

```

> with(OreModules):
> with(Stafford):
> Alg := DefineOreAlgebra(diff=[D,t], polynom=[t]):
```

Let us consider the ordinary differential linear control system defined by the following matrix of operators:

```

> R:=evalm([[D, 0, -t, 0],[0, D, 0, -1]]);
```

$$R := \begin{bmatrix} D & 0 & -t & 0 \\ 0 & D & 0 & -1 \end{bmatrix}$$

We then obtain the following system:

```

> ApplyMatrix(R,[x[1](t),x[2](t),u[1](t),u[2](t)],Alg)=evalm([[0]$2]);
\left[\begin{array}{c} \left(\frac{d}{dt}x_1(t)\right)-tu_1(t) \\ \left(\frac{d}{dt}x_2(t)\right)-u_2(t) \end{array}\right]=\left[\begin{array}{c} 0 \\ 0 \end{array}\right]
```

Let us check whether or not the time-varying linear control system is flat, namely, whether or not the left Alg -module $M = Alg^{\{1*4\}}/(Alg^{\{1*2\}} R)$ is free.

In order to do that, we can first check whether the full row rank matrix R admits a right-inverse T :

```

> T:=RightInverse(R,Alg);
> Mult(R,T,Alg);
T := \begin{bmatrix} t & 0 \\ 0 & 0 \\ D & 0 \\ 0 & -1 \end{bmatrix}
```

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As R admits a right-inverse, we obtain that the left Alg -module M is projective, and thus, stably free. Let us compute the rank of M over Alg :

```

> OreRank(R,Alg);
2
```

As the rank of M over Alg is 2, we know that M is a free left Alg -module due to a classical result of J. T. Stafford. Let us compute a basis of the left Alg -module M .

In order to do that, we can first compute an injective parametrization of the system $R(x_1, x_2, u_1, u_2)^T = 0$:

```

> st:=time(): Q:=InjectiveParametrization(R,Alg); time()-st;
Q := \begin{bmatrix} 1-tD+t^2D^2+Dt^2 & -t^3D-t^3 \\ -D & t \\ D^2t+2D+D^2+D^3t & -t^2D^2-3tD-3t-Dt^2 \\ -D^2 & 1+tD \end{bmatrix}
```

0.500

Hence, all the solutions of $R(x_1, x_2, u_1, u_2)^T = 0$ in any left A_1 -module F (e.g., smooth functions) are parametrized by means of B , i.e., we have:

```

> evalm([[x[1](t)], [x[2](t)], [u[1](t)], [u[2](t)]])
> =ApplyMatrix(Q, [xi[1](t), xi[2](t)], Alg);


$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} =$$


$$\begin{bmatrix} \xi_1(t) - (\frac{d}{dt} \xi_1(t)) t + (\frac{d}{dt} \xi_1(t)) t^2 + t^2 \%1 - t^3 \xi_2(t) - t^3 (\frac{d}{dt} \xi_2(t)) \\ [-(\frac{d}{dt} \xi_1(t)) + t \xi_2(t)] \\ [2 (\frac{d}{dt} \xi_1(t)) + \%1 t + \%1 + t (\frac{d^3}{dt^3} \xi_1(t)) - 3 t \xi_2(t) - 3 t (\frac{d}{dt} \xi_2(t)) - (\frac{d}{dt} \xi_2(t)) t^2 \\ - t^2 (\frac{d^2}{dt^2} \xi_2(t))] \\ [-\%1 + \xi_2(t) + t (\frac{d}{dt} \xi_2(t))] \\ \%1 := \frac{d^2}{dt^2} \xi_1(t) \end{bmatrix}$$


```

Let us check that Q is a parametrization of the system $R(x_1, x_2, u_1, u_2)^T = 0$ by computing the syzygy module of the left Alg -module generated by the rows of Q .

```

> SyzygyModule(Q,Alg);

$$\begin{bmatrix} D & 0 & -t & 0 \\ 0 & D & 0 & -1 \end{bmatrix}$$


```

As the syzygy module of Q is generated by the rows of R , we find that Q is a parametrization of the system $R(x_1, x_2, u_1, u_2)^T = 0$. Let us now check whether or not this parametrization is injective, i.e., $Q(\xi_1, \xi_2)^T = 0$ implies $(\xi_1, \xi_2)^T = (0, 0)^T$.

```

> B:=LeftInverse(Q,Alg);

$$B := \begin{bmatrix} 1 & -t + t^2 & 0 & t^2 \\ 0 & 2 & 1 & tD + t + 1 \end{bmatrix}$$


