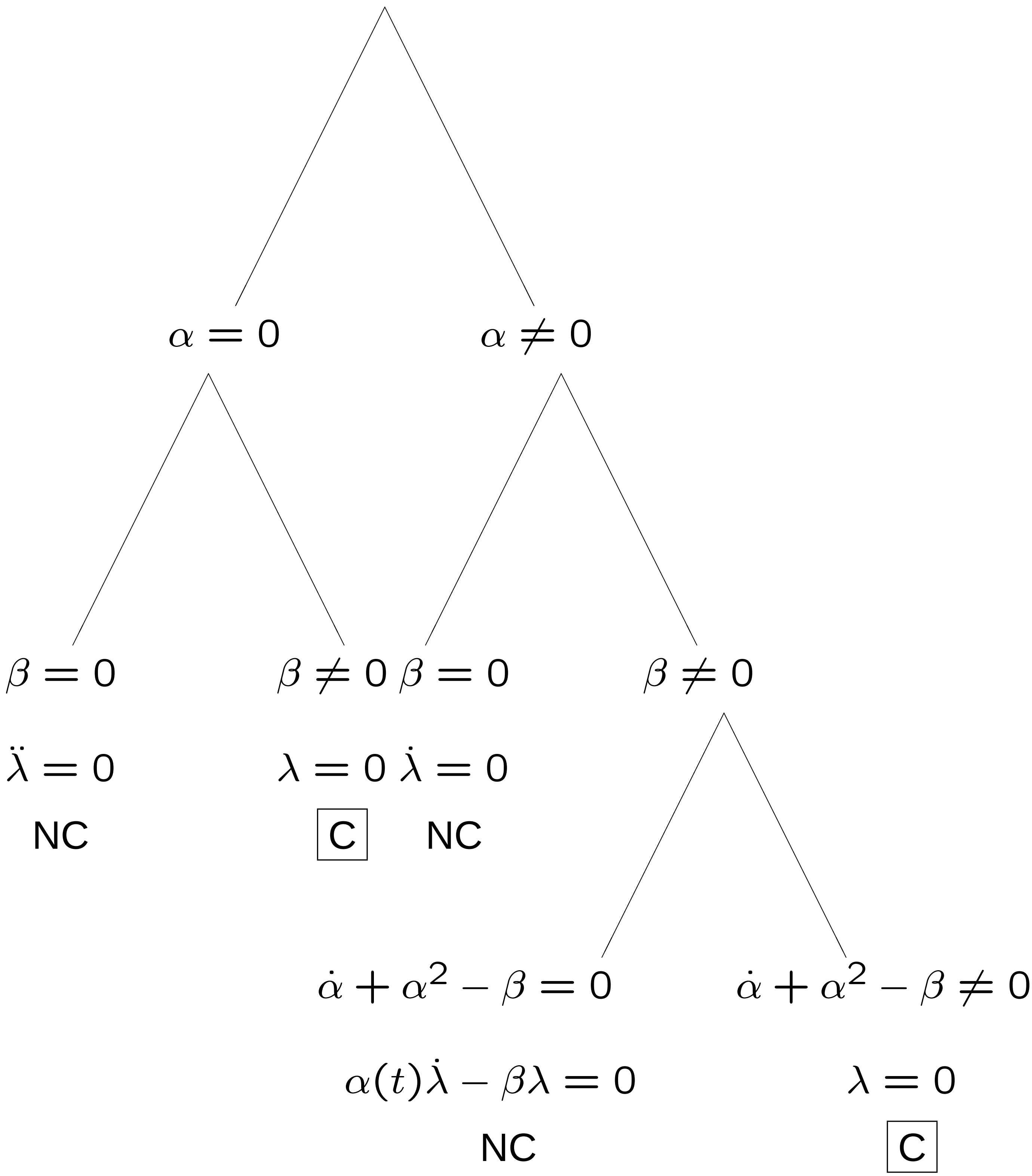
$$\begin{cases} \ddot{\lambda} - \beta\,\lambda = 0, \\ \alpha(t)\,\dot{\lambda} - \beta\,\lambda = 0, \\ \beta\,(\dot{\alpha}(t) + \alpha(t)^2 - \beta)\,\lambda = 0. \end{cases} \tag{7}$$

   (a) If $\beta = 0$, then $(7) \Leftrightarrow \dot{\lambda} = 0 \Rightarrow N \neq 0$
   $\Rightarrow$ **system not controllable**.

