```
> with(OreModules):
> with(OreMorphisms):
> with(Stafford):
> with(linalg):
```

Let us consider the second Weyl algebra $A = A_2(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers.

```
> A := DefineOreAlgebra(diff=[d1,x1], diff=[d2,x2], polynom=[x1,x2]):
```

More precisely, the procedures of the OreModules, Stafford, and OreMorphisms packages perform multiplication of polynomials in x1, x2, d1, d2 subject to the following commutation rules:

$$\forall\, i, j = 1, 2: \quad di\, xj = xj\, di + \delta(i, j), \quad di\, dj = dj\, di, \quad xi\, xj = xj\, xi,$$

where $\delta(i, j)$ is the Kronecker delta. Since Maple's data structure for polynomials is used here and this data structure does not retain the order of symbols in a product in general, the following results are to be understood with the *convention* that a monomial in xi and dj represents the element of the Weyl algebra that is defined by writing all xi to the left of all the dj. For instance, even when a procedure returns the polynomial d1 x1, the element x1 d1 of A is referred to here (and not x1 d1 + 1).

We demonstrate the procedure *TwoStrongGenerators*. Given elements a, b, c of A and non-zero elements d, e of A, this procedure computes two elements u, v of A such that

$$A\, a + A\, b + A\, c = A\, (a + d\, u\, c) + A\, (b + e\, v\, c).$$

We apply *TwoStrongGenerators* to the following elements a, b, c and d := x2, e := x1.

```
> a := d1; b := d2; c := x1;
```

$$a := d1$$
$$b := d2$$
$$c := x1$$

Using the above notations, the result of the procedure TwoStrongGenerators is a list of the form [a + d u c, b + e v c, [u, v]].

```
> G := TwoStrongGenerators(a,b,c,x2,x1,A);
```

$$G := [d1,\ d2 + x1^2,\ [0,\ 1]]$$

Let us check the result. Let us consider the following two matrices:

```
> R := evalm([[a],[b],[c]]);
```

$$R := \begin{bmatrix} d1 \\ d2 \\ x1 \end{bmatrix}$$

```
> S := evalm([[G[1]],[G[2]]]);
```

$$S := \begin{bmatrix} d1 \\ d2 + x1^2 \end{bmatrix}$$

The next command computes (if it exists) a (2 x 3) matrix F_1 with entries in A satisfying S = F_1 R.

```
> F1 := Factorize(S,R,A);
```

$$F1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x1 \end{bmatrix}$$

Similarly, we can check whether or not there exists a (3 x 2) matrix F_2 with entries in A such that R = F_2 S.

```
> F2 := Factorize(R,S,A);
```

$$F2 := \begin{bmatrix} 1 & 0 \\ \frac{1}{2}\, d2\, x1 + \frac{1}{2}\, x1^3 & 1 - \frac{x1\, d1}{2} \\ -\frac{x1^2}{2} - \frac{d2}{2} & \frac{d1}{2} \end{bmatrix}$$

Hence, a, b, c can be expressed as left A-linear combinations of d1 and d2 + x1^2, and vice versa. In other words, we have

$$A\, a + A\, b + A\, c = A\, d1 + A\, (d2 + x1^2).$$

We continue with another example.

```
> a := d1^2; b := d2^2; c := d1*d2;
```

$$a := d1^2$$
$$b := d2^2$$
$$c := d1\, d2$$

Now, for d = d1, e = d2, using the procedure *TwoStrongGenerators*, we compute elements u, v of A such that

$$A\, a + A\, b + A\, c = A\, (a + d\, u\, c) + A\, (b + e\, v\, c).$$

```
> G := TwoStrongGenerators(a,b,c,d1,d2,A);
```

$$G := [d1^2,\ d2^2 + d2^2\, (x1^3 + x2\, x1^3 + x2\, x1^5)\, d1 + (x1^3 + x1^5)\, d1\, d2,$$
$$[0,\ x1^3 + (x1^3 + x1^5)\, x2]]$$