```

As Q admits a left-inverse B , we deduce that $(x_1, x_2, u_1, u_2)^T = Q(\xi_1, \xi_2)^T$ is an injective parametrization of the system $R(x_1, x_2, u_1, u_2)^T = 0$.

```

> evalm([[xi[1](t)], [xi[2](t)]])=ApplyMatrix(B,[x[1](t),x[2](t),u[1](t),u[2](t)],Alg);

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_2(t)t + x_2(t)t^2 + t^2 u_2(t) \\ 2x_2(t) + u_1(t) + u_2(t)t + u_2(t) + t(\frac{d}{dt} u_2(t)) \end{bmatrix}$$


```

$(\xi_1, \xi_2)^T$ is then called a flat output of the system $R(x_1, x_2, u_1, u_2)^T = 0$. In terms of the associated module, this previous result shows that the residue classes of the rows of B form a basis of the left Alg -module M . This last result can be directly obtained by:

```

> BasisOfModule(R,Alg);

$$\begin{bmatrix} 1 & -t + t^2 & 0 & t^2 \\ 0 & 2 & 1 & tD + t + 1 \end{bmatrix}$$


```

The command *InjectiveParametrization* uses an heuristic which generally speeds up the computations, which is particularly helpful in the case of partial differential equations.

However, we can also use the straightforward algorithm described in A. Quadrat and D. Robertz, *Constructive computation of bases of free modules over the Weyl algebra*; INRIA Report 5786, 2005.

This can be done by using the command *InjectiveParametrization2*. We then obtain:

```

> Qbis:=InjectiveParametrization2(R,Alg);
Qbis :=

$$[-8Dt^2 + 2t^2 - 1 + 7t^2D^2 + tD + 2D^3t^3 - 3t^3D^2 + t^3D,$$


$$D^3t - D^2 + 2D^4t^2 - 3D^3t^2 - D^2t + D + t^2D^2]$$


$$[t + 2Dt^2 - t^2, -D + 2D^2t + 1 - tD]$$


$$[4 - 17D^2t - 16D + 5tD + 2D^4t^2 + 13D^3t + 15D^2 - 3D^3t^2 + t^2D^2,$$


$$-7D^3 + 2D^5t + 5D^4 - 3D^4t + 2D^2 + D^3t]$$


$$[1 + 2t^2D^2 + 5tD - 2t - Dt^2, 2D^3t + D^2 - D^2t]$$


```

We check that $Qbis$ is another injective parametrization of the system R (x_1, x_2, u_1, u_2) $^T=0$ as we have:

```

> SyzygyModule(Qbis,Alg);

$$\begin{bmatrix} D & 0 & -t & 0 \\ 0 & D & 0 & -1 \end{bmatrix}$$

> Bbis:=LeftInverse(Qbis,Alg);
Bbis := 
$$\begin{bmatrix} 0 & 0 & 2tD - t + 3 & -2D^3t + 3D^2t - 3D^2 - tD + 5D - 1 \\ t + 1 & 1 & -2t^2 & 2t^2D^2 - 3Dt^2 - tD + t^2 + 1 \end{bmatrix}$$


```

Equivalently, the residue classes of the rows of $Bbis$ define a basis of the left Alg -module M .

Using the algorithms described in F. Chyzak, A. Quadrat, D. Robertz, *Effective algorithms for parametrizing linear control systems over Ore algebras*, Applicable Algebra in Engineering, Communications and Computing (AAECC), 16 (2005), 319-376, we can obtain the following minimal parametrization P of the system $R(x_1, x_2, u_1, u_2)^T = 0$:

```

> P:=MinimalParametrization(R,Alg);
P := 
$$\begin{bmatrix} -t^2 & 0 \\ 0 & -1 \\ -2 - tD & 0 \\ 0 & -D \end{bmatrix}$$


```

which is not injective over Alg as we have

```

> LeftInverse(P,Alg);

$$\square$$


```

but which is injective if we allow rational coefficients in t , i.e., P admits a left-inverse over the algebra B_1 of the ordinary differential operators with rational coefficients in t :

```

> BRat:=LeftInverseRat(P,Alg);
BRat := 
$$\begin{bmatrix} -\frac{1}{t^2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$


```

Hence, P defines an injective parametrization of all the solutions of the system $R(x_1, x_2, u_1, u_2)^T = 0$ over any left B_1 -module (e.g., rational functions, meromorphic functions), i.e., we have $R(x_1, x_2, u_1, u_2)^T = 0 \iff (x_1, x_2, u_1, u_2)^T = P(\zeta_1, \zeta_2)^T$.