   (b) If $\beta \neq 0$, then:

      i. If $\dot{\alpha}(t) + \alpha(t)^2 - \beta = 0$, then $(7) \Leftrightarrow \alpha(t)\dot{\lambda} - \beta\lambda = 0 \Rightarrow N \neq 0$
      $\Rightarrow$ **system not controllable**.

      ii. If $\dot{\alpha}(t) + \alpha(t)^2 - \beta \neq 0$, then $(7) \Leftrightarrow \lambda = 0 \Rightarrow N = 0 \Rightarrow$ **system controllable**.



Tree of integrability conditions

C=controllable   NC=not controllable

Polynomial/rational/exponential solutions of underdetermined linear systems of ODE

- Let $D = K[\frac{d}{dt}]$ and $R \in D^{q\times p}$ be full row rank.

- **Theorem**: There exist $R' \in D^{q\times p}$, $R'' \in D^{q\times q}$, $R_{-1} \in D^{p\times(p-q)}$, $S \in D^{p\times q}$ and $S_{-1} \in D^{(p-q)\times p}$:

$$\left\{\begin{array}{l} R = R''\,R', \\ \begin{pmatrix} R' \\ S_{-1} \end{pmatrix} \begin{pmatrix} S & R_{-1} \end{pmatrix} = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}, \\ \begin{pmatrix} S & R_{-1} \end{pmatrix} \begin{pmatrix} R' \\ S_{-1} \end{pmatrix} = I_p, \end{array}\right. \quad (\star).$$

Moreover, we have:

$$\left\{\begin{array}{l} t(M) = D^q/D^q\,R'', \\ M/t(M) = D^p/D^q\,R', \end{array}\right. \quad \text{where } M = D^p/D^q\,R.$$

- **Algorithm:** solutions of $R\,y = 0$ in a D-module X (e.g. $X = k[x],\ k(x)\ldots$):

1. Compute the **Hermite (Smith) form** of $R \Rightarrow (\star)$.

2. Find a **basis of solutions in** X of $R''\,z = 0$ using the algorithms of Abramov, Barkatou, Bronstein ...

3. **All the solutions** of $R\,y = 0$ in X are given by:

$$y = S\,z + R_{-1}\,u,\ \forall\, u \in X^{p-q}.$$

Example

- Let us consider the following system

$$\ddot{y}(t) + \alpha(t)\,\dot{y}(t) + \dot{\alpha}(t)\,y(t) + \ddot{u}(t) - u(t) = 0,$$

where α is a nowhere zero function satisfying the Riccati equation $\dot{\alpha}(t) + \alpha(t)^2 - 1 = 0$.

- There exists an **autonomous (torsion) element**:

$$\begin{cases} z(t) = \dot{y}(t) - \alpha(t)\,y(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}\,(y(t) + u(t)) \\ \qquad\qquad + \dot{u}(t) - u(t), \\ \alpha(t)\,\dot{z}(t) + z(t) = 0. \end{cases}$$

- $\alpha(t)\,\dot{z}(t) + z(t) = 0 \Rightarrow z(t) = C\,e^{-\int_0^t \frac{ds}{\alpha(s)}}$, where $C \in \mathbb{R}$ is a constant.

$$\Rightarrow \dot{y}(t) - \alpha(t)\,y(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}\,(y(t) + u(t)) = C\,e^{-\int_0^t \frac{ds}{\alpha(s)}}.$$

- The **controllable part** of the system

$$\dot{y}(t) - \alpha(t)\,y(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}\,(y(t) + u(t)) = 0$$

admits the **parametrization**

$$\begin{cases} y(t) = \frac{1}{\alpha(t)}\,\dot{\xi}(t) - \frac{\dot{\alpha}(t)}{\alpha^2(t)}\,\xi(t), \\ u(t) = -\frac{1}{\alpha(t)}\,\dot{\xi}(t) + \frac{1}{\alpha^2(t)}\,\xi(t), \end{cases}$$

with $\xi(t) = y(t) + u(t)$.

