```
> with(OreModules):
> with(Stafford):
> Alg:=DefineOreAlgebra(diff=[D1,x1],diff=[D2,x2],polynom=[x1,x2]):
```

In 2-dimensional linear elasticity, in the case without force, it is known that the stress tensor can be parametrized by means of the Airy function and the following differential operator:

```
> R:=evalm([[D1^2],[D1*D2],[D2^2]]);
```

$$
R:=\left[\begin{array}{c}
\mathrm{D} 1^{2} \\
\mathrm{D} 1 \mathrm{D} 2 \\
\mathrm{D} 2^{2}
\end{array}\right]
$$

The corresponding linear system of partial differential equations is then defined by:

```
> ApplyMatrix(R,[lambda(x1,x2)],Alg)=evalm([[0]$3]);
```

$$
\left[\begin{array}{c}
\frac{\partial^{2}}{\partial x 1^{2}} \lambda(x 1, x 2) \\
\frac{\partial^{2}}{\partial x 2 \partial x 1} \lambda(x 1, x 2) \\
\frac{\partial^{2} x}{\partial x 2^{2}} \lambda(x 1, x 2)
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

For more details, see for instance J.-F. Pommaret, Partial Differential Control Theory, Vol. II: Control Systems, Kluwer Academic Press, 2001, and the references therein.

Over the Weyl algebra $A_{-} n(k)$, where $k$ is a field of characteristic 0 , a well-known result due to J. T. Stafford asserts that there always exists a system equivalent to $R \lambda=0$ which is defined by means of only two equations. Moreover, the two differential operators appearing in these two equations can be chosen of the form $R[1,1]+a^{*} R[3,1]$ and $R[2,1]+b^{*} R[3,1]$, where $a$ and $b$ are two elements of $A \_n(k)$. We can compute them as follows:

```
> st:=time(): Gen := TwoGenerators(R[1,1], R[2,1], R[3,1], Alg); time()-st;
    Gen }:=[\textrm{D}\mp@subsup{1}{}{2},\textrm{D}1\textrm{D}2+(x\mp@subsup{1}{}{3}x2+x\mp@subsup{1}{}{2}+x\mp@subsup{1}{}{5})\textrm{D}\mp@subsup{2}{}{2},[0,x\mp@subsup{1}{}{3}x2+x\mp@subsup{1}{}{2}+x\mp@subsup{1}{}{5}]
```

The first two elements in Gen are exactly the two differential operators that we want and the two elements of Gen $[3]$ are the differential operators $a$ and $b$. Hence, if we form the matrix $S$ defined by the first two entries of $g$, we obtain that the system $R \lambda=0$ is equivalent to:

$$
\begin{aligned}
&>S:=\operatorname{evalm}([[\operatorname{Gen}[1]],[\operatorname{Gen}[2]]]): \quad \text { ApplyMatrix }(\mathrm{S},[\operatorname{lambda}(\mathrm{x} 1, \mathrm{x} 2)], \mathrm{Alg})=\operatorname{evalm}([[0] \$ 2]) ; \\
& {\left[\frac{\partial^{2}}{\partial x 1^{2}} \lambda(x 1, x 2)\right] } \\
& {\left[\left(\frac{\partial^{2}}{\partial x 2^{2}} \lambda(x 1, x 2)\right) x 1^{3} x 2+\left(\frac{\partial^{2}}{\partial x 2^{2}} \lambda(x 1, x 2)\right) x 1^{2}+\left(\frac{\partial^{2}}{\partial x 2^{2}} \lambda(x 1, x 2)\right) x 1^{5}\right.} \\
&\left.+\left(\frac{\partial^{2}}{\partial x 2 \partial x 1} \lambda(x 1, x 2)\right)\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
\end{aligned}
$$

