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Generalized Bezout Identity ⋆

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Abstract. We describe a new approach of the generalized Bezout identity for linear time-varying ordinary differential control systems. We also explain when and how it can be extended to linear partial differential control systems. We show that it only depends on the algebraic nature of the differential module determined by the equations of the system. This formulation shows that the generalized Bezout identity is equivalent to the splitting of an exact differential sequence formed by the control system and its parametrization. This point of view gives a new algebraic and geometric interpretation of the entries of the generalized Bezout identity.

Keywords: Generalized Bezout identity, Controllability, Parametrization, Janet sequence, Formal integrability, D-module, Commutative algebra.

1 Introduction

Let us denote $s = \frac{d}{dt}$, $\mathbb{R}[s]$ the polynomial algebra in s and M_{mp} the set of $m \times p$ matrices with entries in $\mathbb{R}[s]$. It is well known that if

$$P(s)y + Q(s)u = 0, \tag{1}$$

is a left-coprime polynomial system, i.e. controllable [2, 9], where $P \in M_{mm}$, $\det P(s) \neq 0$ and $Q \in M_{mp}$, then we can find four polynomial matrices $X \in M_{mm}$, $\overline{X}$, $Y \in M_{pm}$, $\overline{P} \in M_{mp}$, $\overline{Y}$, $\overline{Q} \in M_{pp}$ such that

$$\begin{bmatrix} P(s) & Q(s) \\ \overline{X}(s) & \overline{Y}(s) \end{bmatrix} \begin{bmatrix} X(s) & \overline{P}(s) \\ Y(s) & \overline{Q}(s) \end{bmatrix} = I, \tag{2}$$

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where I is the $(m+p)\times(m+p)$ identity matrix. This identity, generally called *generalized Bezout identity*, is useful in control theory [16, 39, 42].

Recently, it has been shown in [8, 9, 10, 19, 24, 27, 29] that controllability of a control system was a "built-in" property of the system and thus did not depend on a separation of the system variables between inputs and outputs. So, we are led to revisit the generalized Bezout identity with a more intrinsic point of view. For controllable *surjective* linear time-varying control system, the generalized Bezout identity is reformulated in terms of *the splitting of a short exact differential sequence* formed by the system and its parametrization. Moreover, in [11, 22, 27, 29, 30], the algebraic and geometric concepts of ordinary differential control theory (OD control theory) have been extended within the framework of partial differential control theory (PD control theory) or delay control theory, that is, linear or nonlinear input/output relations defined by systems of partial differential equations or differential and delay equations. See also [19, 24] for a n-dimensional control systems theory. Then, we can wonder if such a generalized Bezout identity exists for PD control systems. However, the existence of the generalized Bezout identity for (1) is deeply based on Bezout theorem which is not true in general for multivariable polynomial algebra. So (2) does not seem to have a generalization for PD control systems. We shall show that its existence only depends on the algebraic nature of the differential module determined by the equations of the system. Such a generalized Bezout identity exists for a *surjective* linear PD control system generating *a free* differential module. In this case, the generalized Bezout identity can be reformulated in terms of the splitting of a short exact differential sequence made by the system and its parametrization. In case the differential module is no longer free but *projective*, then only the upper part of (2) is satisfied, or in other words, the system admits a *parametrization* and a *right-inverse*. Finally, if the system is controllable, i.e. if it generates *a torsion-free* differential module, we only have the right upper part of (2), that is, the system admits a parametrization. Tests are known for checking whether a finitely generated differential module is torsion-free, projective or free [29, 30, 40, 41]. Thus for linear PD control systems, we are able to know which parts of the generalized Bezout identity exist and to compute them. Moreover, the extension of the generalized Bezout identity in the case of non surjective linear OD and PD control systems is obtained. In this case, we have to build and split a long exact differential sequence. Many explicit examples will illustrate the main results.

2 Controllability

The use of the module language for control systems was initiated by Kalman thirty years ago [17] and it took a new insight with Blomberg and Ylinen [2]. Recently, its use seems to have given new results on structural properties of the system like controllability, observability, poles and zeros, motion planning... See for example [5, 8, 9, 10, 22, 24, 27, 29]. We recall a few results.

Definition 1 A *differential field* K with n commuting derivations $\partial_1, \ldots, \partial_n$ is a field which satisfies $\forall\, a, b \in K,\ \forall\, i, j = 1, \ldots, n$:

- $\partial_i\, a \in K$,
- $\partial_i (a + b) = \partial_i\, a + \partial_i\, b$,
- $\partial_i (ab) = (\partial_i\, a)\, b + a\, \partial_i\, b$,
- $\partial_i\, \partial_j = \partial_j\, \partial_i$.

See [20, 27, 36] for more details.

In the course of the text, we will always consider differential field K containing $\mathbb{Q}$. We form the ring of linear differential operators with coefficients in K and denote it by $D = K[d_1, \ldots, d_n]$. Any element of D has the form $\sum_{\text{finite}} a^\mu\, d_\mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with length $\mid \mu \mid = \mu_1 + \ldots + \mu_n$ and $a^\mu \in K$. D is a non commutative integral domain which satisfies

$$\forall\, a, b \in K\, :\, a d_i\, (b\, d_j) = ab\, d_i\, d_j + a\, (\partial_i\, b)\, d_j,$$

and possesses the Ore property: $\forall\, (p, q) \in D^2,\ \exists\, (u, v) \in D^2$ such that $u\, p = v\, q$.

Example 1 The field of rational functions $\mathbb{R}(t)$ is a differential field with derivative $\dfrac{d}{dt}$. Indeed, $\forall\, a(t), 0 \neq b(t) \in \mathbb{R}(t)$, we have:

$$\frac{d}{dt}\left(\frac{a(t)}{b(t)}\right) = \frac{\dot{a}(t)\, b(t) - a(t)\, \dot{b}(t)}{b^2(t)} \in \mathbb{R}(t).$$

When $D = \mathbb{R}(t)\left[\frac{d}{dt}\right]$ is the ring of linear operators with coefficients in $\mathbb{R}(t)$, every element $p \in D$ has the form $p = \sum_{\text{finite}} a_i(t) \left(\frac{d}{dt}\right)^i$, with $a_i \in \mathbb{R}(t)$.

We introduce the differential indeterminates $y = \{y^k \mid k = 1, \ldots, m\}$ and denote by $Dy = Dy^1 + \ldots + Dy^m$ or by $[y] = [y^1, ..., y^m]$ the left D-module spanned by the set y. Every element of Dy has the form $\sum_{\text{finite}} a_k^\mu\, d_\mu\, y^k$. If we have a finite set $\mathscr{R}$ of linear OD or PD equations (ODE or PDE), we form the finitely generated left D- module $[\mathscr{R}]$ of linear differential consequences of the system generators and the differential residual D-module $M = [y]/[\mathscr{R}]$. See [1, 21, 26] for much details on D-modules.

Remark 1. In the examples, we shall use either the language of jet theory for systems of PDE or the language of section for operators [27]. In the first case, we have $d_i\, y_\mu^k = y_{\mu+1_i}^k$ while in the second case d_i must be replaced by ∂_i on sections.

Definition 2 We call *observable* any element of M, or in other words, any linear combination of the system variables and their derivatives satisfying the equations of the control system.

Only two possibilities may happen for an observable: it may or may not verify an OD or a PD equation by itself. An observable which does not satisfy any OD or PD equation is called *free*. We find in [27] the following definition of controllability.

Definition 3 A system is controllable if every observable is free.

A characterization of the controllability in terms of *differential closure* is shown in [27]. In [8, 9, 19, 22, 24, 27], the equivalent notion of *torsion-free* D-module has been used for linear time-varying OD, delay, n-dimensional and PD control systems. We recall the definition.

Definition 4 A *torsion* element m of a $D-$module M is an element which satisfies $\exists\ 0 \neq a \in D$ such that $a\, m = 0$ [37]. We denote by $t(M)$ the submodule of M made by all the torsion elements of M. A module is *torsion-free* if $t(M) = 0$.

From Definition 3 and Definition 4, we obtain the following theorem.

Theorem 1 *A linear OD or PD system is controllable iff the D-module M determined by its equations is torsion-free. In any case, $M/t(M)$ is a torsion-free module, a result leading to the concept of minimal realization [12, 19, 24, 27].*

Let us give an illustrating example.

Example 2 We take $D = \mathbb{R}\left[\frac{d}{dt}\right]$ and let M be the residual D-module of $D\, y^1 + D\, y^2$ with respect to the D-submodule generated by $\ddot{y}^1 + y^1 + y^1 - y^2 + \alpha\, y^3,\ \ddot{y}^2 + y^2 - y^1 - y^3$. In the language of systems of ODE (see remark 1), we have the two equations:

$$\begin{cases} \ddot{y}^1 + y^1 - y^2 + \alpha\, y^3 = 0, \\ \ddot{y}^2 + y^2 - y^1 - y^3 = 0. \end{cases} \tag{3}$$

- For $\alpha = -1$, if we substract the first equation from the second of (3) and set $z = y^1 - y^2$, we find $\ddot{z} + 2z = 0$. The image of z in M is a torsion element.
- For $\alpha = 1$, if we add the first equation to the second of (3) and set $z = y^1 + y^2$, we find $\ddot{z} = 0$ and thus the image of z in M is a torsion element.

We recall two other definitions of module properties which will be at the core of this paper (see [37] for more details).

Definition 5 1. A D-module M is *free* if there is a set of elements which generate M and are independent on D. 2. A D-module M is *projective* if there exists a free D-module F and a D-module N such that $F \cong M \oplus N$. Hence, the module N is also a projective D-module.

Remark 2. It is quite easy to verify that a free D-module is a projective D-module and that a projective D-module is a torsion-free D-module, which can be summed up by the following module inclusions:

$$\text{free} \subseteq \text{projective} \subseteq \text{torsion-free}.$$

Moreover, any submodule of a free D-module is a torsion-free D-module. We will see that the reciprocity is true and we will describe a way to construct a free D-module containing a given torsion-free D-module (see the *Torsion-free Test* described in the section *Formal Tests* and p. 18 of [18]).