Example

- The **differential operator**

$$\begin{pmatrix} y(t) \\ u(t) \end{pmatrix} \longrightarrow \dot{y}(t) - \alpha(t)\, y(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}\,(y(t)+u(t)) = z(t)$$

admits the following **right-inverse**:

$$z(t) \longrightarrow \begin{pmatrix} y(t) = -\frac{1}{\alpha(t)}\, z(t) \\ u(t) = \frac{1}{\alpha(t)}\, z(t) \end{pmatrix}.$$

- **A particular solution** of the system

$$\dot{y}(t) - \alpha(t)\, y(t) - \frac{\dot{\alpha}(t)}{\alpha(t)}\,(y(t)+u(t)) = C\, e^{-\int_0^t \frac{ds}{\alpha(s)}}$$

is defined by:

$$\begin{cases} \underline{y}(t) = -\frac{C}{\alpha(t)}\, e^{-\int_0^t \frac{ds}{\alpha(s)}}, \\ \underline{u}(t) = \frac{C}{\alpha(t)}\, e^{-\int_0^t \frac{ds}{\alpha(s)}}. \end{cases}$$

- **All the solutions** in a $K[\frac{d}{dt}]$-module S ($\ni \underline{y}, \underline{u}$) of

$$\ddot{y}(t) + \alpha(t)\, \dot{y}(t) + \dot{\alpha}(t)\, y(t) + \ddot{u}(t) - u(t) = 0,$$

have the following **form** with ξ is any element in S:

$$\begin{cases} y(t) = -\frac{C}{\alpha(t)}\, e^{-\int_0^t \frac{ds}{\alpha(s)}} + \frac{1}{\alpha(t)}\, \dot{\xi}(t) - \frac{\dot{\alpha}(t)}{\alpha^2(t)}\, \xi(t), \\ u(t) = \frac{C}{\alpha(t)}\, e^{-\int_0^t \frac{ds}{\alpha(s)}} - \frac{1}{\alpha(t)}\, \dot{\xi}(t) + \frac{1}{\alpha^2(t)}\, \xi(t). \end{cases}$$

First integrals of the motion

• Let us consider the **differential ring** $D = K[\frac{d}{dt}]$.

• If $R \in D^{q \times p}$, the **adjoint** $\widetilde{R} \in D^{p \times q}$ is defined by integrations by part:

$$< z,\ R\,y >=< \widetilde{R}\,z,\ y > + \frac{d}{dt}(\cdot) \qquad (\star).$$

• The D-module $M = D^p/D^q\,R$ is **torsion-free** iff:

$$\widetilde{N} = D^q/D^p\,\widetilde{R} = 0.$$

• Let us suppose that M is **not torsion-free**, then:

$$\widetilde{N} \neq 0 \Leftrightarrow (\widetilde{R}\,z = 0 \not\Rightarrow z = 0).$$

$\Rightarrow \widetilde{R}\,z = 0$ **admits a non-zero solution** $\overline{z}$.

• Let y be a solution of $R\,y = 0$, i.e. **an element of the system**, then, using $(\star)$, we have:

$$< \overline{z},\ R\,y >=< \widetilde{R}\,\overline{z},\ y > + \frac{d}{dt}(f(\overline{z}, \dot{\overline{z}}, \ldots \overline{z}^{(r)}, y, \dot{y}, \ldots, y^{(s)})) = 0,$$

$$\Rightarrow \frac{d}{dt}(f(\overline{z}, \dot{\overline{z}}, \ldots \overline{z}^{(r)}, y, \dot{y}, \ldots, y^{(s)})) = 0,$$

$$\Rightarrow f(\overline{z}, \dot{\overline{z}}, \ldots \overline{z}^{(r)}, y, \dot{y}, \ldots, y^{(s)})) = \text{cste},$$

is a **first integral of the motion**.

