Let us consider the first Weyl algebra $A = A_1(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers,

$$A := \text{DefineOreAlgebra}(\text{diff}=[d,t], \text{polynom}=[t])$$

and the left $A$-module $M$ finitely presented by the matrix $R$ defined by

$$R := \text{evalm([[d,0,-t]])};$$

and the left $A$-module $M'$ finitely presented by the matrix $R'$ defined by

$$R' := \text{evalm([[d,-t,0,0,0,-1], [0,d,0,-t,0,0], [0,0,d,0,-t,0]])};$$

Let us also consider the matrix $P$ defined by

$$P := \text{evalm([[0,0,1,0,0,0], [1,0,0,0,0,1], [0,0,0,0,1,0]])};$$

and the matrix $P'$ defined by

$$P' := \text{evalm([[0,0,1]])};$$

which are such that $R P = P' R'$. Thus, they define the left $A$-homomorphism $\iota$ from $M$ to $M'$ induced by $P$. We can check that $\iota$ is injective:

$$\text{TestInj}(R,R',P,A);$$

Hence, we get that $M$ is isomorphic to the left $A$-submodule $\iota(M) = (A^{1\times3} P + A^{1\times3} R') / (A^{1\times3} R')$ of $M'$.

Let us now compute an element $m^*$ of $M$ such that $\iota(m^*)$ is a unimodular element of $M'$.

$$U := \text{UnimodularElementInSubmodule}(R,R',P,A);$$

The output $U$ of the command $\text{UnimodularElementInSubmodule}$ contains two entries.

$$\text{nops}(U);$$

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the first one $U[1]$, namely,

> $U[1];$

$\begin{bmatrix}
0 & 1 & 0
\end{bmatrix}$

represents an element $m^*$ of $M$ which is such that $\iota(m^*)$ is a unimodular element of $M'$. The second entry $U[2]$ of $U$, namely,

> `map(collect,U[2],[d,t]);`

\[
\begin{bmatrix}
-\frac{2}{9} t^2 + t - \frac{1}{3} d t^2 + \frac{2}{27} t^3 \\
-\frac{t d^2}{3} + \left(\frac{1}{3} - \frac{5}{9} t + \frac{2}{27} t^2\right) d + \frac{5}{9} + \frac{2 t^2}{27} \\
\frac{4}{27} - \frac{d^3}{3} + \left(-\frac{5}{9} + \frac{2 t}{27}\right) d^2 + \left(\frac{2 t}{27} + \frac{4}{27}\right) d \\
\frac{1}{3} d t^2 - t + 1 + \frac{2}{9} t^2 - \frac{2}{27} t^3
\end{bmatrix}
\]

defines a left $A$-homomorphism $\phi$ from $M'$ to $A$ which is such that $\phi(\iota(m^*)) = U[1] P U[2] = 1$. Indeed, if $\lambda^* = U[1] P$, i.e.,

> `lambda_star := Mult(U[1],P,A);`

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

then $\lambda^* U[2] = U[1] P U[2]$ is equal to:

> `Mult(lambda_star,U[2],A);`

$\begin{bmatrix}
1
\end{bmatrix}$

Finally, let us check that $\phi$ is a well-defined left $A$-homomorphism from $M'$ to $A$, i.e., $R' U[2] = 0$:

> `Mult(Rp,U[2],A);`

$\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}$