In [10, 22], the basis of a free D-module determined by a control system is called *flat output* or *linearizing output* and plays an important role for the motion planning. We have the useful theorem [37].

Theorem 2 *1. If D is a principal ideal ring (for example $K[\frac{d}{dt}]$) the D-module M is torsion-free iff M is* free*. 2. Over a polynomial ring $k[\chi_1, \ldots, \chi_n]$, where k is a field, any projective module is also a free module.*

The last part of the previous theorem has been conjecturated in 1950 by Serre and demonstrated independently in 1976 by Quillen and Suslin [37, 41]. We can find in [22, 40, 41] tests permitting to know if a finitely generated $k[d_1, ..., d_n]-$module M, with k a field of constants (i.e. $\forall a \in k : \forall i = 1, ..., n, \ \partial_i a = 0$), is respectively torsion-free, projective and free (see [25] for more deeper results). Remark that in this case, we can use the Quillen-Suslin theorem and any projective module is a free module. Recently, formal tests have been found in [27, 30] permitting to treat the more general situation of $D = K[d_1, ..., d_n]$ where K is a differential field with subfield of constants k (for example $D = \mathbb{R}(x^1, ..., x^n)[d_1, ..., d_n]$). We will recall these tests.

3 Linear Differential Operators

From a geometric point of view, a linear PD control system with n derivatives may be defined as a linear PD operator $\mathscr{D}_1 : F_0 \to F_1$ where F_0, F_1 are vector bundles over a manifold X of dimension n, with local coordinates $x = (x^1, ..., x^n)$. In other words, $\mathscr{D}_1$ is a PD linear operator acting on sections of F_0, i.e. acting on functions $\eta : X \to F_0$. We define its sheaf of solutions by $\mathscr{D}_1 \eta = 0$ (see remark 1).

Remark 3. A fundamental idea is to associate to each operator $\mathscr{D}_1 : \eta \to \zeta$ the D-module $M = [\eta]/[\mathscr{D}_1\eta]$ and we shall say that the operator $\mathscr{D}_1$ determines the D-module M.

Example 3 Let us take the operator $\mathscr{D}_1 : \eta \to \zeta$ defined by:

$$\begin{cases} \ddot{\eta}^1 + \eta^1 - \eta^2 + \alpha\,\eta^3 = \zeta^1, \\ \ddot{\eta}^2 + \eta^2 - \eta^1 - \eta^3 = \zeta^2. \end{cases}$$

The operator $\mathscr{D}_1$ determined the $D = \mathbb{R}\left[\frac{d}{dt}\right]$-module $M = [\eta^1, \eta^2, \eta^3]/[\ddot{\eta}^1 + \eta^1 - \eta^2 + \alpha\,\eta^3, \ddot{\eta}^2 + \eta^2 - \eta^1 - \eta^3]$.

By an abuse of language, we shall say that an operator $\mathscr{D}_1$ is controllable if the D-module M determined by $\mathscr{D}_1$ is torsion-free.

Definition 6 1. An operator $\mathscr{D}_1$ is *formally injective* if $\mathscr{D}_1\,\eta = 0 \Rightarrow \eta = 0$. 2. The operator $\mathscr{D}_1$ is *formally surjective* if the equations $\mathscr{D}_1\eta = 0$ are differentially independent (see [20, 36]), i.e. independent on D, or equivalently if $\mathscr{D}_1\,\eta = \zeta$ has no compatibility conditions, that is, if there does not exist an operator $\mathscr{D}_2$ such that $\mathscr{D}_1\eta = \zeta \Rightarrow \mathscr{D}_2\,\zeta = 0$.

In the course of the text, we shall say injective (resp. surjective) operator for formally injective (resp. formally surjective) operator. Moreover, a control system defined by $\mathscr{D}_1$ will be called *surjective* (resp. *injective*) if $\mathscr{D}_1$ is a surjective (resp. injective) operator.

Example 4

- The operator $\mathscr{D}_1 : \eta \to \zeta$ defined by

$$\begin{cases} x^2\partial_1\,\eta + \eta = \zeta^1, \\ -\partial_2\,\eta = \zeta^2, \end{cases} \tag{4}$$

  where (x^1, x^2) are local coordinates on X, is an injective operator as we may easily verified that $\eta = -(x^2)^2\partial_1\,\zeta^2 - x^2\partial_2\,\zeta^1 + \zeta^1 - x^2\zeta^2$. Thus, $(\zeta^1, \zeta^2) = (0, 0) \Rightarrow \eta = 0$.
- We take the *Spencer operator* (see [34] for more details) $\mathscr{D}_1 : \eta \to \zeta$ defined by:

$$\begin{cases} \partial_1\,\eta^1 - \eta^2 = \zeta^1, \\ \partial_2\,\eta^1 - \eta^3 = \zeta^2, \\ \partial_2\,\eta^2 - \partial_1\,\eta^3 = \zeta^3. \end{cases} \tag{5}$$

  It is not a surjective operator. Indeed, if differentiating ζ^1 with respect to ∂_2 and ζ^2 to ∂_1 and substracting them, we find $\partial_1\,\zeta^2 - \partial_2\,\zeta^1 - \zeta^3 = 0$. The operator $\mathscr{D}_2 : \zeta \to \chi$, defined by the compatibility condition $\partial_1\,\zeta^2 - \partial_2\,\zeta^1 - \zeta^3 = \chi$ of $\mathscr{D}_1$, is surjective because it has only one equation.

- The operator $\mathscr{D}_1 : (y, u)^t \to \zeta$ defined by

$$P(s)\, y + Q(s)\, u = \zeta,$$

with det $P(s) \neq 0$ is a surjective operator.

We recall that a differential sequence of operators $\{\mathscr{D}_i,\ i = 0, \ldots, l\}$ is *locally exact* if ker $\mathscr{D}_{i+1}$ = im $\mathscr{D}_i$. We have in particular $\forall\, i = 0, \ldots, l$: $\mathscr{D}_{i+1} \circ \mathscr{D}_i = 0$. A differential sequence is called *formally exact* if each operator generates all the compatibility conditions of the preceding one. Such a situation is met in particular if all the corresponding sequences on the jet level at any order are exact [27, 34], but the converse may not be true (see example 12). An injective operator $\mathscr{D}$ will be denoted by the following formally exact differential sequence $0 \longrightarrow E \stackrel{\mathscr{D}}{\longrightarrow} F$, whereas the formally exact differential sequence $E \stackrel{\mathscr{D}}{\longrightarrow} F \longrightarrow 0$ will mean that $\mathscr{D}$ is a surjective operator.

Definition 7 The formally exact sequence $0 \longrightarrow E \stackrel{\mathscr{D}_0}{\longrightarrow} F_0 \stackrel{\mathscr{D}_1}{\longrightarrow} F_1 \longrightarrow 0$ is said to be a *split exact differential sequence* if we have one of the following equivalent properties [37]:

- there exists an operator $\mathscr{P}_1 : F_1 \longrightarrow F_0$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$,
- there exists an operator $\mathscr{P}_0 : F_0 \longrightarrow E$ such that $\mathscr{P}_0 \circ \mathscr{D}_0 = id_E$,
- $F_0 \cong E \oplus F_1$ (on the level of sections).

A system of partial differential equations $\mathscr{D}_1 \eta = 0$ is said to be *formally integrable* whenever the formal power series of the solutions can be determined step by step by successive derivations without obtaining backwards new informations on lower-order derivatives [27, 32, 34]. For a sufficiently regular operator $\mathscr{D}_1$, we are always able to add to its equations new equations, made by differential consequences of the given ones, in order to have a formally integrable system with an *involutive* symbol (see [27, 32, 34] for more details). Such an operator is called *involutive*. In the course of the text, we shall always suppose that these regularity conditions are satisfied. We can find in [38] a symbolic package which completes a system of PDE to an involutive one, using algorithms based on the formal integrability theory or some other packages, based on Janet-Riquier theory [15, 35], can be found in [13, 14, 33]. See also the symbolic packages in [3, 6, 7] using the effective methods of differential algebra [20, 36]. Now, if $\mathscr{D}_1$ is an involutive operator, then the sequence starting with $\mathscr{D}_1$ and in which each operator exactly describes the compatibility conditions of the preceding one, is finite and stops after at most $n + 1$ operators where n is the dimension of X. This sequence

$$F_0 \stackrel{\mathscr{D}_1}{\longrightarrow} F_1 \stackrel{\mathscr{D}_2}{\longrightarrow} \ \ldots\ \stackrel{\mathscr{D}_n}{\longrightarrow} F_n \stackrel{\mathscr{D}_{n+1}}{\longrightarrow} F_{n+1} \longrightarrow 0,$$

is formally exact and it is usually called the *Janet sequence* of $\mathscr{D}_1$ [27].

Remark 4. As both the initial (not involutive) and final (involutive) operators have the same solutions, if the original operator $\mathscr{D}_1$ is injective, then the final equations contain the zero order equation $\eta = 0$ which can therefore be a differential consequence of the initial equations. Thus, we have obtained a *left-inverse* $\eta = \mathscr{P}_1 \zeta$ which is however not uniquely determined. Indeed, we can take $\mathscr{P}'_1 = \mathscr{P}_1 + \mathscr{Q} \circ \mathscr{D}_2$ with $\mathscr{P}_1 \circ \mathscr{D}_1 = id_{F_1}$ and $\mathscr{Q} : F_2 \to F_0$ any operator and we easily verify that $\mathscr{P}'_1 \circ \mathscr{D}_1 = \mathscr{P}_1 \circ \mathscr{D}_1 = id_{F_1}$ ($\mathscr{D}_2 \circ \mathscr{D}_1 = 0$).