Example

- Let us consider the **differential ring** $D = k[\frac{d}{dt}]$.

- Let us consider the following system:

$$\begin{cases} \dot{x}_1 = x_2 + u, \\ \dot{x}_2 = x_1 - u. \end{cases}$$

- The system is **not controllable** because we have the **torsion element** (**non controllable element**):

$$\begin{cases} z = x_1 + x_2, \\ \dot{z} - z = 0. \end{cases}$$

- We have:

$$R = \begin{pmatrix} \frac{d}{dt} & -1 & -1 \\ -1 & \frac{d}{dt} & 1 \end{pmatrix}, \quad \widetilde{R} = \begin{pmatrix} -\frac{d}{dt} & -1 \\ -1 & -\frac{d}{dt} \\ -1 & 1 \end{pmatrix}.$$

$\Rightarrow \widetilde{N} = D^2/D^3\,\widetilde{R}$ is defined by:

$$\begin{cases} -\dot{\lambda}_1 - \lambda_2 = 0, \\ -\dot{\lambda}_2 - \lambda_1 = 0, \\ -\lambda_1 + \lambda_2 = 0, \end{cases} \Rightarrow \lambda_1(t) = \lambda_2(t) = c\,e^{-t}.$$

$$\Rightarrow \frac{d}{dt}(\lambda_1\,x_1 + \lambda_2\,x_2) = \frac{d}{dt}(c\,e^{-t}\,(x_1 + x_2)) = 0$$

$\Rightarrow Z(t) = c\,e^{-t}\,(x_1(t) + x_2(t))$ is a **first integral of the motion**.

Applications to optimal control

- **Problem**: Let us minimize the cost function

$$\frac{1}{2}\int_0^T (x(t)^2 + u(t)^2)\,dt$$

where $\dot{x}(t) + x(t) - u(t) = 0$, $x(0) = x_0$.

- $\dot{x}(t) + x(t) - u(t) = 0$ is **parametrized** by:

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases} \tag{8}$$

- By substitution of (8) in the cost, we are led to the following **variational problem without constraints**:

minimize $\frac{1}{2}\int_0^T (\xi(t)^2 + (\dot{\xi}(t) + \xi(t))^2)\,dt$ with:

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t). \end{cases}$$

We obtain the following system

$$\begin{cases} \xi(t) = x(t), \\ \dot{\xi}(t) + \xi(t) = u(t), \\ \ddot{\xi}(t) - 2\,\xi(t) = 0, \\ \dot{\xi}(T) + \xi(T) = 0, \\ \xi(0) = x_0, \end{cases}$$

which, by integrations, gives the **controller**:

$$u(t) = \frac{-e^{\sqrt{2}(t-T)} + e^{-\sqrt{2}(t-T)}}{(1-\sqrt{2})\,e^{\sqrt{2}(t-T)} - (1+\sqrt{2})\,e^{-\sqrt{2}(t-T)}}\,x(t).$$

nD linear control systems

Wind tunnel model

• Let us consider the wind tunnel model (Manitius 84):

$$\left\{\begin{array}{l} \dot{x}_1(t) = -a\, x_1(t) + k\, a\, x_2(t-h), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -\omega^2\, x_2(t) - 2\,\zeta\,\omega\, x_3(t) + \omega^2\, u(t). \end{array}\right. \qquad (\star)$$

• System $(\star)$ is **controllable** $\Leftrightarrow$ **parametrizable**:

$$\left\{\begin{array}{ll} x_1(t) = & -\omega^2\, k\, a\, \xi(t-h), \\ x_2(t) = & -\omega^2\, \dot{\xi}(t) + a\,\omega^2\, \xi(t), \\ x_3(t) = & \omega^2\, \ddot{\xi}(t) - \omega^2\, a\, \dot{\xi}(t), \\ u(t) = & -\xi(t)^{(3)} + (2\,\zeta\,\omega + a)\, \ddot{\xi}(t) \\ & -(\omega^2 + 2\,a\,\omega\,\zeta)\, \dot{\xi}(t) + a\,\omega\, \xi(t). \end{array}\right. \qquad (\star\star)$$