Let us check the equivalence between the two systems $R \lambda=0$ and $S \lambda=0$. In order to do that, we compute a Groebner basis for the left $A l g$-ideals defined by the entries of $R$ and $S$ respectively and compare them. In order to do that, we introduce the following command Gbasis defined by:

```
> Gbasis:=proc(L,Alg::list)
> local i,lambda,G,Algc,Ord,vl,vr;
> vl:=Alg[3];
> vr:=Alg[4];
> Algc:=Ore_algebra[diff_algebra](seq([vr[i],vl[i]],i=1..nops(vl)),
> polynom=[lambda],comm=[lambda]);
> Ord:=Groebner[termorder](Algc,tdeg(lambda,op(vr)));
> G:=Groebner[gbasis](map(i->i*lambda,L),Ord);
> map(i->simplify(i/lambda),G);
> end:
```

Then, we obtain

```
> Gbasis([Gen[1],Gen[2]],Alg);
```

$$
\left[\mathrm{D} 2^{2}, \mathrm{D} 1 \mathrm{D} 2, \mathrm{D} 1^{2}\right]
$$

and

```
> Gbasis([R[1,1],R[2,1],R[3,1]],Alg);
```

                                    \(\left[\mathrm{D} 2^{2}, \mathrm{D} 1 \mathrm{D} 2, \mathrm{D} 1^{2}\right]\)
    which proves that the two systems $R \lambda=0$ and $S \lambda=0$ are equivalent.
Let us consider Example 1.15 in page 802 of J.-F. Pommaret, Partial Differential Control Theory, Vol. II: Control Systems, Kluwer Academic Press, 2001. The system of partial differential equations is defined over the Weyl algebra $A_{-4}$, i.e.:

```
> Alg2 := DefineOreAlgebra(diff=[D1,x1], diff=[D2,x2], diff=[D3,x3], diff=[D4,x4],
> polynom=[x1,x2,x3,x4]):
```

The matrix of operators which defines the system is given by:

$$
\begin{array}{r}
>\mathrm{R} 2:=\operatorname{evalm}([[\mathrm{D} 4-\mathrm{x} 3 * \mathrm{D} 2-1],[\mathrm{D} 3-\mathrm{x} 4 * \mathrm{D} 1],[\mathrm{D} 2-\mathrm{D} 1]]) ; \\
\\
R 2:=\left[\begin{array}{c}
\mathrm{D} 4-x 3 \mathrm{D} 2-1 \\
\mathrm{D} 3-x 4 \mathrm{D} 1 \\
\mathrm{D} 2-\mathrm{D} 1
\end{array}\right]
\end{array}
$$

In terms of equations, we have:

```
> x:=x1,x2,x3,x4:
> ApplyMatrix(R2,[y(x)],Alg2)=evalm([[0]$3]);
    [ - y(x1,x2,x3,x4)+(\frac{\partial}{\partialx4}\textrm{y}(x1,x2,x3,x4))-x3(\frac{\partial}{\partialx2}\textrm{y}(x1,x2,x3,x4))}+\mp@code{c
```

The famous theorem of Stafford states that the previous system is equivalent to a system defined by only two equations, i.e., the left ideal $I=A \lg 2 R[1,1]+\operatorname{Alg} 2 R[2,1]+\operatorname{Alg} 2 \mathrm{R}[3,1]$ can be generated by only two differential operators. Let us find such a pair.

```
> st:=time(): Gen2 := TwoGeneratorsRat(R2[1,1], R2[2,1], R2[3,1], Alg2); time()-st;
    Gen2 := [D4 - x3 D2 - 1, D3 - x4 D1, [0, 0]]
    0.111
```

The last result means that the system is in fact equivalent to the one formed by the first two equations. In particular, this means that the first operator $\mathrm{D} 2-\mathrm{D} 1$ is a left $A l g$-linear combination of $R[1,1]$ and $R[2,1]$. Let us check it:

```
> F:=Factorize(evalm([[R2[3,1]]]),linalg[submatrix](R2,1..2,1..1),Alg2);
    F:=[ [x4 D1-D3 D4-x3 D2-1]
```