Let us check the result. Let us consider the following two matrices.

```
> R := evalm([[a],[b],[c]]);
```

$$R := \begin{bmatrix} d1^2 \\ d2^2 \\ d1\, d2 \end{bmatrix}$$

```
> S := evalm([[G[1]],[G[2]]]);
```

$$S := \begin{bmatrix} d1^2 \\ d2^2 + d2^2\, (x1^3 + x2\, x1^3 + x2\, x1^5)\, d1 + (x1^3 + x1^5)\, d1\, d2 \end{bmatrix}$$

The procedure Factorize, applied to S and R, computes (if it exists) a (2 x 3) matrix E with entries in A satisfying S = E R.

```
> E := Factorize(S,R,A);
```

$$E := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x1^3 + d2\, x1^3 + d2\, x2\, x1^3 + x1^5 + d2\, x2\, x1^5 \end{bmatrix}$$

We exchange the roles of S and R to find a (3 x 2) matrix F with entries in A such that R = F S.

```
> F := Factorize(R,S,A);
```

$$F :=$$
$$[1\,,\,0]$$
$$\Big[\frac{14\,d2\,x1^2}{15}+\frac{161\,d1\,d2^2\,x1}{900}+\frac{7\,d2}{75}-\frac{7\,d2^2}{300}+\frac{14\,d2^2\,x2\,x1^2}{15}+\frac{7\,d1\,d2\,x1^3}{10}$$
$$+\frac{7\,d1\,d2\,x1}{50}+\frac{7\,d1^2\,d2^2\,x1^4}{180}+\frac{7\,d1^2\,d2\,x1^4}{60}+\frac{7\,d1^2\,d2\,x1^6}{180}-\frac{d1\,d2^2\,x1^5}{15}$$
$$-\frac{d1\,d2\,x1^7}{15}+\frac{7\,d1^2\,d2^2\,x1^2}{150}+\frac{7\,d1^2\,d2\,x1^2}{150}+\frac{7\,d1^3\,d2^2\,x1^3}{1800}+\frac{7\,d1^3\,d2\,x1^3}{1800}$$
$$+\frac{7\,d1^3\,d2\,x1^5}{1800}+\frac{7\,d1^2\,d2^2\,x2\,x1^4}{60}+\frac{7\,d1^2\,d2^2\,x2\,x1^6}{180}-\frac{d1\,d2^2\,x2\,x1^7}{15}$$
$$+\frac{7\,d1\,d2^2\,x2\,x1}{50}+\frac{7\,d1^2\,d2^2\,x2\,x1^2}{150}+\frac{7\,d1^3\,d2^2\,x2\,x1^5}{1800}+\frac{7\,d1^3\,d2^2\,x2\,x1^3}{1800}$$
$$+\frac{37\,d2^2\,x2\,x1^4}{30}+\frac{2\,d2^2\,d1\,x2\,x1^5}{5}+\frac{7\,d2^2\,d1\,x2\,x1^3}{10}+\frac{2\,d2\,x1^5\,d1}{5}$$