• System $(\star)$ **is not flat** but δ**-free** because:

$$\xi(t) = -\frac{1}{\omega^2\, k\, a}\, \delta^{-1}\, x_1(t) = -\frac{1}{\omega^2\, k\, a}\, x_1(t+h).$$

• If $y(t) = x_1(t)$ is the output of System $(\star)$, then we can solve the **tracking problem**:

$$y_r(t) = x_1(t) \Rightarrow \xi_r(t) = -\frac{1}{\omega^2\, k\, a}\, y_r(t+h)$$

$$\begin{array}{ll} \Rightarrow u_r(t) = & -\frac{1}{\omega^2\, k\, a}\, (-y_r(t+h)^{(3)} + (2\,\zeta\,\omega + a)\, \ddot{y}_r(t+h) \\ & -(\omega^2 + 2\,a\,\omega\,\zeta)\, \dot{y}_r(t+h) + a\,\omega\, y_r(t+h)). \end{array}$$

Differential time-delay system

• **Flexible rod** (Mounier, Fliess & co.):

$$\left\{\begin{array}{l} \frac{\partial^2 z}{\partial t^2}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \\ \frac{\partial z}{\partial x}(0,t) = -u(t), \\ \frac{\partial z}{\partial x}(1,t) = 0, \\ y_1(t) = z(0,t), \\ y_2(t) = z(1,t), \end{array}\right. \Rightarrow \left\{\begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,\dot{y}_1(t-1) - \dot{y}_2(t) - \dot{y}_2(t-2) = 0, \end{array}\right. \quad (\star).$$

• Let $D = \mathbb{R}[x_1, x_2]$ be the **commutative polynomial ring** with $x_1 = \frac{d}{dt}$, $x_2 = \delta$ ($\delta f(t) = f(t-1)$):

$$(\star) \Leftrightarrow \underbrace{\left(\begin{array}{ccc} x_1 & -x_1\,x_2 & -1 \\ 2\,x_1\,x_2 & -x_1\,x_2^2 - x_1 & 0 \end{array}\right)}_{R} \left(\begin{array}{c} y_1 \\ y_2 \\ u \end{array}\right) = 0.$$

1. $M = D^3/D^2\,R$ is not a torsion-free D-module $\Rightarrow$ **the system is not controllable**:

$$\left\{\begin{array}{l} \theta(t) = 2\,y_1(t-1) - y_2(t) - y_2(t-1), \\ \dot{\theta}(t) = 0. \end{array}\right.$$

2. **We have** $M/t(M) = D^3/D^2\,R'$ where:

$$R' = \left(\begin{array}{ccc} x_1 & -x_1\,x_2 & -1 \\ 2\,x_2 & -x_2^2 - 1 & 0 \end{array}\right) \in D^{2\times 3}.$$

3. $M/t(M)$ is a **free D-module** (**$\Rightarrow$ flatness**):

$$\left\{\begin{array}{l} \dot{y}_1(t) - \dot{y}_2(t-1) - u(t) = 0, \\ 2\,y_1(t) - y_2(t) - y_2(t-2) = 0, \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} y_1(t) = \xi(t) + \xi(t-2), \\ y_2(t) = 2\,\xi(t-1), \\ u(t) = \dot{\xi}(t) - \dot{\xi}(t-2), \end{array}\right.$$

with $\xi(t) = y_1(t) - \frac{1}{2}\,y_2(t-1)$.