Hence, we have $\mathrm{D} 2-\mathrm{D} 1=F(R[1,1], R[2,1])^{\wedge} \mathrm{T}$, which proves the last statement. Equivalently, we can also use the command Gbasis introduced above:

```
> Gbasis([seq(R2[i,1],i=1..3)],Alg2);
    [x3 D3 - x4 D4 + x4, -D4 + x3 D2 + 1, x3 D1 - D4 + 1]
> Gbasis([Gen2[1],Gen2[2]],Alg2);
    [x3 D3-x4 D4 + x4, -D4 + x3 D2 + 1, x3 D1 - D4 + 1]
```

Let us now consider the gradient operator in three-dimensional space. It is defined by the following matrix of differential operators:

```
> Alg3 := DefineOreAlgebra(diff=[D1,x1], diff=[D2,x2], diff=[D3,x3],
> polynom=[x1,x2,x3]):
> R3 := evalm([[D1],[D2],[D3]]);
```

$$
R 3:=\left[\begin{array}{l}
\mathrm{D} 1 \\
\mathrm{D} 2 \\
\mathrm{D} 3
\end{array}\right]
$$

The corresponding system of partial differential equations is then defined by:

```
> ApplyMatrix(R3,[y(x1,x2,x3)],Alg3)=evalm([[0]$3]);
```

$$
\left[\begin{array}{l}
\frac{\partial}{\partial x^{1}} \mathrm{y}(x 1, x 2, x 3) \\
\frac{\partial}{\partial x^{2}} \mathrm{y}(x 1, x 2, x 3) \\
\frac{\partial}{\partial x 3} \mathrm{y}(x 1, x 2, x 3)
\end{array}\right]=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]
$$

Let us compute an equivalent system with only two equations over the Weyl algebra $A_{-} 3$ :

```
> Gen3 := TwoGenerators(R3[1,1], R3[2,1], R3[3,1], Alg3);
\[
\text { Gen3 }:=\left[\mathrm{D} 1, \mathrm{D} 2+\left(x 1^{2} x 3+x 1^{3}\right) \mathrm{D} 3,\left[0, x 1^{2} x 3+x 1^{3}\right]\right]
\]
```

Therefore, an equivalent system of partial differential equations is defined by:

```
> S3:=evalm([[Gen3[1]],[Gen3[2]]]);
```

$$
S 3:=\left[\begin{array}{c}
\mathrm{D} 1 \\
\mathrm{D} 2+\left(x 1^{2} x 3+x 1^{3}\right) \mathrm{D} 3
\end{array}\right]
$$

> ApplyMatrix(S3,[y(x1,x2,x3)],Alg3)=evalm([[0]\$2]);

$$
\left[\begin{array}{c}
\frac{\partial}{\partial x 1} \mathrm{y}(x 1, x 2, x 3) \\
\left(\frac{\partial}{\partial x 2} \mathrm{y}(x 1, x 2, x 3)\right)+\left(\frac{\partial}{\partial x 3} \mathrm{y}(x 1, x 2, x 3)\right) x 1^{2} x 3+\left(\frac{\partial}{\partial x 3} \mathrm{y}(x 1, x 2, x 3)\right) x 1^{3}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

Indeed, let us check that both system have the same Groebner basis using the command Gbasis:

```
> Gbasis([seq(R3[i,1],i=1..3)],Alg3);
```

[D3, D2, D1]
[D3, D2, D1]
In particular, there exists a matrix X3 over $A \_3$ which satisfies $R 3=$ X3 S3:

```
> X3:=Factorize(R3,S3,Alg3);
```