We recall the duality for differential operators [27, 29]. We denote by E a vector bundle over a manifold X, $T^\star$ the cotangent bundle of X, $E^\star$ the dual bundle of E and $\tilde{E} = \wedge^n T^\star \otimes E^\star$ its adjoint bundle. This is the right generalization of the concept of tensor density in physics.

Definition 8 If $\mathscr{D}_1 : F_0 \to F_1$ is a linear differential operator, its formal adjoint $\tilde{\mathscr{D}}_1 : \tilde{F}_1 \to \tilde{F}_0$ is defined by the following formal rules equivalent to the integration by parts:

- the adjoint of a matrix (zero order operator) is the transposed matrix,
- the adjoint of ∂_i is $-\partial_i$,
- for two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We easily verified that $\tilde{\tilde{\mathscr{D}}}_1 = \mathscr{D}_1$. It can be proved that, for any section λ of $\tilde{F}_1$, we have the relation

$$< \lambda,\ \mathscr{D}_1 \eta > - < \tilde{\mathscr{D}}_1 \lambda,\ \eta >= d(\cdot),$$

expressing a difference of n-forms ($\lambda \in \wedge^n T^\star \otimes F_1^\star \Rightarrow < \lambda,\ \mathscr{D}_1 \eta > \in \wedge^n T^\star$), where d is the standard exterior derivative. Equivalently, we can directly compute the adjoint of an operator by multiplying it by test functions on the left and integrating by parts.

Example 5 We compute the adjoint operator of the Spencer operator (5). We multiply $\mathscr{D}_1 \eta$ on the left by a row vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and integrate the result by parts, we obtain the operator $\tilde{\mathscr{D}}_1 : \lambda \to \mu$ defined by:

$$\left\{ \begin{array}{l} -\partial_1\lambda_1 - \partial_2\lambda_2 = \mu_1, \\ -\partial_2\lambda_3 - \lambda_1 = \mu_2, \\ \partial_1\lambda_3 - \lambda_2 = \mu_3. \end{array} \right. \tag{6}$$

Definition 9 We call an operator $\mathscr{D}_1$ *parametrizable* if there exists a set of arbitrary functions $\xi = (\xi^1, \ldots, \xi^r)$ or "*potentials*" and a linear operator $\mathscr{D}_0$ such that all the compatibility conditions of the inhomogenous system $\mathscr{D}_0 \xi = \eta$ are *exactly* generated by $\mathscr{D}_1 \eta = 0$, i.e., if the sequence $E \xrightarrow{\mathscr{D}_0} F_0 \xrightarrow{\mathscr{D}_1} F_1$ is formally exact.

4 Formal Tests

We can find in [22, 40, 41] tests to know whether a finitely generated $k[\chi_1, ..., \chi_n]$-module M (k is a field) has torsion elements or if it is respectively torsion-free or free. We give formal tests which can check those module properties over rings D of the form $D = K[d_1, ..., d_n]$ where K is a differential field containing $\mathbb{Q}$ (for example $\mathbb{R}(x^1, ..., x^n)[d_1, ..., d_n]$). All the calculations can be effectively performed by means of existing symbolic packages.

4.1 Torsion-free D-modules

We describe a formal test checking if the operator $\mathscr{D}_1$ determines a torsion-free D-module M or not (compare with p. 18 of [18]):

Torsion-free Test

1. Start with $\mathscr{D}_1$.
2. Construct its adjoint $\tilde{\mathscr{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathscr{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathscr{D}}_0$.
4. Construct its adjoint $\mathscr{D}_0$ $(= \tilde{\tilde{\mathscr{D}}}_0)$.
5. Find the compatibility conditions of $\mathscr{D}_0\,\xi = \eta$ and call this operator $\mathscr{D}'_1$.

We are led to two different cases. If $\mathscr{D}_1$ is exactly the compatibility conditions $\mathscr{D}'_1$ of $\mathscr{D}_0$, then the system $\mathscr{D}_1$ determines a torsion-free D-module M and $\mathscr{D}_0$ is a parametrization of $\mathscr{D}_1$ in sense of the definition 9. Otherwise, the operator $\mathscr{D}_1$ is among, but not exactly, the compatibility conditions of $\mathscr{D}_0$. The torsion elements of M are all the new compatibility conditions modulo the equations $\mathscr{D}_1\eta = 0$.

Proof. The operator $\tilde{\mathscr{D}}_0$ describes exactly the compatibility conditions of the operator $\tilde{\mathscr{D}}_1$ and we have in particular $\tilde{\mathscr{D}}_0 \circ \tilde{\mathscr{D}}_1 = 0 \Rightarrow \mathscr{D}_1 \circ \mathscr{D}_0 = 0$. Hence, $\mathscr{D}_1$ is among the compatibility conditions of $\mathscr{D}_0$, which are described by the operator $\mathscr{D}'_1$. Now, computing the rank of the operators $\mathscr{D}'_1$ and $\mathscr{D}_1$, we find that rank $\mathscr{D}'_1$ = rank $\mathscr{D}_1$ (see [30] for more details). If $\mathscr{D}_1$ is strictly among the compatibility conditions of $\mathscr{D}_0$, then any new single compatibility condition ζ' in $\mathscr{D}'_1$ is a differential consequence of $\mathscr{D}_1$ (rank $\mathscr{D}'_1$ = rank $\mathscr{D}_1$) and we can find an operator $q \in D$ such that $q\,\zeta' = 0$ whenever $\mathscr{D}_1\,\eta = 0$. Hence, any new single compatibility condition of $\mathscr{D}_0$ (not in $\mathscr{D}_1$) determines a torsion element. If $\mathscr{D}_1$ describes exactly the compatibility conditions of $\mathscr{D}_0$, i.e. the sequence $E \xrightarrow{\mathscr{D}_0} F_0 \xrightarrow{\mathscr{D}_1} F_1$ is formally exact, then the D-module M determined by $\mathscr{D}_1$ is torsion-free because $M \subseteq D\,\xi$ and $D\,\xi$ is a free D-module (see remark 2).

We can represent the test by the following differential sequences where the number indicates the different stages:

$$\begin{array}{ccccc} & & 5 & & \\ & & \xrightarrow{\mathscr{D}_1'} & F_1', & \\ E & \xrightarrow{\mathscr{D}_0} & F_0 \xrightarrow{\mathscr{D}_1} & F_1, & \\ & 4 & & 1 & \\ \tilde{E} & \xleftarrow{\tilde{\mathscr{D}}_0} & \tilde{F}_0 \xleftarrow{\tilde{\mathscr{D}}_1} & \tilde{F}_1. & \\ & 3 & & 2 & \end{array}$$

In the preceding sequences, only the dual sequence and the sequence made with $\mathscr{D}_0$ and $\mathscr{D}_1'$ are formally exact. Thus, the defect of controllability of the operator $\mathscr{D}_1$ may be seen as a defect of the formal exactness of the upper sequence formed by $\mathscr{D}_0$ and $\mathscr{D}_1$. This fact will lead to a future introduction of the functor Ext (see [4, 18, 25, 37]) in control theory, organized around the following two central results obtained one from the other by reversing the arrows:

1. *Controllability* of $\mathscr{D}_1$ amounts to the cancellation of the first extension of the D-module determined by $\tilde{\mathscr{D}}_1$ with value in the ring D of differential operators.
2. *Obervability* of $\mathscr{D}_1$ amounts to the cancellation of $\text{ext}^1(M, D)$, the first extension of the D-module M determined by $\mathscr{D}_1$ with value in the ring D of differential operators.

In this framework, an operator is naturally controllable (observable) iff its formal adjoint is observable (controllable).

Using theorem 1, we obtain the following useful corollary.

Theorem 3 *A linear PD control system is controllable iff it is parametrizable.*

Proof. The operator $\mathscr{D}_1$ is controllable iff it determines a torsion-free D-module. By the previous test, $\mathscr{D}_1$ determines a torsion-free D-module iff there exists an operator $\mathscr{D}_0 : E \to F_0$ parametrizing $\mathscr{D}_1$, i.e. the sequence $E \xrightarrow{\mathscr{D}_0} F_0 \xrightarrow{\mathscr{D}_1} F_1$ is formally exact.

We now illustrate the test by an example.

Example 6 We wonder if the Spencer operator (5) determines a torsion-free D-module M (see example 3). The adjoint operator of the Spencer operator is (6). Differentiating the second equation of $\tilde{\mathscr{D}}_1$ with respect to ∂_1, the third with respect to ∂_2 and adding them, we obtain the operator $\tilde{\mathscr{D}}_0 : \mu \to \nu$ defined by $-\partial_1\mu_2 - \partial_2\mu_3 + \mu_1 = \nu$. We multiply $\tilde{\mathscr{D}}_0$ by ξ and after one integration by parts, we obtain the operator $\mathscr{D}_0 : \xi \to \eta$ defined by:

$$\begin{cases} \xi = \eta^1, \\ \partial_1 \xi = \eta^2, \\ \partial_2 \xi = \eta^3. \end{cases} \tag{7}$$

We find the compatibility conditions of $\mathscr{D}_0$ by differentiating the second equation by ∂_2, the third by ∂_1 and subtracting them, we obtain the third equation of $\mathscr{D}_1$. Differentiating the first equation of $\mathscr{D}_0$ by respectively ∂_1 and ∂_2 and subtracting it by respectively the second and the third equation, we obtain the first and the second equation of $\mathscr{D}_1$. Thus, all the compatibility conditions of $\mathscr{D}_0$ are exactly generated by $\mathscr{D}_1$ and the Spencer operator determines a torsion-free D-module M.

We now describe how to compute the torsion elements if $\mathscr{D}_1$ does not determine a torsion-free D-module M.