$$+\frac{7\,d2^2\,d1\,x1^3}{30}-\frac{d2^2\,x1^2}{15}+\frac{37\,d2\,x1^4}{30}+\frac{d2^2\,x1^4}{15}-\frac{d2\,x1^6}{5}+\frac{7\,d1^2\,d2^2}{1800}+\frac{7\,d2^2\,x2}{75}$$
$$-\frac{d2^2\,x2\,x1^6}{5},\,1+\frac{7}{60}\,d1^2-\frac{7}{15}\,d1\,x1-\frac{7}{180}\,x1\,d1^3+\frac{1}{15}\,d1^2\,x1^2-\frac{7}{1800}\,d1^4\Big]$$
$$\Big[-d2\,x1+\frac{19\,d2\,x1^3}{2}-\frac{d2^2\,x2^2\,x1^3}{2}-\frac{13\,d2^2\,x2\,x1}{8}+\frac{5\,d1^2\,d2\,x1^3}{6}+\frac{3\,d1\,d2\,x1^2}{2}$$
$$+\frac{d1\,d2^2\,x1^4}{2}+8\,d1\,d2\,x1^4+\frac{5\,d1\,d2\,x1^6}{4}+\frac{13\,d1\,d2^2\,x1^2}{8}+\frac{5\,d1^2\,d2^2\,x1^3}{6}$$
$$+\frac{13\,d1^2\,d2\,x1^5}{8}+\frac{d1^2\,d2^2\,x1^5}{8}+\frac{d1^2\,d2\,x1^7}{8}+\frac{d1^3\,d2^2\,x1^4}{12}+\frac{d1^3\,d2\,x1^6}{12}$$
$$+\frac{d1^3\,d2\,x1^4}{12}+\frac{d1^2\,d2^2\,x1}{12}-\frac{d2\,x2\,x1^3}{2}+\frac{9\,d2\,x2\,x1^5}{4}+\frac{5\,d1\,d2^2\,x2\,x1^6}{4}$$
$$+\frac{17\,d1\,d2^2\,x2\,x1^4}{2}+\frac{13\,d1\,d2^2\,x2\,x1^2}{8}+\frac{7\,d1^2\,d2^2\,x2\,x1^5}{4}+\frac{5\,d1^2\,d2^2\,x2\,x1^3}{6}$$
$$+\frac{x2\,x1^4\,d1\,d2}{2}+\frac{5\,x2\,x1^6\,d1\,d2}{4}+\frac{5\,x2^2\,x1^6\,d1\,d2^2}{4}+\frac{x2^2\,x1^4\,d1\,d2^2}{2}$$
$$+\frac{x2\,d1^2\,d2\,x1^5}{8}+\frac{d1^2\,d2^2\,x2^2\,x1^5}{8}+\frac{d1^2\,d2^2\,x2\,x1^7}{8}+\frac{9\,d2^2\,x2\,x1^5}{4}-\frac{d1\,d2^2}{6}$$
$$+\frac{9\,d2^2\,x2^2\,x1^5}{4}+\frac{x1^7\,x2\,d1^2\,d2}{8}+\frac{x1^7\,d1^2\,d2^2\,x2^2}{8}+\frac{d1^3\,d2^2\,x2\,x1^6}{12}$$
$$+\frac{d1^3\,d2^2\,x2\,x1^4}{12}-\frac{d2^2\,x1^3}{2}+\frac{9\,d2\,x1^5}{4}-\frac{13\,d2^2\,x1}{8}+9\,d2^2\,x2\,x1^3,-d1$$
$$+\frac{5}{8}\,x2\,x1\,d1^2+\frac{5}{8}\,x1\,d1^2+\frac{1}{6}\,d1^3-x2\,d1-\frac{1}{8}\,x1^2\,x2\,d1^3-\frac{1}{8}\,x1^2\,d1^3-\frac{1}{12}\,x1\,d1^4\Big]$$