X3 :=

$$
[1,0]
$$

$$
\left[\frac{\mathrm{D} 1 \mathrm{D} 3 x 1^{4} x 3}{9}+\frac{x 3^{2} \mathrm{D} 1 \mathrm{D} 3 x 1^{3}}{9}+\frac{\mathrm{D} 3 x 1^{4}}{3}+\frac{8 \mathrm{D} 3 x 1^{3} x 3}{9}+\frac{\mathrm{D} 3 x 1^{2} x 3^{2}}{3}\right.
$$

$$
\left.+\frac{x 1 x 3 \mathrm{D} 1 \mathrm{D} 2}{9}+\frac{\mathrm{D} 2 x 1}{3}-\frac{x 3 \mathrm{D} 2}{9},-\frac{1}{9} x 1 x 3 \mathrm{D} 1^{2}-\frac{1}{3} \mathrm{D} 1 x 1+\frac{1}{9} x 3 \mathrm{D} 1+1\right]
$$

$$
\left[-3 \mathrm{D} 3 x 1-\frac{3 \mathrm{D} 3 x 1^{2} \mathrm{D} 1}{2}-\frac{\mathrm{D} 1^{2} \mathrm{D} 3 x 1^{3}}{6}-\frac{\mathrm{D} 1^{2} \mathrm{D} 3 x 1^{2} x 3}{6}-\frac{\mathrm{D} 2 \mathrm{D} 1^{2}}{6}\right.
$$

$$
\left.-x 3 \mathrm{D} 3 x 1 \mathrm{D} 1-x 3 \mathrm{D} 3, \frac{\mathrm{D} 1^{3}}{6}\right]
$$

Similarly, there exists a matrix $Y 3$ over $A \_3$ which satisfies $S 3=Y 3 R 3$ :

```
> Y3:=Factorize(S3,R3,Alg3);
```

$$
Y 3:=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & x 1^{2} x 3+x 1^{3}
\end{array}\right]
$$

Finally, we trivially check that the solutions of the gradient operator $R 3 y=0$ are constant, a fact which can be checked as follows:

```
> pdsolve({seq(ApplyMatrix(R3[i,1],[y(x1,x2,x3)],Alg3)[1]=0,i=1..3)}, {y(x1,x2,x3)});
    {y (x1, x2, x3) = _C1}
```

Hence, we deduce that the solution $S 3 y=0$ must be the same, as we can check:

```
> pdsolve({seq(ApplyMatrix(S3[i,1],[y(x1,x2,x3)],Alg3)[1]=0,i=1..2)}, {y(x1,x2,x3)});
    {y(x1,x2,x3) = _C1}
```

Finallly, let us consider the Example 2 of A. Leykin, Algorithmic proofs of two theorems of Stafford, Journal of Symbolic Computation 38 (2004), 1535-1550, defined by the following three differential operators:

```
> P1:=D1*D3^2; P2:=D1*D2; P3:=D2*D3^2;
\[
\begin{aligned}
P 1 & :=\mathrm{D} 1 \mathrm{D} 3^{2} \\
P 2 & :=\mathrm{D} 1 \mathrm{D} 2 \\
P 3 & :=\mathrm{D} 2 \mathrm{D} 3^{2}
\end{aligned}
\]
```

By Stafford's result, we know that the left ideal $J$ of $\operatorname{Alg} 3$ generated by the three previous operators can be generated by only two elements of Alg3. Let us compute some pairs:
$>\mathrm{G}:=$ TwoGenerators(P1, P2, P3, Alg3);

$$
G:=\left[\mathrm{D} 1 \mathrm{D} 3^{2}, \mathrm{D} 1 \mathrm{D} 2+\left(x 1 x 3^{2}+x 1^{2} x 3+x 1^{3}\right) \mathrm{D} 2 \mathrm{D} 3^{2},\left[0, x 1 x 3^{2}+x 1^{2} x 3+x 1^{3}\right]\right]
$$