Computation of torsion elements

1. Compute $\mathscr{D}'_1$ and check that $\mathscr{D}_1$ is strictly among $\mathscr{D}'_1$.
2. For any new single compatibility condition of the form $\mathscr{D}'_1\eta = \zeta'$ of $\mathscr{D}'_1$, compute the compatibility conditions of the following system:

$$\begin{cases} \mathscr{D}_1 \, \eta = 0, \\ \mathscr{D}'_1 \, \eta = \zeta' \text{ (one equation only)}. \end{cases}$$

3. We find that ζ' is a torsion element of M satisfying $q\,\zeta' = 0$ with $0 \neq q \in D$.

We now give a theoretical but non-trivial example of a computation of a torsion element.

Example 7 We consider the system

$$\ddot{\eta}^2 + \alpha(t)\,\dot{\eta}^2 + \dot{\alpha}(t)\,\eta^2 + \ddot{\eta}^1 - \eta^1 = 0,$$

where $\alpha(t)$ is a non zero function satisfying $\dot{\alpha}(t) + \alpha(t)^2 - 1 = 0$. See [31] for the general situation. We let the reader check that the operator $\mathscr{D}'_1 : \eta \to \zeta'$ is:

$$\dot{\eta}^2 + \dot{\eta}^1 - \alpha(t)\,\eta^1 - \frac{\dot{\alpha}(t)}{\alpha(t)}(\eta^2 + \eta^1) = \zeta'$$

(be careful, the adjoint of $\alpha(t)\,\dot{y}$ is $-\alpha(t)\,\dot{\lambda} - \dot{\alpha}(t)\,\lambda$). The compatibility condition of $\mathscr{D}_0$ is not the operator $\mathscr{D}_1$ and thus the system is not controllable. If we want to find the torsion element of the associated D-module M, we only have to compute the compatibility conditions of the system:

$$\begin{cases} \ddot{\eta}^2 + \alpha(t)\,\dot{\eta}^2 + \dot{\alpha}(t)\,\eta^2 + \ddot{\eta}^1 - \eta^1 = 0, \\ \dot{\eta}^2 + \dot{\eta}^1 - \alpha(t)\,\eta^1 - \frac{\dot{\alpha}(t)}{\alpha(t)}(\eta^2 + \eta^1) = \zeta'. \end{cases}$$

After straightforward but tedious computations, we find that the torsion element ζ' satisfies $\alpha(t)\dot{\zeta}' + \zeta' = 0$.

4.2 Projective D-modules

Let $\mathscr{D}_1$ be a surjective operator with an injective adjoint $\tilde{\mathscr{D}}_1$. As $\tilde{\mathscr{D}}_1$ is an injective operator, among the differential consequences of the equations $\tilde{\mathscr{D}}_1 \lambda = \mu$, we must find $\lambda = \tilde{\mathscr{P}}_1 \mu$ (see remark 4). A natural way to compute $\tilde{\mathscr{P}}_1$ is to bring $\tilde{\mathscr{D}}_1$ to formal integrability, that is, roughly speaking to saturate the system by lower order consequences of the equations [27]. Thus, bringing $\tilde{\mathscr{D}}_1$ to formal integrability, we form an operator $\tilde{\mathscr{P}}_1$ satisfying $\tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 = id_{\tilde{F}_1}$ where $id_{\tilde{F}_1}$ is the identity operator of $\tilde{F}_1$. The operator $\tilde{\mathscr{P}}_1$ is then a left-inverse of $\tilde{\mathscr{D}}_1$. Dualizing $\tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 = id_{\tilde{F}_1}$, we obtain $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$ or in other words, $\mathscr{D}_1$ admits a right-inverse $\mathscr{P}_1$. It is equivalent to say that the D-module M, determined by the surjective operator $\mathscr{D}_1$, is a projective D-module [22, 30, 41].

Theorem 4 *A surjective differential operator $\mathscr{D}_1 : F_0 \to F_1$ determines a projective D-module iff its adjoint is injective, i.e., if there exists $\mathscr{P}_1 : F_1 \to F_0$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$.*

We shall represent the operator $\mathscr{P}_1 : F_1 \to F_0$ by the following upper left arrow:

$$\begin{array}{c} \overset{\mathscr{P}_1}{\longleftarrow} \\ F_0 \overset{\mathscr{D}_1}{\longrightarrow} F_1 \longrightarrow 0. \end{array}$$

Example 8 To illustrate what has been said, we show that the system

$$\partial_2 \eta^2 - x^2 \partial_1 \eta^1 + \eta^1 = 0, \qquad (8)$$

where (x^1, x^2) are local coordinates on X, determines a projective D-module and we find a right-inverse. Its adjoint $\tilde{\mathscr{D}}_1 : \lambda \to \mu$ is just (4), i.e.,

$$\begin{cases} x^2 \partial_1 \lambda + \lambda = \mu_1, \\ -\partial_2 \lambda = \mu_2. \end{cases}$$

Bringing this system to formal integrability, we obtain by derivating the first equation with respect to ∂_2 and the second by ∂_1, the new lower order equation:

$$\lambda = -(x^2)^2 \partial_1 \mu_2 - x^2 \partial_2 \mu_1 + \mu_1 - x^2 \mu_2.$$

Thus $\tilde{\mathscr{D}}_1$ is an injective operator and $\mathscr{D}_1$ determines a projective D-module. If we denote by $\tilde{\mathscr{P}}_1 : \mu \to \lambda$ the operator defined by $-(x^2)^2 \partial_1 \mu_2 - x^2 \partial_2 \mu_1 + \mu_1 - x^2 \mu_2 = \lambda$ then $\tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 = id_{\tilde{F}_1}$, and its adjoint $\mathscr{P}_1 : \zeta \to \eta$, given by

$$\begin{cases} x^2 \partial_2 \zeta + 2\zeta = \eta^1, \\ (x^2)^2 \partial_1 \zeta - x^2 \zeta = \eta^2, \end{cases}$$

is a right-inverse of (8). Indeed, we easily verify that $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$.

In the general case, $\mathscr{D}_1$ is no longer a surjective operator and a characterization of projective module in the language of operator can be found in [4, 23, 30]. We recall it.

Theorem 5 *An operator $\mathscr{D}_1 : F_0 \to F_1$ determines a projective D-module if there exists an operator $\mathscr{P}_1 : F_1 \to F_0$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1$. The operator $\mathscr{P}_1$ is then called a lift-operator.*

Proof. As $\mathscr{D}_1$ is not a surjective operator, there exists an operator $\mathscr{D}_2 : F_1 \to F_2$ describing the compatibility conditions of $\mathscr{D}_1$. The operator $\mathscr{D}_1$ defines a projective D-module M iff there exists an operator $\mathscr{P}_1 : F_1 \longrightarrow F_0$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{\mathrm{im}\,\mathscr{D}_1} = id_{F_1}$ modulo $\mathscr{D}_2$, i.e., if there exists $\mathscr{P}_2 : F_2 \to F_1$ such that $\mathscr{D}_1 \circ \mathscr{P}_1 + \mathscr{P}_2 \circ \mathscr{D}_2 = id_{F_1}$. However, $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{\mathrm{im}\,\mathscr{D}_1}$ is equivalent to:

$$\mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1. \tag{9}$$

Indeed, the direct way is trivial whereas the reciprocity can be demonstrated as follows. From (9), we have $(id_{F_1} - \mathscr{D}_1 \circ \mathscr{P}_1) \circ \mathscr{D}_1 = 0$ and thus $id_{F_1} - \mathscr{D}_1 \circ \mathscr{P}_1$ must factorize through $\mathscr{D}_2$ (see p. 150 of [28]), that is to say, there exists an operator $\mathscr{P}_2$ such that

$$\mathscr{D}_1 \circ \mathscr{P}_1 + \mathscr{P}_2 \circ \mathscr{D}_2 = id_{F_1}, \tag{10}$$

which proves the inverse way.

Moreover, the identity (10) implies $\mathscr{D}_2 \circ \mathscr{P}_2 \circ \mathscr{D}_2 = \mathscr{D}_2$ and $\mathscr{D}_2$ defines, in its turn, a projective D-module. In a similar way, all the successive operators of compatibility conditions define a projective D-module. Now, if we dualize (9), we obtain $\tilde{\mathscr{D}}_1 \circ \tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 = \tilde{\mathscr{D}}_1$ and thus $\tilde{\mathscr{D}}_1$ defines a projective D-module. Moreover, we have $\tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 + \tilde{\mathscr{D}}_2 \circ \tilde{\mathscr{P}}_2 = id_{\tilde{F}_1}$ and $\tilde{\mathscr{D}}_1 \circ \tilde{\mathscr{D}}_2 = 0$. The first identity shows that $\mathrm{im}\,\tilde{\mathscr{D}}_2 \subseteq \ker \tilde{\mathscr{D}}_1$ whereas if we take $\lambda \in \ker \tilde{\mathscr{D}}_1$, the second shows that $\tilde{\mathscr{D}}_2\,(\tilde{\mathscr{P}}_2\lambda) = \lambda$ and thus $\lambda \in \mathrm{im}\,\tilde{\mathscr{D}}_2 \Rightarrow \mathrm{im}\,\tilde{\mathscr{D}}_2 = \ker \tilde{\mathscr{D}}_1$. Hence, we have the following locally exact sequence:

$$\tilde{F}_0 \xleftarrow{\tilde{\mathscr{D}}_1} \tilde{F}_1 \xleftarrow{\tilde{\mathscr{D}}_2} \tilde{F}_2.$$

For a non surjective operator $\mathscr{D}_1$, we give a test checking whether the module M determines by the operator $\mathscr{D}_1$ is a projective D-module or not [30].

Projective Test

1. Construct the Janet sequence starting with $\mathscr{D}_1$.
2. Check if the adjoint of the last operator of the sequence is injective.
3. Check if the backward sequence, made with the adjoint of the operators of the Janet sequence of $\mathscr{D}_1$, is a formally exact sequence.

