with(OreModules):
with(OreMorphisms):
with(Stafford):
with(linalg):

Let us consider the third Weyl algebra \( A = A_3(\mathbb{Q}) \), where \( \mathbb{Q} \) is the field of rational numbers,

\[
A := \text{DefineOreAlgebra}(\text{diff}=[x[1],d[1]], \text{diff}=[x[2],d[2]],
\text{diff}=[x[3],d[3]], \text{polynom}=[x[1],x[2],x[3]]);
\]

and the left \( A \)-module \( M \) finitely presented by the following matrix:

\[
[-x[2]*d[2]/2-3/2,0,d[2]/2],[-d[1]-x[2]*d[3]/2,-d[2],-d[3]/2]]);
\]

The left \( A \)-module \( M \) corresponds to a system defining the infinitesimal transformations of the Lie pseudogroup formed by the contact transformations.

Let us compute the rank of \( M \):

\[
\text{OreRank}(R,A);
\]

1

Thus, we get \( \text{rank}_A(M) = 1 \). Let us now study \( \text{hom}_A(M, A) \). We first compute \( \ker_A(R) \).

\[
Q := \text{Involution}(\text{SyzygyModule}(\text{Involution}(R,A),A),A);
\]

\[
Q := \begin{bmatrix}
-d_2 \\
d_1 + x_2 d_3 \\
-2 - x_2 d_2
\end{bmatrix}
\]

We obtain \( \ker_A(R) = Q A^3 \). In particular, let us check that \( R Q = 0 \):

\[
\text{Mult}(R,Q,A);
\]

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Hence, we get \( \text{hom}_A(M, A) \) is isomorphic to \( \ker_A(R) = Q A \). A form of \( M \) is then defined by means of a right multiple \( Q \xi \) of \( Q \), where \( \xi \) is an element of \( A \).

A unimodular element of \( M \) represented by some row vector \( \lambda^* \) in \( A^{1 \times 3} \) satisfies \( \lambda^* (Q \xi^*) = 1 \) for a certain element \( \xi^* \) of \( A \). Since \( Q \) admits a left inverse

\[
T := \text{LeftInverse}(Q,A);
\]

\[
T := \begin{bmatrix}
\frac{1}{2} x_2 & 0 & \frac{-1}{2}
\end{bmatrix}
\]
i.e., $TQ = 1$, if $\lambda^* = T$, then $\lambda^*$ represents a unimodular element $m^*$ of $M$ and $Q$ induces a left $A$-homomorphism $\phi$ from $M$ to $A$ which satisfies $\phi(m^*) = TQ = 1$.

Serre’s Splitting-off theorem cannot be used since $\text{rank}_A(M) < 2$.

> UnimodularElement(R,A);

Error, (in Stafford/UnimodularElementInSubmodule) expecting that the rank of the left module presented by the first matrix is at least 2.

But using the option "checkrank"=false, we can try to detect a unimodular element of $M$ by means of a different method.

> U := UnimodularElement(R,A,"checkrank"=false);

$$U := \begin{bmatrix}
\frac{1}{2} x_2 & 0 & -\frac{1}{2} \\
1 & x_2 & d_2 \\
-2 & x_2 & d_2
\end{bmatrix}, \begin{bmatrix}
-d_2 \\
d_1 + x_2 d_3 \\
-2 - x_2 d_2
\end{bmatrix}$$

We find again the unimodular element $m^*$ of $M$ represented by $U[1]$ and the left $A$-homomorphism $\phi$ from $M$ to $A$ induced by $U[2]$ which satisfies $\phi(m^*) = 1$:

> Mult(U[1],U[2],A);

$$\begin{bmatrix}
1
\end{bmatrix}$$

Finally, we can check that $\phi$ is a well-defined left $A$-homomorphism from $M$ to $A$ since $R U[2] = 0$:

> Mult(R,U[2],A);

$$\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}$$

Now, let us consider a new left $A$-module $M$ finitely presented by the following matrix:


$$R := \begin{bmatrix}
d_1 + x_2 \\
d_2 \\
d_3 + x_1
\end{bmatrix}$$

The rank of $M$ is clearly 2. Let us compute a unimodular element based on Serre’s Splitting-off theorem.

> U := UnimodularElement(R,A);

$$U := \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
-(d_3 + x_1 + d_2)(d_3 + x_1) \\
2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2
\end{bmatrix}$$

We obtain a unimodular element $m^*$ of $M$ represented by $U[1]$ and a left $A$-homomorphism $\phi$ from $M$ to $A$ induced by $U[2]$ which satisfies $\phi(m^*) = 1$:

> Mult(U[1],U[2],A);

$$\begin{bmatrix}
1
\end{bmatrix}$$

Let us check that $\phi$ is a well-defined left $A$-homomorphism from $M$ to $A$, i.e., $R U[2] = 0$:
Since $M$ admits a unimodular element, $M$ can be decomposed as a direct sum of $A$ and another left $A$-module $M'$ up to isomorphism. A presentation of $M'$ can be obtained using the command `FreeDirectSummand` with the option "presentation".

```plaintext
> F := FreeDirectSummand(R,A,"presentation");
> nops(F);
4
```