Poles placement

- Let us consider the system:

$$D\left(\frac{d}{dt}, \underline{\delta}\right)\, y(t) = N\left(\frac{d}{dt}, \underline{\delta}\right)\, u(t) \quad (1).$$

- Let us consider the following feedback law:

$$A\left(\frac{d}{dt}, \underline{\delta}\right)\, u(t) = B\left(\frac{d}{dt}, \underline{\delta}\right)\, y(t) \quad (2)$$

- If System (1) is **parametrizable**, then:

$$(1) \Leftrightarrow \left(\begin{array}{c} y(t) \\ u(t) \end{array}\right) = \left(\begin{array}{c} \widetilde{N}\left(\frac{d}{dt}, \underline{\delta}\right) \\ \widetilde{D}\left(\frac{d}{dt}, \underline{\delta}\right) \end{array}\right) \xi(t) \quad (4).$$

- The **closed-loop dynamic** is given by:

$$(B\, \widetilde{N} - A\, \widetilde{D})\, \xi(t) = 0.$$

- **Proposition**: Let us consider a dynamic S. Then, there exists a feedback law (2) satisfying

$$(B : \, -A) \left(\begin{array}{c} \widetilde{N} \\ \widetilde{D} \end{array}\right) = S \quad (5)$$

iff $S_i \in k[\frac{d}{dt}, \underline{\delta}]^{m+n} \left(\begin{array}{c} \widetilde{N} \\ \widetilde{D} \end{array}\right)$, where $S = \left(\begin{array}{c} S_1 \\ \vdots \\ S_{m+n} \end{array}\right)$.

$\Rightarrow$ (2) can be computed by means of **Gröbner bases**.

- If (1) is a **flat system**, then (5) is always feasible:

$$(B : \, -A) = S\, (-\widetilde{Y} : \, \widetilde{X}) + Q\, (D : \, -N), \;\; \forall\, Q,$$

$$\text{with} \quad \xi(t) = -\widetilde{Y}\, y(t) + \widetilde{X}\, u(t).$$

OreModules

• ***OreModules*** is a tool-box developed in *Maple*.

• ***OreModules*** uses *Mgfun* developed by F. Chyzak (INRIA Rocquencourt, ALGO):

http://algo.inria.fr/chyzak/mgfun.html.

• ***OreModules*** is developed by Chyzak-Q.-Robertz.

• ***Oremodules*** can handle linear systems of ODEs, PDEs, differential time-delay systems, multidimensional discrete systems...

• ***OreModules*** computes:

1. autonomous elements, non-controllable elements,

2. parametrizations of under-determined systems,

3. left-right-generalized inverses,

4. flat outputs of a flat system,

5. polynomial, rational or exponential first integrals of the motion...

• A **first release is available** on the web page:

http://wwwb.math.rwth-aachen.de/OreModules

Extended Bézout Identities

• A **multidimensional system** is defined by means of a matrix R in the ring $D = k[\chi_1, \ldots, \chi_n]$ of polynomials in χ_i with coefficients in $k = \mathbb{R}, \mathbb{C}$.

• It is known since the works of Youla (1979) that the **primeness** of a multidimensional system, defined by a full row rank matrix R, **is linked with extended Bézout identities**, namely the existence of a matrix S and $\pi \in D$ such that:

$$R\,S = \pi\,I.$$

• **Example**: If R is **zero left prime**, i.e. there exists no common zero in all the minors of R, then $\pi = 1$.

• **Example**: If R is **minor left prime**, i.e. there exists no common factor in all the minors of R, then π contains $n-1$ variables χ_i.

• Recently, **the introduction of algebraic analysis** (Oberst, Pommaret...) has allowed to develop new powerful results on multidimensional systems.

The aim of this talk is to study the extended Bézout identities in the algebraic analysis framework.

$\Rightarrow$ We introduce the new concept of **torsion-free degree** and we show how to pass from one torsion-free degree to another by inverting a polynomial $\pi \in D$.

Torsion-free degree

• **Definition**: Let M be a finitely generated D-module and $N = T(M)$ its transposed. The **torsion-free degree** of M is the number defined by:

$$i(M) = \min_{k \geq 1} \{\, k - 1 \mid \ \mathrm{ext}^k_D(N, D) \neq 0 \,\}.$$

$$\begin{cases} t(M) \neq 0 \Leftrightarrow i(M) = 0, \\ \\ M \text{ is torsion-free } \Leftrightarrow i(M) = 1, \\ \\ M \text{ is reflexive } \Leftrightarrow i(M) = 2, \\ \dots \\ M \text{ is projective } \Leftrightarrow i(M) = +\infty. \end{cases}$$

Let S_n be the group of permutations of n elements.