Example 9 The Spencer operator $\mathscr{D}_1$ is not a surjective operator as we have seen in the example 4. The operator $\mathscr{D}_2 : \zeta \rightarrow \chi$ defining the compatibility conditions of $\mathscr{D}_1$ is

$$\partial_1 \zeta^2 - \partial_2 \zeta^1 - \zeta^3 = \chi, \tag{11}$$

and it is surjective. Dualizing the operator $\mathscr{D}_2$ by multiplying it by β and integrating the result by parts, we obtain the injective operator $\tilde{\mathscr{D}}_2 : \beta \rightarrow \lambda$ defined by:

$$\begin{cases} \partial_2 \beta = \lambda_1, \\ -\partial_1 \beta = \lambda_2, \\ -\beta = \lambda_3. \end{cases}$$

Thus, we only have to verify that all the compatibility conditions of the operator $\tilde{\mathscr{D}}_2$ are exactly defined by the operator $\tilde{\mathscr{D}}_1$. Up to a change of sign, it is the same as verifying that all the compatibility conditions of $\mathscr{D}_0$ are defined by $\mathscr{D}_1$ (see the example 6). We conclude that the Spencer operator determines a projective D-module M. As the Spencer operator is a PD system with constant coefficients, then according to the theorem of Quillen-Suslin, it determines a free D-module. Indeed, the D-module M determined by the Spencer operator is equal to the module $D\,\xi = D\,\eta^1$ which is a free D-module (see the parametrization (7) of $\mathscr{D}_1$).

Let us describe now how to compute the lift-operators $\mathscr{P}_i$. Let $\mathscr{D}_1$ be an operator defining a projective D-module. Thus, we have the two following locally exact sequences:

$$\begin{array}{l} F_0 \xrightarrow{\mathscr{D}_1} F_1 \xrightarrow{\mathscr{D}_2} \ldots \xrightarrow{\mathscr{D}_n} F_n \xrightarrow{\mathscr{D}_{n+1}} F_{n+1} \longrightarrow 0, \\ \tilde{F}_0 \xleftarrow{\tilde{\mathscr{D}}_1} \tilde{F}_1 \xleftarrow{\tilde{\mathscr{D}}_2} \ldots \xleftarrow{\tilde{\mathscr{D}}_n} \tilde{F}_n \xleftarrow{\tilde{\mathscr{D}}_{n+1}} \tilde{F}_{n+1} \longleftarrow 0. \end{array}$$

As $\mathscr{D}_{n+1}$ is a surjective operator with an injective adjoint $\tilde{\mathscr{D}}_{n+1}$, there exists an operator $\mathscr{P}_{n+1} : F_{n+1} \longrightarrow F_n$ such that $\mathscr{D}_{n+1} \circ \mathscr{P}_{n+1} = id_{F_{n+1}} \Rightarrow \mathscr{D}_{n+1} \circ \mathscr{P}_{n+1} \circ \mathscr{D}_{n+1} = \mathscr{D}_{n+1}$. Let us denote $\mathscr{Q}_n = id_{F_n} - \mathscr{P}_{n+1} \circ \mathscr{D}_{n+1}$. We have $\mathscr{D}_{n+1} \circ \mathscr{Q}_n = \mathscr{D}_{n+1} - \mathscr{D}_{n+1} \circ \mathscr{P}_{n+1} \circ \mathscr{D}_{n+1} = 0$ and thus $\tilde{\mathscr{Q}}_n \circ \tilde{\mathscr{D}}_{n+1} = 0$. However, we have $\tilde{\mathscr{D}}_n \circ \tilde{\mathscr{D}}_{n+1} = 0$ which implies that $\tilde{\mathscr{Q}}_n$ factorizes through $\tilde{\mathscr{D}}_n$: $\tilde{\mathscr{Q}}_n = \tilde{\mathscr{P}}_n \circ \tilde{\mathscr{D}}_n \Rightarrow \mathscr{Q}_n = \mathscr{D}_n \circ \mathscr{P}_n \Rightarrow \mathscr{D}_n \circ \mathscr{P}_n + \mathscr{P}_{n+1} \circ \mathscr{D}_{n+1} = id_{F_n} \Rightarrow \mathscr{D}_n \circ \mathscr{P}_n \circ \mathscr{D}_n = \mathscr{D}_n$. In a similar way, we can find $\mathscr{P}_i$ for $i \in \{1, \ldots, n\}$ satisfying $\mathscr{D}_i \circ \mathscr{P}_i \circ \mathscr{D}_i = \mathscr{D}_i$. Hence, the lift-operator $\mathscr{P}_{i-1}$ can be computed as follows:

Computation of the lift-operators $\mathscr{P}_i$

1. Compute an operator $\tilde{\mathscr{P}}_{n+1}$ such that $\tilde{\mathscr{P}}_{n+1} \circ \tilde{\mathscr{D}}_{n+1} = id_{\tilde{F}_{n+1}}$ and take its adjoint $\mathscr{P}_{n+1}$.

For $i = n + 1, \ldots, 2$:

2. Compute $\mathscr{Q}_{i-1} = id_{F_{i-1}} - \mathscr{P}_i \circ \mathscr{D}_i$ and $\tilde{\mathscr{Q}}_{i-1}$.
3. As before, $\tilde{\mathscr{Q}}_{i-1}$ must factorize through $\tilde{\mathscr{D}}_{i-1}$ and we find $\tilde{\mathscr{P}}_{i-1}$ such that $\tilde{\mathscr{Q}}_{i-1} = \tilde{\mathscr{P}}_{i-1} \circ \tilde{\mathscr{D}}_{i-1}$ and dualizing, we have $\mathscr{P}_{i-1}$.

Example 10 We have seen that the Spencer operator defined a projective D-module. We show how to compute $\mathscr{P}_1$. We easily find that $\mathscr{P}_2 : \chi \to \zeta$ defined by

$$\begin{cases} 0 = \zeta^1, \\ 0 = \zeta^2, \\ -\chi = \zeta^3, \end{cases} \tag{12}$$

is a right-inverse of $\mathscr{D}_2$. We start by defining $Q_1 = id_{F_1} - \mathscr{P}_2 \circ \mathscr{D}_2$. The operator $Q_1 : \zeta \to \gamma$ is thus:

$$\begin{cases} \zeta^1 = \gamma^1, \\ \zeta^2 = \gamma^2, \\ \partial_1 \zeta^2 - \partial_2 \zeta^1 = \gamma^3. \end{cases}$$

Taking its adjoint, we obtain $\tilde{Q}_1 : \lambda \to \phi$

$$\begin{cases} \partial_2 \lambda_3 + \lambda_1 = \phi_1, \\ -\partial_1 \lambda_3 + \lambda_2 = \phi_2, \\ 0 = \phi_3, \end{cases}$$

whereas $\tilde{\mathscr{D}}_1$ is given by (6). We easily find that $\tilde{\mathscr{P}}_1$ is defined by

$$\begin{cases} -\mu_2 = \phi_1, \\ -\mu_3 = \phi_2, \\ 0 = \phi_3, \end{cases}$$

and we have the operator $\mathscr{P}_1 : \zeta \to \eta$ given by:

$$\begin{cases} 0 = \eta^1, \\ -\zeta^1 = \eta^2, \\ -\zeta^2 = \eta^3. \end{cases} \tag{13}$$

We let the reader check that $\mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1$.

4.3 Free D-modules

We have seen in 1. of definition 8 that for a principal ring D (for example $K[\frac{d}{dt}]$), a torsion-free D-module is a free D-module. Hence, we state a very useful theorem [27, 29].

Theorem 6 *A surjective linear time-varying OD control system is controllable iff its adjoint is injective.*

Proof. In a principal ring, the notions of torsion-free and projective module are equivalent. Thus, a linear OD control system is controllable iff the D-module M is projective. The operator $\mathscr{D}_1$ is surjective and its adjoint $\tilde{\mathscr{D}}_1$ is injective, then $\mathscr{D}_1$ determines a projective D-module M and the system is controllable. Conversely, if $\tilde{\mathscr{D}}_1$ is not injective then we can find a test vector $\lambda \neq 0$ which satisfies $\tilde{\mathscr{D}}_1 \lambda = 0$. Thus $< \lambda, \mathscr{D}_1 \eta >$ is a total derivative of an observable which is therefore a torsion element, as its derivative is null as soon as η is a solution of the system and the system is not controllable.

Remark 5. Even in the case of time-varying system, we deduce that there is a bijective correspondence between torsion elements and first integrals of motions.

Example 11 We take again the first example. Multiplying it by a row vector $\lambda = (\lambda_1, \lambda_2)$ and integrating the result by parts, we obtain $\tilde{\mathscr{D}}_1 : \lambda \to \mu$ defined by:

$$\begin{cases} \ddot{\lambda}_1 + \lambda_1 - \lambda_2 = \mu_1, \\ \ddot{\lambda}_2 + \lambda_2 - \lambda_1 = \mu_2, \\ -\lambda_2 + \alpha\, \lambda_1 = \mu_3. \end{cases}$$

Differentiating twice the zero-order equation and substituting it, we obtain

$$(\alpha + 1)(\alpha - 1)\lambda_1 = 0,$$

and the operator $\tilde{\mathscr{D}}_1$ is injective i.e. controllable iff $\alpha \neq -1$ and $\alpha \neq 1$.

Theorem 7 *An operator $\mathscr{D}_1$ determines a free D-module M iff there exists an injective parametrization of $\mathscr{D}_1$.*

Proof. Let $\mathscr{D}_0 \xi = \eta$ be a parametrization of $\mathscr{D}_1 \eta = \zeta$ then we have $M \subseteq D\, \xi$. Now, if $\mathscr{D}_0 \xi = \eta$ is an injective parametrization of $\mathscr{D}_1$, then there exists a left-inverse $\mathscr{P}_0$ of $\mathscr{D}_0$ such that $\xi = \mathscr{P}_0 \circ \mathscr{D}_0 \xi \Leftrightarrow \xi = \mathscr{P}_0\, \eta \Rightarrow D\, \xi \subseteq \mathscr{M}$. Thus, $M = D\, \xi$ and M is a free D-module. The reciprocity is obvious.