The output of the command `FreeDirectSummand` with the option "presentation" is a list with four entries. The first one $F[1]$ is a representative of a unimodular element of $M$

```plaintext
> F[1];
[0 1 0]
```
i.e., $m^*$ represented by $F[1]$ is a unimodular element of $M$. The second entry of $F$

```plaintext
> F[2];
\[
\begin{bmatrix}
-(d_3 + x_1 + d_2)(d_3 + x_1) \\
1 \\
2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2
\end{bmatrix}
\]
```
induces a left $A$-homomorphism $\phi$ from $M$ to $A$ which satisfies $\phi(m^*) = 1$. The third entry of $F$, namely,

```plaintext
> F[3];
\[
\begin{bmatrix}
1 & 0 & 0 \\
-1 & d_1 + x_2 & -d_3 - x_1
\end{bmatrix}
\]
```
is a presentation matrix of $M'$ which is isomorphic to $\ker(\phi)$. Finally, the last entry of $F$, namely,

```plaintext
> F[4];
\[
\begin{bmatrix}
d_1 + x_2 & d_2 & d_3 + x_1 \\
1 & d_2 d_3 + d_3^2 + x_1^2 + 2 d_3 x_1 + d_2 x_1 & 0 \\
0 & 2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2 & -1
\end{bmatrix}
\]
```
induces an injective left $A$-homomorphism $i$ from $M'$ to $M$. Using the `OreMorphism` package, we can check again that $i$ is injective:

```plaintext
> TestInj(F[3],R,F[4],A);
true
```

Let us compute $\text{rank}_A(M')$:

```plaintext
> OreRank(F[3],A);
1
```

Since $\text{rank}_A(M') = 1$, we cannot use Serre’s Splitting-off theorem again to decompose the left $A$-module $M'$. 

```plaintext
3
```
Using the option "isomorphism" of the command \texttt{FreeDirectSummand},

\begin{verbatim}
> G := FreeDirectSummand(R,A,"isomorphism");
\end{verbatim}

we can get another representation of the above splitting. The output G contains

\begin{verbatim}
> nops(G); 4
\end{verbatim}

four entries. The first one

\begin{verbatim}
> G[1];
[ 0 1 0 ]
\end{verbatim}

represents the unimodular element $m^*$ of $M$. The second one, namely,

\begin{verbatim}
> G[2];
\begin{bmatrix}
-(d_3 + x_1 + d_2)(d_3 + x_1) \\
1 \\
2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2 \\
\end{bmatrix}
\end{verbatim}

induces a left $A$-homomorphism $\phi$ from $M$ to $A$ such that $\phi(m^*) = 1$. The third one

\begin{verbatim}
> G[3];
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -1 & d_1 + x_2 & -d_3 - x_1 \\
\end{bmatrix}
\end{verbatim}

is a presentation matrix of the left $A$-module $M_1$ which is isomorphic to the direct sum of $A$ and $M'$, and isomorphic to $M$. Indeed, we note that $G[3] = (0 \ F[3])$. The last entry of G

\begin{verbatim}
> G[4];
\begin{bmatrix}
0 & d_1 + x_2 & 1 & 0 \\
1 & d_2 & d_3 + x_1 & 0 \\
0 & 2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2 & -1 \\
\end{bmatrix}
\end{verbatim}

induces a left $A$-isomorphism $g$ from $M_1$ to $M$. This last result can be checked again using the OreMor-phisms package:

\begin{verbatim}
> TestIso(G[3],R,G[4],A);
true
\end{verbatim}

We can simplify the presentation $G[3]$ of $M_1$. Indeed, $M_1$ is isomorphic to $M_2$ which is finitely presented by the second entry of S defined by:

\begin{verbatim}
> with(PurityFiltration):
> S := ReducedPresentation(G[3],A);
\end{verbatim}

\begin{verbatim}
S := \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -1 & d_1 + x_2 & -d_3 - x_1 \\
\end{bmatrix}, \begin{bmatrix}
0 & d_1 + x_2 & -d_3 - x_1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\end{verbatim}
The left $A$-isomorphism $h$ from $M_2$ to $M_1$ is induced by $S[4]$.

> TestIso(S[2],S[1],S[4],A);

true

Hence, if we define $P = S[4] \cdot G[4]$, namely,

> P := Mult(S[4],G[4],A);

\[
P := \begin{bmatrix}
0 & 1 & 0 \\
1 & d_2 d_3 + d_3^2 + x_1^2 + 2 d_3 x_1 + d_2 x_1 & 0 \\
0 & 2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2 & -1
\end{bmatrix}
\]

then the composition $i$ of $g$ and $h$, which is induced by $P$, is a left $A$-isomorphism.

> TestIso(S[2],R,P,A);

true

Let us now compute $i^{-1}$.

> Q := InverseMorphism(S[2],R,P,A);

\[
Q := \begin{bmatrix}
-d_2 d_3 - d_3^2 - x_1^2 - 2 d_3 x_1 - d_2 x_1 & 1 & 0 \\
1 & 0 & 0 \\
2 + d_1 d_3 + x_1 d_1 + x_2 d_3 + x_1 x_2 + d_2 d_1 + x_2 d_2 & 0 & -1
\end{bmatrix}, \begin{bmatrix}
1 
\end{bmatrix}
\]

We obtain that $i^{-1}$ is induced by $Q[1]$. Finally, let us check again that $i^{-1}$ is a well-defined left $A$-isomorphism between $M$ and $M_2$:

> TestIso(R,S[2],Q[1],A);

true