• **Theorem**: Let M be a finitely generated D-module, $\sigma \in S_n$ and:

$$\begin{cases} D^\sigma_{n-i(M)} = k[\chi_{\sigma(1)}, \dots, \chi_{\sigma(n-i(M))}], \\ \qquad\qquad\qquad\qquad\qquad 0 \leq i(M) \leq n-1, \\ D^\sigma_{-\infty} = k, \ \ i(M) = +\infty. \end{cases}$$

Then, $\forall\, k \geq 0$, $\exists\, \pi^\sigma_{n-i(M)} \in D^\sigma_{n-i(M)}$ such that:

$$i(D_{\pi^\sigma_{n-i(M)}} \otimes_D M) \ \geq \ i(M) + k,$$

with $S_{\pi^\sigma_{n-i(M)}} = \{1, \pi^\sigma_{n-i(M)}, (\pi^\sigma_{n-i(M)})^2, \dots\}$ and:

$$D_{\pi^\sigma_{n-i(M)}} = \left\{ \frac{P}{Q} \ \mid \ P \in D, \ Q \in S_{\pi^\sigma_{n-i(M)}} \right\}.$$

Algorithm

1. **Start** with the D-module $M = D^{l_0}/D^{l_1}\, R_1$.

2. **Define its tranposed** D-module $N = D^{l_1}/D^{l_0}\, R_1^T$.

3. **Compute a free resolution of** N.

4. **Compute** $\text{ext}^i_D(N, D)$ for $i \geq 1$.

5. **Compute the torsion-free degree** $i(M)$ of M.

5. **For** $i(M) + 1 \leq j \leq i(M) + k$**, compute**:

$$I^{\sigma j}_{n-i(M)} = \text{ann}(\text{ext}^j_D(N, D)) \cap k[\chi_{\sigma(1)}, \ldots, \chi_{\sigma(n-i(M))}].$$

6. **For** $i(M) + 1 \leq j \leq i(M) + k$**, choose**

$$\pi^{\sigma\, j}_{n-i(M)} \in I^{\sigma j}_{n-i(M)}$$

and **define:**

$$\pi^{\sigma}_{n-i(M)} = \prod_{\{i(M)+1 \leq j \leq i(M)+k,\ \pi^{\sigma\, j}_{n-i(M)} \neq 0\}} \pi^{\sigma\, j}_{n-i(M)}.$$

Example

• The $D = k[\chi_1, \chi_2, \chi_3]$-module $M = D^3 / D^3\, R_1$ defined by the **curl operator** has the free resolution

$$0 \longrightarrow D \xrightarrow{.R_2} D^3 \xrightarrow{.R_1} D^3 \longrightarrow M \longrightarrow 0,$$

where $R_2 = (\chi_1 \,:\, \chi_2 \,:\, \chi_3)$ is the divergence.

• The D-module $N = T(M)$ is defined by:

$$0 \longleftarrow N \longleftarrow D^3 \xleftarrow{.R_1^T} D^3 \xleftarrow{.R_2} D \longleftarrow 0.$$

• $\begin{cases} \mathrm{ext}^1_D(N, D) = 0, \\ \mathrm{ext}^2_D(N, D) = D/D^3 R_2^T, \\ \mathrm{ext}^j_D(N, D) = 0, \ \ \forall\, j \geq 3. \end{cases}$

$$\Rightarrow i(M) = 2 - 1 = 1 \Rightarrow 3 - i(M) = 2.$$

• $\mathrm{ext}^2_D(N, D)$ is defined by the equations

$$\chi_1\, z = 0, \ \chi_2\, z = 0, \ \chi_3\, z = 0.$$

$$\begin{aligned} \Rightarrow I_2^{\sigma\,2} \ &= \mathrm{ann}(\mathrm{ext}^2_D(N, D)) \cap k[\chi_{\sigma(1)}, \chi_{\sigma(2)}] \\ &= (\chi_{\sigma(1)}, \chi_{\sigma(2)}), \ \forall\, \sigma \in S_3. \end{aligned}$$