Example 12 The operator $\mathscr{D}_2 : \zeta \to \chi$, generating the compatibility condition of the Spencer operator, is defined by $\partial_1 \zeta^2 - \partial_2 \zeta^1 - \zeta^3 = \chi$. We know that the operator $\mathscr{D}_2$ determines a free D-module (see example 9). However, the operator $\mathscr{D}_1$ is a parametrization of $\mathscr{D}_2$ which is not injective. We have in (6) the relation $\mu_1 = \partial_1 \mu_2 + \partial_2 \mu_3$ and if we take only the second and the third equations of $\tilde{\mathscr{D}}_1$ as a new operator, we easily see that its adjoint $\mathscr{D}_1^\sharp : \theta \to \zeta$, defined by

$$\begin{cases} -\theta^1 = \zeta^1, \\ -\theta^2 = \zeta^2, \\ -\partial_1\, \theta^2 + \partial_2\, \theta^1 = \zeta^3, \end{cases}$$

is an injective parametrization of $\mathscr{D}_2$.

5 Split Exact Sequences

5.1 Main Result

The following theorem is at the core of the generalization of the generalized Bezout identity for non surjective operators. It shows how we can construct the lift-operators in order to split a long formally exact sequence of differential operators.

Theorem 8 *Let $\mathscr{D}_1 : F_0 \longrightarrow F_1$ be an operator determining a projective D-module and let*

$$F_0 \xrightarrow{\mathscr{D}_1} F_1 \xrightarrow{\mathscr{D}_2} \dots \xrightarrow{\mathscr{D}_n} F_n \xrightarrow{\mathscr{D}_{n+1}} F_{n+1} \longrightarrow 0,$$

be its Janet sequence. Then there exists lift-operators $\mathscr{P}_i : F_i \to F_{i-1}$ such that the sequence

$$F_0 \xleftarrow{\mathscr{P}_1} F_1 \xleftarrow{\mathscr{P}_2} \dots \xleftarrow{\mathscr{P}_n} F_n \xleftarrow{\mathscr{P}_{n+1}} F_{n+1} \longleftarrow 0,$$

is locally exact and $\forall\, i = 1...n+1 : \mathscr{P}_i \circ \mathscr{D}_i \circ \mathscr{P}_i = \mathscr{P}_i$.

Proof. Let $\mathscr{D}_1 : F_0 \longrightarrow F_1$ be an operator determining a projective D-module M and let

$$F_0 \xrightarrow{\mathscr{D}_1} F_1 \xrightarrow{\mathscr{D}_2} \dots \xrightarrow{\mathscr{D}_n} F_n \xrightarrow{\mathscr{D}_{n+1}} F_{n+1} \longrightarrow 0,$$

be the Janet sequence of $\mathscr{D}_1$. Let us suppose that we have found operators $\mathscr{P}_i$ such that $\mathscr{D}_i \circ \mathscr{P}_i \circ \mathscr{D}_i = \mathscr{D}_i$ for $i = 1 \dots n$ and $\mathscr{D}_{n+1} \circ \mathscr{P}_{n+1} = id_{F_{n+1}}$. Let us focus, only for the moment, on the locally exact differential sequence $F_{i-1} \xrightarrow{\mathscr{D}_i} F_i \xrightarrow{\mathscr{D}_{i+1}} F_{i+1}$ with $\mathscr{D}_{i+1} \circ \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1} = \mathscr{D}_{i+1}$ and $\mathscr{D}_i \circ \mathscr{P}_i \circ \mathscr{D}_i = \mathscr{D}_i$. We have $\forall\, \eta \in F_i : \mathscr{D}_{i+1} \circ (id_{F_i} - \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1})\, \eta = 0 \Rightarrow \exists\, \xi \in F_{i-1} : (id_{F_i} - \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1})\, \eta = \mathscr{D}_i\, \xi$ as the sequence formed by $\mathscr{D}_i$ and $\mathscr{D}_{i+1}$ is locally exact. However, we have $\forall\, \xi \in F_{i-1} : (\mathscr{D}_i \circ \mathscr{P}_i - id_{F_i}) \circ \mathscr{D}_i\, \xi = 0 \Rightarrow (\mathscr{D}_i \circ \mathscr{P}_i - id_{F_i}) \circ (id_{F_i} - \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1})\, \eta = 0, \forall\, \eta \in F_i$. Finally, we obtain the new identity $id_{F_i} = \mathscr{D}_i \circ \mathscr{P}_i + \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1} - \mathscr{D}_i \circ \mathscr{P}_i \circ \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1}$. This identity can be rewritten under the two different following forms:

$$\begin{cases} \mathscr{P}'_i = \mathscr{P}_i \circ (id_{F_i} - \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1}), \\ id_{F_i} = \mathscr{D}_i \circ \mathscr{P}'_i + \mathscr{P}_{i+1} \circ \mathscr{D}_{i+1}, \end{cases} \tag{14}$$

or

$$\begin{cases} \mathscr{P}''_{i+1} = (id_{F_i} - \mathscr{D}_i \circ \mathscr{P}_i) \circ \mathscr{P}_{i+1}, \\ id_{F_i} = \mathscr{D}_i \circ \mathscr{P}_i + \mathscr{P}''_{i+1} \circ \mathscr{D}_{i+1}. \end{cases} \tag{15}$$

Now, let us suppose that $\mathscr{P}_{i+1} \circ \mathscr{D}_{i+1} \circ \mathscr{P}_{i+1} = \mathscr{P}_{i+1}$, then we have $\mathscr{P}'_i \circ \mathscr{P}_{i+1} = 0 \Rightarrow \operatorname{im} \mathscr{P}_{i+1} \subseteq \ker \mathscr{P}'_i$. Let us take $\eta \in \ker \mathscr{P}'_i$ then, from the second equation of (14), we have $\eta = \mathscr{P}_{i+1}(\mathscr{D}_{i+1} \eta) \Rightarrow \eta \in \operatorname{im} \mathscr{P}_{i+1}$ showing that

$$F_{i-1} \overset{\mathscr{P}'_i}{\longleftarrow} F_i \overset{\mathscr{P}_{i+1}}{\longleftarrow} F_{i+1},$$

is a locally exact differential sequence. Moreover, from the second equation of (14), we have $\mathscr{P}'_i \circ \mathscr{D}_i \circ \mathscr{P}'_i = \mathscr{P}'_i$. For showing that $\mathscr{P}_{i+1} \circ \mathscr{D}_{i+1} \circ \mathscr{P}_{i+1} = \mathscr{P}_{i+1}$, we only have to prove it for $i = n$ (the above demonstration has shown that $\mathscr{P}_n \circ \mathscr{D}_n \circ \mathscr{P}_n = \mathscr{P}_n \Rightarrow \mathscr{P}'_{n-1} \circ \mathscr{D}_{n-1} \circ \mathscr{P}'_{n-1} = \mathscr{P}'_{n-1}$ ...). However, $\mathscr{D}_{n+1} \circ \mathscr{P}_{n+1} = id_{F_{n+1}} \Rightarrow \mathscr{P}_{n+1} \circ \mathscr{D}_{n+1} \circ \mathscr{P}_{n+1} = \mathscr{P}_{n+1}$.

Finally, we have shown that, starting with $\mathscr{P}_{n+1}$, we can change $\mathscr{P}_n$ into another lift-operator $\mathscr{P}'_n$, according to (14), in order to have the sequence formed by $\mathscr{P}_{n+1}$ and $\mathscr{P}'_n$ locally exact. Now, starting with $\mathscr{P}'_n$, we can change $\mathscr{P}_{n-1}$ using (14), in which we have substituted $\mathscr{P}_n$ by $\mathscr{P}'_n$, to have the sequence made by $\mathscr{P}'_n$ and $\mathscr{P}'_{n-1}$ locally exact. We can do similary for all the lift-operators $\mathscr{P}_i$ and we obtain the following locally exact differential sequence

$$F_0 \overset{\mathscr{P}'_1}{\longleftarrow} F_1 \overset{\mathscr{P}'_2}{\longleftarrow} \dots \overset{\mathscr{P}'_n}{\longleftarrow} F_n \overset{\mathscr{P}_{n+1}}{\longleftarrow} F_{n+1} \longleftarrow 0,$$

with $\mathscr{D}_i \circ \mathscr{P}'_i \circ \mathscr{D}_i = \mathscr{D}_i$ and $\mathscr{P}'_i \circ \mathscr{D}_i \circ \mathscr{P}'_i = \mathscr{P}'_i$.

Corollary 1 *Let $\mathscr{D}_1 : F_0 \to F_1$ be an operator determining a projective D-module, then there exists an operator $\mathscr{D}_0 : E \to F_0$ and lift-operators $\mathscr{P}_i : F_i \to F_{i-1}$ such that the sequence*

$$E \overset{\mathscr{P}_0}{\longleftarrow} F_0 \overset{\mathscr{P}_1}{\longleftarrow} F_1 \overset{\mathscr{P}_2}{\longleftarrow} \dots \overset{\mathscr{P}_n}{\longleftarrow} F_n \overset{\mathscr{P}_{n+1}}{\longleftarrow} F_{n+1} \longleftarrow 0,$$

is locally exact and $\forall\, i = 0 \dots n+1 : \mathscr{P}_i \circ \mathscr{D}_i \circ \mathscr{P}_i = \mathscr{P}_i$.