Hence, we have:

$$A\,d1^2+A\,d2^2+A\,d1\,d2=Ad1^2+A\,(d2^2+d2^2\,(x1^3+x1^3\,x2+x1^5\,x2)\,d1+(x1^3+x1^5)\,d1\,d2).$$

The next example deals with the third Weyl algebra $A=A_3(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers.

```
> A := DefineOreAlgebra(diff=[d1,x1], diff=[d2,x2], diff=[d3,x3],
> polynom=[x1,x2,x3]):
```

We use the procedure *TwoStrongGenerators* to compute two elements u, v of A such that

$$A\,a + A\,b + A\,c = A\,(a + d\,u\,c) + A\,(b + e\,v\,c),$$

where

```
> a := d1; b := d2; c := d3;
```

$$a := d1$$
$$b := d2$$
$$c := d3$$

and d = e = d1.

```
> G := TwoStrongGenerators(a,b,c,d1,d1,A);
```

$$G := [d1,\ d2 + d1\,(x1^3 + x3\,x1^3 + x3\,x1^5)\,d3 + (3\,x1^2 + 3\,x3\,x1^2 + 5\,x3\,x1^4)\,d3,$$
$$[0,\ x1^3 + (x1^3 + x1^5)\,x3]]$$

Let us check the result. Let us consider the following two matrices:

```
> R := evalm([[a],[b],[c]]);
```

$$R := \begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$$

```
> S := evalm([[G[1]],[G[2]]]);
```

$$S := \begin{bmatrix} d1 \\ d2 + d1\,(x1^3 + x3\,x1^3 + x3\,x1^5)\,d3 + (3\,x1^2 + 3\,x3\,x1^2 + 5\,x3\,x1^4)\,d3 \end{bmatrix}$$

We can check whether or not there exists a (2 x 3) matrix E with entries in A satisfying S = E R:

```
> E := Factorize(S,R,A);
```

$$E := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & x3\,x1^5\,d1 + x3\,x1^3\,d1 + 5\,x3\,x1^4 + x1^3\,d1 + 3\,x3\,x1^2 + 3\,x1^2 \end{bmatrix}$$

Similarly, we can compute a (3 x 2) matrix F with entries in A such that R = F S:

```
> F := Factorize(R,S,A);
```

$F :=$
$[1\,,\,0]$

$$\left[\frac{d3\ x1^3}{2} + \frac{5\ d2\ x1}{8} - \frac{x1^7\ d1^2\ d3\ x3}{8} - \frac{d1\ d2\ x1^2}{8} - \frac{x1^5\ d1^2\ d3}{8} - \frac{9\ d3\ x3\ x1^5}{4}\right.$$
$$+ \frac{d3\ x3\ x1^3}{2} - \frac{d1\ d3\ x1^4}{2} - \frac{d1\ d3\ x3\ x1^4}{2} - \frac{d1^2\ d3\ x3\ x1^5}{8} - \frac{5\ d1\ d3\ x3\ x1^6}{4},$$
$$\left.-\frac{5}{8}\ d1\ x1 + 1 + \frac{1}{8}\ d1^2\ x1^2\right]$$

$$\left[- d3\ x1 - \frac{d1\ d2}{6} + \frac{7\ d1^3\ d3\ x3\ x1^4}{24} + \frac{d1^3\ d3\ x3\ x1^6}{12} + \frac{d1^3\ d3\ x1^4}{12} + \frac{d1^2\ d2\ x1}{12}\right.$$
$$+ \frac{d1\ d3}{2} + \frac{d1^2\ d3\ x1}{2} + \frac{d1^3\ d3\ x1^2}{8} + \frac{d1^4\ d3\ x1^3}{120} + 10\ d3\ x3\ x1^3 + 4\ d3\ x3\ x1$$
$$+ \frac{3\ d1\ d3\ x1^2}{2} + \frac{5\ d1^2\ d3\ x1^3}{6} + \frac{d1^3\ d2}{120} + \frac{d1\ d3\ x3}{2} + \frac{d1^2\ d3\ x3\ x1}{2}$$
$$+ \frac{d1^3\ d3\ x3\ x1^2}{8} + \frac{d1^4\ d3\ x3\ x1^3}{120} + \frac{d1^4\ d3\ x3\ x1^5}{120} + \frac{13\ d1\ d3\ x3\ x1^2}{2}$$
$$+ \frac{15\ d1\ d3\ x3\ x1^4}{2} + \frac{5\ d1^2\ d3\ x3\ x1^3}{2} + \frac{3\ d1^2\ d3\ x3\ x1^5}{2},$$
$$\left.-\frac{1}{120}\ d1^4 + \frac{1}{6}\ d1^2 - \frac{1}{12}\ x1\ d1^3\right]$$

We conclude that

$$A\,d1 + A\,d2 + A\,d3 = A\,d1 + A\,(d2 + (x1^3 + x1^3\,x3 + x1^5\,x3)\,d1\,d3 + (3\,x1^2 + 3\,x1^2\,x3 + 5\,x1^4\,x3)\,d3).$$