$\Rightarrow$ the $D_{\pi_2^\sigma}$-module $D_{\pi_2^\sigma} \otimes_D M$ is a free $D_{\pi_2^\sigma}$-module, where $\pi_2^\sigma = \chi_{\sigma(1)}$, $S_{\pi_2^\sigma} = \{1, \pi_2^\sigma, (\pi_2^\sigma)^2, \ldots\}$ and:

$$D_{\pi_2^\sigma} = \left\{ \frac{P}{Q} \ | \ P \in D, \ Q \in S_{\pi_2^\sigma} \right\}.$$

$$\Rightarrow y_{\sigma(i)} = \left(\frac{\chi_{\sigma(i)}}{\chi_{\sigma(1)}} \right) y_{\sigma(1)}, \ \ i = 2, 3.$$

More precise results

- **Theorem**: Let M be a finitely generated D-module, $N = T(M)$, $\sigma \in S_n$ and:

$$h(M) = i(M) + i(N) \in \{0, \ldots, n-1, +\infty\}.$$

Then, $\forall\, k \geq 0$, $\exists\, \pi^{\sigma}_{n-h(M)} \in D^{\sigma}_{n-h(M)}$ such that

$$i(D_{\pi^{\sigma}_{n-h(M)}} \otimes_D M) \;\geq\; i(M) + k,$$

with $S_{\pi^{\sigma}_{n-h(M)}} = \{1, \pi^{\sigma}_{n-h(M)}, (\pi^{\sigma}_{n-h(M)})^2, \ldots\}$ and:

$$D_{\pi^{\sigma}_{n-h(M)}} = \left\{ \frac{P}{Q} \;\middle|\; P \in D,\; Q \in S_{\pi^{\sigma}_{n-h(M)}} \right\}.$$

- **Example**: Let $M = D^3/D^3\, R_1$ be the D-module defined by the curl operator and $N = D^3/D^3\, R_1^T = M$.

$$\left\{ \begin{array}{l} i(M) = 1, \\ \\ i(N) = 1, \end{array} \right. \Rightarrow h(M) = 2 \Rightarrow 3 - h(M) = 1,$$

$\Rightarrow \forall\, \sigma \in S_3$, $\exists\, \pi^{\sigma}_1$ such that $D_{\pi^{\sigma}_1} \otimes_D M$ is free.

- **Theorem**: Let $R \in D^{l_1 \times l_0}$ ($1 \leq l_1 \leq l_0$) be **a full rank matrix** and $M = D^{l_0}/D^{l_1}\, R$. Then, we have:

$$h(M) = i(M) = j(N) - 1 = n - d(N) - 1,$$

$$\Rightarrow n - h(M) = \left\{ \begin{array}{ll} d(N) + 1, & N \neq 0, \\ -\infty, & N = 0. \end{array} \right.$$

Extended Bézout identities

• **Theorem**: Let $R \in D^{q \times p}$ ($1 \leq q \leq p$) be a full rank matrix, $M = D^p / D^q\, R$ and $N = T(M)$.

Then, for all $\sigma \in S_n$, there exist

$$\left\{ \begin{array}{l} \pi^{\sigma}_{d(N)+1} \in D^{\sigma}_{d(N)+1}, \\ R_{-1} \in D^{p \times (p-q)}, \\ S \in D^{p \times q}, \\ S_{-1} \in D^{(p-q) \times p}, \\ \nu \in \mathbb{Z}_+, \end{array} \right.$$

such that we have the **extended Bézout identities**:

- $$\left(\begin{array}{cc} S & R_{-1} \end{array} \right) \left(\begin{array}{c} R \\ S_{-1} \end{array} \right) = (\pi^{\sigma}_{d(N)+1})^{\nu}\, I_p,$$

- $$\left(\begin{array}{c} R \\ S_{-1} \end{array} \right) \left(\begin{array}{cc} S & R_{-1} \end{array} \right) = (\pi^{\sigma}_{d(N)+1})^{\nu} \left(\begin{array}{cc} I_q & 0 \\ 0 & I_{p-q} \end{array} \right).$$

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