Proof. As $\mathscr{D}_1$ determines a projective D-module, there exists a parametrization $\mathscr{D}_0 : E \to F_1$ and $\tilde{\mathscr{D}}_1$ determines a projective D-module. Thus $\tilde{\mathscr{D}}_0$ and $\mathscr{D}_0$ determine also a projective D-module, which implies the existence of a lift-operator $\mathscr{P}_0$ such that $\mathscr{D}_0 \circ \mathscr{P}_0 \circ \mathscr{D}_0 = \mathscr{D}_0$. Applying the previous theorem, we have the locally exact differential sequence

$$F_0 \overset{\mathscr{P}'_1}{\longleftarrow} F_1 \overset{\mathscr{P}'_2}{\longleftarrow} \dots \overset{\mathscr{P}'_n}{\longleftarrow} F_n \overset{\mathscr{P}_{n+1}}{\longleftarrow} F_{n+1} \longleftarrow 0,$$

with $\mathscr{P}'_i \circ \mathscr{D}_i \circ \mathscr{P}'_i = \mathscr{P}'_i$. We can prolong, as in the previous proof, this differential sequence to have the following locally exact differential sequence

$$E \overset{\mathscr{P}'_0}{\longleftarrow} F_0 \overset{\mathscr{P}'_1}{\longleftarrow} F_1 \overset{\mathscr{P}'_2}{\longleftarrow} \dots \overset{\mathscr{P}'_n}{\longleftarrow} F_n \overset{\mathscr{P}_{n+1}}{\longleftarrow} F_{n+1} \longleftarrow 0,$$

with $\mathscr{D}_0 \circ \mathscr{P}'_0 \circ \mathscr{D}_0 = \mathscr{D}_0$ and $\mathscr{P}'_0 \circ \mathscr{D}_0 \circ \mathscr{P}'_0 = \mathscr{P}'_0$.

5.2 Applications to the Generalized Bezout Identity

We now explain the link of the preceding result with the generalized Bezout identity.

5.2.1 PD Control Systems with Variable Coefficients. The following theorem is the generalization of the generalized Bezout identity for non surjective operator $\mathscr{D}_1$ describing the control system. We insist on the fact that everything that follows can be computed explicitly using symbolic packages. See the examples illustrating the main results.

Theorem 9 *Let $\mathscr{D}_1 : F_0 \to F_1$ be a PD control system with variable coefficients.*

1. *If $\mathscr{D}_1$ determines a free D-module M then there exists three operators $\mathscr{D}_0 : E \to F_0$, $\mathscr{P}_0 : F_0 \to E$ and $\mathscr{P}_1 : F_1 \to F_0$ such that:*

$$\begin{cases} \mathscr{D}_1 \circ \mathscr{D}_0 = 0, \\ \mathscr{P}_0 \circ \mathscr{D}_0 = id_E, \\ \mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1, \\ \mathscr{P}_1 \circ \mathscr{D}_1 \circ \mathscr{P}_1 = \mathscr{P}_1, \\ \mathscr{P}_0 \circ \mathscr{P}_1 = 0. \end{cases}$$

   The differential sequences $0 \longrightarrow E \stackrel{\mathscr{D}_0}{\longrightarrow} F_0 \stackrel{\mathscr{D}_1}{\longrightarrow} F_1$ and $0 \longleftarrow E \stackrel{\mathscr{P}_0}{\longleftarrow} F_0 \stackrel{\mathscr{P}_1}{\longleftarrow} F_1$ are locally exact.

2. *If $\mathscr{D}_1$ determines a projective D-module M, then there exists three operators $\mathscr{D}_0 : E \to F_0$, $\mathscr{P}_0 : F_0 \to E$ and $\mathscr{P}_1 : F_1 \to F_0$ such that:*

$$\begin{cases} \mathscr{D}_1 \circ \mathscr{D}_0 = 0, \\ \mathscr{D}_0 \circ \mathscr{P}_0 \circ \mathscr{D}_0 = \mathscr{D}_0, \\ \mathscr{P}_0 \circ \mathscr{D}_0 \circ \mathscr{P}_0 = \mathscr{P}_0, \\ \mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1, \\ \mathscr{P}_1 \circ \mathscr{D}_1 \circ \mathscr{P}_1 = \mathscr{P}_1, \\ \mathscr{P}_0 \circ \mathscr{P}_1 = 0. \end{cases}$$

   The differential sequences $E \stackrel{\mathscr{D}_0}{\longrightarrow} F_0 \stackrel{\mathscr{D}_1}{\longrightarrow} F_1$ and $E \stackrel{\mathscr{P}_0}{\longleftarrow} F_0 \stackrel{\mathscr{P}_1}{\longleftarrow} F_1$ are locally exact.

3. *If $\mathscr{D}_1$ determines a torsion-free D-module M, i.e. $\mathscr{D}_1$ is controllable, then there exists one operator $\mathscr{D}_0 : E \to F_0$ such that:*

$$\mathscr{D}_1 \circ \mathscr{D}_0 = 0.$$

   The differential sequence $E \stackrel{\mathscr{D}_0}{\longrightarrow} F_0 \stackrel{\mathscr{D}_1}{\longrightarrow} F_1$ is formally exact.

Example 13 Let us take again the Spencer operator $\mathscr{D}_1$ defined by (5). We have shown that $\mathscr{D}_1$ determined a free D-module and that the following differential sequence

$$0 \longrightarrow E \stackrel{\mathscr{D}_0}{\longrightarrow} F_0 \stackrel{\mathscr{D}_1}{\longrightarrow} F_1 \stackrel{\mathscr{D}_2}{\longrightarrow} F_2 \longrightarrow 0, \tag{16}$$

was locally exact, where $\mathscr{D}_0$ and $\mathscr{D}_2$ are defined respectively by (7) and by (11). The operator $\mathscr{P}_2$ defined by (12) is a right-inverse of $\mathscr{D}_2$ and $\mathscr{P}_1$, defined by (13), satisfies $\mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1$. We let the reader check that $\mathscr{P}'_1 = \mathscr{P}_1$ and $\mathscr{P}'_0 = \mathscr{P}_0 : \eta \rightarrow \xi$ defined by $\eta^1 = \xi$ satisfies $\mathscr{P}_0 \circ \mathscr{D}_0 = id_{F_0}$. Thus, we have:

$$\begin{cases} \mathscr{D}_1 \circ \mathscr{D}_0 = 0, \\ \mathscr{D}_2 \circ \mathscr{D}_1 = 0, \\ \mathscr{P}_0 \circ \mathscr{D}_0 = id_E, \\ \mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1, \\ \mathscr{P}_1 \circ \mathscr{D}_1 \circ \mathscr{P}_1 = \mathscr{P}_1, \\ \mathscr{D}_2 \circ \mathscr{P}_2 = id_{F_2}, \\ \mathscr{P}_1 \circ \mathscr{P}_2 = 0, \\ \mathscr{P}_0 \circ \mathscr{P}_1 = 0. \end{cases}$$

Moreover, the differential sequence

$$0 \longleftarrow E \overset{\mathscr{P}_0}{\longleftarrow} F_0 \overset{\mathscr{P}_1}{\longleftarrow} F_1 \overset{\mathscr{P}_2}{\longleftarrow} F_2 \longleftarrow 0,$$

is locally exact.

In the case $\mathscr{D}_1$ is a surjective operator, the previous theorem leads to the existence of the generalized Bezout identity.

Corollary 2 *Let $\mathscr{D}_1 : F_0 \longrightarrow F_1$ be a surjective PD control system with variable coefficients.*

1. *If the operator $\mathscr{D}_1$ determines a free D-module M, then we have*

$$\begin{bmatrix} \mathscr{D}_1 \\ \mathscr{P}_0 \end{bmatrix} \circ \begin{bmatrix} \mathscr{P}_1 & \mathscr{D}_0 \end{bmatrix} = \begin{bmatrix} id_{F_1} & 0 \\ 0 & id_E \end{bmatrix},$$

and the generalized Bezout identity is equivalent to the splitting of the following locally exact differential sequence:

$$0 \longrightarrow E \underset{\overset{\longleftarrow}{\mathscr{P}_0}}{\overset{\mathscr{D}_0}{\longrightarrow}} F_0 \underset{\overset{\longleftarrow}{\mathscr{P}_1}}{\overset{\mathscr{D}_1}{\longrightarrow}} F_1 \longrightarrow 0.$$

In such a situation, we notice that the use of the formula (14) for i=0 in Corollary 1 is nothing else but what is called in the literature as reversed Bezout identity (16, lemma 6.3–9) which has thus been extended to variable coefficients case.

2. *If $\mathscr{D}_1$ determines a projective D-module M, then we have:*

$$\begin{bmatrix} \mathscr{D}_1 \end{bmatrix} \circ \begin{bmatrix} \mathscr{P}_1 & \mathscr{D}_0 \end{bmatrix} = \begin{bmatrix} id_{F_1} & 0 \end{bmatrix}.$$

3. *If $\mathscr{D}_1$ determines a torsion-free D-module M, then we have:*

$$\mathscr{D}_1 \circ \mathscr{D}_0 = 0.$$

In the next example, we illustrate each situation for a surjective operator.

Example 14 1. The system $\partial_2\,\eta^1 - x^2\partial_1\,\eta^2 - \eta^3 = 0$ determines a free-module and we have:

$$\begin{bmatrix} \partial_2 & -x^2\partial_1 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -\partial_2 & -x^2\partial_1 \end{bmatrix} = I.$$

2. We have seen that the system (8), defined by $\partial_2\,\eta^2 - x^2\partial_1\,\eta^1 + \eta^1 = 0$, was generating a projective module and we found a right-inverse. We let the reader check that:

$$\left[-x^2\partial_1 + 1 \;\; \partial_2 \right] \circ \begin{bmatrix} x^2\partial_2 + 2 & x^2\partial_2^2 + 2\partial_2 \\ (x^2)^2\partial_1 - x^2 & (x^2)^2\partial_1\partial_2 - x^2\partial_2 - 1 \end{bmatrix} = [1, 0].$$

3. The system $\partial_2\,\eta^2 - x^1\partial_1\,\eta^1 + \eta^1 = 0$ determines only a torsion-free module and we have:

$$\left[-x^1\partial_1 + 1 \;\; \partial_2 \right] \circ \begin{bmatrix} -\partial_2 \\ -x^1\partial_1 + 1 \end{bmatrix} = 0.$$

Remark 6. Projective module and right-inverse are useful if we want to know whether a system of polynomial equations admits solutions. Indeed, the Hilbert theorem claims that the system of polynomial equations

$$\begin{cases} P_1(\chi_1, ..., \chi_n) = 0, \\ P_2(\chi_1, ..., \chi_n) = 0, \\ \cdots \\ \\ \cdots \\ P_m(\chi_1, ..., \chi_n) = 0, \end{cases} \tag{17}$$

where $P_1, \ldots, P_m \in k\,[\chi_1, \ldots, \chi_n]$, has no solution iff there exists $Q_1, \ldots, Q_m \in k\,[\chi_1, \ldots, \chi_n]$ such that:

$$P_1\,Q_1 + P_2\,Q_2 + \ldots + P_m\,Q_m = 1.$$

We can reformulate the Hilbert theorem saying that the system of polynomial equations (17) has no solution iff the adjoint of the surjective operator $\mathscr{D}_1 : \eta \to \zeta$ defined by

$$P_1(\partial_1, ..., \partial_n)\,\eta^1 + \ldots + P_m(\partial_1, ..., \partial_n)\,\eta^m = \zeta,$$

where we have substituted χ_i by ∂_i in P_j, is injective, i.e., the module M determined by $\mathscr{D}_1$ is a projective and thus a free D-module (Quillen-Suslin theorem).