Therefore, the left ideal $J$ of $\operatorname{Alg} 3$ generated by $P 1, P 2$ and $P 3$ is also generated by $G[1]$ and $G[2]$. Let us verify this result by computing Groebner bases of $J$ and the left ideal of $A l g 3$ defined by $G[1]$ and $G[2]$ and checking whether they define the same Groebner basis. We get:

```
> Gbasis([P1,P2,P3],Alg3); Gbasis([G[1],G[2]],Alg3);
    [D1D2, D2 D3 ', D1 D3 }\mp@subsup{}{}{2}\mathrm{ ]
    [D1 D2, D2 D3 }\mp@subsup{}{}{2}, D1 D3 2]
```

Moreover, the left ideal $J$ of $\operatorname{Alg} 3$ can also be generated by $G 2[1]$ and $G 2[2]$ defined by:

```
> G2 := TwoGenerators(P3, P1, P2, Alg3);
    G2 := [D2 D3 2},\textrm{D}1\textrm{D}\mp@subsup{3}{}{2}+(x\mp@subsup{3}{}{2}x2+x3+x\mp@subsup{3}{}{4})\textrm{D}1\textrm{D}2,[0,x\mp@subsup{3}{}{2}x2+x3+x\mp@subsup{3}{}{4}]
```

Let us check this result by computing a Groebner basis of the left ideal of $\operatorname{Alg} 3$ generated by G2[1] and G2[2]:

```
> Gbasis([G2[1],G2[2]],Alg3);
```

[D1 D2, D2 D3 $\left.{ }^{2}, ~ D 1 D 3^{2}\right]$
Finally, $J$ can also be generated by $G 3[1]$ and $G 3[2]$ defined by:

```
> G3:=TwoGeneratorsRat(P2, P3, P1, Alg3);
```

$$
G 3:=\left[\mathrm{D} 1 \mathrm{D} 2, \mathrm{D} 2 \mathrm{D} 3^{2}+\left(x 1 x 2+x 2^{2}\right) \mathrm{D} 1 \mathrm{D} 3^{2},\left[0, x 1 x 2+x 2^{2}\right]\right]
$$

Indeed, we have:

```
> Gbasis([G3[1],G3[2]],Alg3);
```

$\left[\mathrm{D} 1 \mathrm{D} 2, \mathrm{D} 2 \mathrm{D} 3^{2}, \mathrm{D} 1 \mathrm{D} 3^{2}\right]$
Finally we give a short remark on the way in which two generators of a given left ideal are found. Let a left ideal of the Weyl algebra $A_{-} n$ be given by three generators $a, b, c$. The algorithm presented in A. Leykin, Algorithmic proofs of two theorems of Stafford, Journal of Symbolic Computation 38 (2004), 1535-1550, essentially computes, for a given integer $0 \leq r \leq n$, some elements $q_{-} r, d_{-} r, e_{-} r$ in the Weyl algebra $A \_n$ such that:

1. q_r contains only the last $r$ indeterminates $D_{-} i$ of the Weyl algebra (and the last $r$ indeterminates $x_{\_} i$, depending on whether the Weyl algebra is considered with polynomial or rational coefficients),
2. $\quad q_{\_} r c$ is a left $A_{-} n$-linear combination of $\left(a+d \_r c\right)$ and $\left(b+e \_r c\right)$, i.e., $q_{-} r c$ is in the left ideal generated by $\left(a+d \_r c\right)$ and $\left(b+e \_r c\right)$.

This algorithm is applied for descreasing $r$, i.e. the program has to eliminate more and more variables in $q_{-} r$ in order to finally achieve a representation of $q_{-} 0 c$ as left $A_{-} n$-linear combination of $\left(a+d_{-} 0 c\right)$ and ( $b+e_{-} 0 c$ ), where $q_{-} 0$ is invertible in the Weyl algebra. Then the last two elements are the two generators we were looking for. The process of elimination alluded to above is very difficult to perform in general. The present implementation uses some heuristics to speed up this computation, but note that small changes to the input may lead to very different computation times.