We give an example.

Example 15 We search the common solutions of the following set of polynomials:

$$\left\{\begin{array}{l} P_1 = \chi_3^3 + \chi_1\chi_3 + 1, \\ P_2 = \chi_3^2 + \chi_2\chi_3, \\ P_3 = \chi_2^2 + \chi_1. \end{array}\right. \tag{18}$$

We define the operator $\mathscr{D}_1 : \eta \to \zeta$ by:

$$(\partial_3^3 + \partial_1\partial_3 + 1)\eta^1 + (\partial_3^2 + \partial_2\partial_3)\eta^2 + (\partial_2^2 + \partial_1)\eta^3 = \zeta.$$

It is quite easy to see that we obtain from $\tilde{\mathscr{D}}_1\lambda = \mu$ the equation:

$$\lambda = \mu_1 + \partial_3\,\mu_2 - \partial_2\,\mu_2 + \partial_3\,\mu_3.$$

Thus, the operator $\mathscr{P}_1 : \zeta \to \eta$, defined by

$$\left\{\begin{array}{l} \zeta = \eta^1, \\ \partial_2\,\zeta - \partial_3\,\zeta = \eta^2, \\ -\partial_3\,\zeta = \eta^3, \end{array}\right.$$

is a right-inverse of $\mathscr{D}_1$, i.e. $\mathscr{D}_1 \circ \mathscr{P}_1 = id_{F_1}$. We have

$$P_1 + P_2\,(\chi_2 - \chi_3) - P_1\,\chi_3 = 1,$$

and the system (18) has no solution.

5.2.2 Time-varying OD Control Systems. The following theorem leads to the existence of the generalized Bezout identity.

Theorem 10 • *Let $\mathscr{D}_1 : F_0 \to F_1$ be a controllable time-varying OD control system, then there exists three operators $\mathscr{D}_0 : E \to F_0$, $\mathscr{P}_0 : F_0 \to E$ and $\mathscr{P}_1 : F_1 \to F_0$ such that:*

$$\left\{\begin{array}{l} \mathscr{D}_1 \circ \mathscr{D}_0 = 0, \\ \mathscr{P}_0 \circ \mathscr{D}_0 = id_E, \\ \mathscr{D}_1 \circ \mathscr{P}_1 \circ \mathscr{D}_1 = \mathscr{D}_1, \\ \mathscr{P}_1 \circ \mathscr{D}_1 \circ \mathscr{P}_1 = \mathscr{P}_1, \\ \mathscr{P}_0 \circ \mathscr{P}_1 = 0. \end{array}\right.$$

Moreover, the differential sequences $0 \longrightarrow E \xrightarrow{\mathscr{D}_0} F_0 \xrightarrow{\mathscr{D}_1} F_1$ and $0 \longleftarrow E \xleftarrow{\mathscr{P}_0} F_0 \xleftarrow{\mathscr{P}_1} F_1$ are locally exact.

- *If $\mathscr{D}_1$ is a surjective operator, then we have*

$$\begin{bmatrix} \mathscr{D}_1 \\ \mathscr{P}_0 \end{bmatrix} \circ \begin{bmatrix} \mathscr{P}_1 & \mathscr{D}_0 \end{bmatrix} = \begin{bmatrix} id_{F_1} & 0 \\ 0 & id_E \end{bmatrix},$$

and the generalized Bezout identity is equivalent to the splitting of the locally exact differential sequence:

$$0 \longrightarrow E \underset{\overset{\longleftarrow}{\mathscr{P}_0}}{\overset{\mathscr{D}_0}{\longrightarrow}} F_0 \underset{\overset{\longleftarrow}{\mathscr{P}_1}}{\overset{\mathscr{D}_1}{\longrightarrow}} F_1 \longrightarrow 0.$$

We now give examples of a computation of the generalized Bezout identity for a time-varying OD control system.

Example 16 1. If we start with the left-coprime system (1), i.e. controllable [2, 5, 9], we can rewrite it under the form $\mathscr{D}_1\eta = \zeta$ where $\mathscr{D}_1 = [P(s), Q(s)]$, $s = \frac{d}{dt}$ and $\eta = (y, u)^t$. The assumption of $\det P(s) \neq 0$ amounts to the surjectivity of $\mathscr{D}_1$ (see example 4) and by theorem 10, we have (2). 2. We compute a generalized Bezout identity for the following time-varying OD control system

$$\ddot{\eta}^1 + \alpha(t)\dot{\eta}^1 + \eta^1 - \dot{\eta}^2 - \alpha(t)\eta^2 = 0,$$

with α a function of time. We take the surjective operator $\mathscr{D}_1$ associated with the previous system and dualizing it, we obtain the operator $\tilde{\mathscr{D}}_1 : \lambda \to \mu$ defined by:

$$\begin{cases} \ddot{\lambda} - \alpha(t)\dot{\lambda} - \dot{\alpha}(t)\lambda + \lambda = \mu_1, \\ \dot{\lambda} - \alpha(t)\lambda = \mu_2. \end{cases}$$

It is easy to see that $\tilde{\mathscr{D}}_1$ is an injective operator as we obtain, by saturating the preceding OD system by low-order equations, the zero-order new equation:

$$\lambda = -\dot{\mu}_2 + \mu_1.$$

The operator $\tilde{\mathscr{P}}_1 : \mu \to \lambda$ defined by

$$-\dot{\mu}_2 + \mu_1 = \lambda,$$

satisfies $\tilde{\mathscr{P}}_1 \circ \tilde{\mathscr{D}}_1 = id_{\tilde{E}}$ and thus the adjoint of $\tilde{\mathscr{P}}_1$ is the operator $\mathscr{P}_1 : \zeta \to \eta$

$$\begin{cases} \zeta = \eta^1, \\ \dot{\zeta} = \eta^2, \end{cases}$$

and we let the reader check that $\mathscr{P}_1$ is a right-inverse of $\mathscr{D}_1$. Substituting $\lambda = \tilde{\mathscr{P}}_1\,\mu$ in $\tilde{\mathscr{D}}_1$, we find the following operator $\tilde{\mathscr{D}}_0 : \mu \to \nu$ defined by:

$$\ddot{\mu}_2 - \alpha(t)\dot{\mu}_2 + \mu_2 - \dot{\mu}_1 + \alpha(t)\mu_1 = \nu.$$

Dualizing $\tilde{\mathscr{D}}_0$, we obtain a parametrization $\mathscr{D}_0 : \xi \to \eta$ of $\mathscr{D}_1$:

$$\begin{cases} \dot{\xi} + \alpha(t)\xi = \eta^1, \\ \ddot{\xi} + \alpha(t)\dot{\xi} + (1 + \dot{\alpha}(t))\xi = \eta^2. \end{cases}$$

This parametrization is injective as we have

$$\xi = -\dot{\eta}^1 + \eta^2,$$

and we have a left-inverse of $\mathscr{D}_0$ given by:

$$\mathscr{P}_0\,\eta = -\dot{\eta}^1 + \eta^2 = \xi.$$

Hence, the module M is a free D-module with basis $\xi = -\dot{\eta}^1 + \eta^2$. We easily check that $\mathscr{P}'_0 = \mathscr{P}_0$ generates exactly the compatibility conditions of $\mathscr{P}_1$. Finally, we have the following generalized Bezout identity

$$\begin{bmatrix} s^2 + \alpha(t)\,s + 1 & -s - \alpha(t) \\ -s & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & s + \alpha(t) \\ s & s^2 + \alpha(t)\,s + (1 + \dot{\alpha}(t)) \end{bmatrix} = I,$$

where the time derivative of the coefficient $\alpha(t)$ explicitly appears. This situation is much general than the classical one (constant coefficients systems), where the composition of matrix operators is just the ordinary multiplication of polynomial matrices.

6 Conclusion

We have seen how the generalized Bezout identity could be extended to non surjective linear time-varying OD control systems. If the linear time-varying OD control system is defined by a surjective operator, we have shown that the generalized Bezout identity was, in fact, the well-known algebraic notion of the splitting of a short exact sequence made with the system and its parametrization. We have seen when and how it could be extended for general linear PD control systems with variable coefficients. We have shown that it only depended on the algebraic nature of the differential module determined by the system. This new formulation has the advantage of bringing the generalized Bezout identity and its computation closer to algebraic and geometric concepts. In particular, we have made clear that it did not depend on a separation of the system variables between inputs and outputs. The formal tests, developped in this paper, can be used for any control systems over the ring of polynomial $k\,[\chi_1, \ldots, \chi_n]$ with k a field (delay control systems, n-dimensional systems, ...). Applications of the generalized Bezout identity for the parametrization of controllers [39, 42] will be treated in future communications.

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