Moreover, the residue classes of the rows of the matrix $BRat$ define a basis of the left B_1 -module $M_{Rat}=B_1^{\{1*4\}}/(B_1^{\{1*2\}} R)$ but we point out the existence of a singularity at $t = 0$. Hence, despite

the simplicity of the parametrization P , the first two parametrizations of the system $R(x_1, x_2, u_1, u_2)^T = 0$ are more relevant when we want to study the system in a neighbourhood of $t = 0$.

We note that the algorithms which compute bases of free modules over the Weyl algebras are very sensitive with respect the permutation of the rows of the presentation matrix.

Let us illustrate this fact by computing a basis of the previous system but where we have permuted the two rows of R , i.e.:

```
> R1:=evalm([[0, D, 0, -1],[D, 0, -t, 0]]);  
R1 := 
$$\begin{bmatrix} 0 & D & 0 & -1 \\ D & 0 & -t & 0 \end{bmatrix}$$

```

Let us compute an injective parametrization of the system defined by $R1$.

```
> st:=time(): Q1:=InjectiveParametrization(R1,Alg); time()-st;  
Q1 := 
$$\begin{bmatrix} -t^2 D^2 + t^2 D + t^3 D & -D + D^2 t + 1 - t D - t^2 D \\ -D - D^2 t + 1 + t D + t^2 D & D^2 - D - t D \\ -2 D^2 + 3 t D - t D^3 + D^2 t + 2 D + t^2 D^2 & D^3 - D^2 - D^2 t - 2 D \\ -2 D^2 + 2 t D - t D^3 + D^2 t + 2 D + t^2 D^2 & D^3 - D^2 - D^2 t - D \end{bmatrix}$$
  
0.630
```

We easily check that $Q1$ is a parametrization of the system defined by $R1$, i.e., by R as we have:

```
> SyzygyModule(Q1,Alg);  

$$\begin{bmatrix} D & 0 & -t & 0 \\ 0 & D & 0 & -1 \end{bmatrix}$$

```

Moreover, $Q1$ is injective as it admits a left-inverse defined by:

```
> LeftInverse(Q1,Alg);  

$$\begin{bmatrix} 0 & 1 & D - t - 1 & -D + t + 1 \\ 1 & 0 & t D - t^2 - t - 1 & -t D + t^2 + t + 1 \end{bmatrix}$$

```

The rows of the left-inverse of $Q1$ then define a basis of the left Alg -module M as we can check by computing a basis:

```
> B1:=BasisOfModule(R1,Alg);  
B1 := 
$$\begin{bmatrix} 0 & 1 & D - t - 1 & -D + t + 1 \\ 1 & 0 & t D - t^2 - t - 1 & -t D + t^2 + t + 1 \end{bmatrix}$$

```

We can also use the straightforward algorithm explained in A. Quadrat and D. Robertz, *Constructive computation of bases of free modules over the Weyl algebra*; INRIA Report 5786, 2005, in order to compute another injective parametrization and basis of M :

```
> st:=time(): Q1bis:=InjectiveParametrization2(R1,Alg); time()-st;  
Q1bis := 
$$\begin{bmatrix} t^2 & 1 - t D \\ t^2 + t & 1 - t D - D \\ 2 + t D & -D^2 \\ t^2 D + 2 t + 1 + t D & -D^2 t - D^2 \end{bmatrix}$$
  
0.471
```

```

> LeftInverse(Q1bis,Alg);

$$\begin{bmatrix} 0 & 0 & t+1 & -1 \\ t+1 & -t & 0 & 0 \end{bmatrix}$$

> B1bis:=BasisOfModule2(R1,Alg);

$$B1bis := \begin{bmatrix} 0 & 0 & t+1 & -1 \\ t+1 & -t & 0 & 0 \end{bmatrix}$$


```

See Example 14 of A. Quadrat and D. Robertz, *Constructive computation of bases of free modules over the Weyl algebra*; INRIA Report 5786, 2005, for the explicit computations.

We note that, in this particular case, *InjectiveParametrization2* gives smaller injective parametrization and basis of M than the one returned by *InjectiveParametrization*.

However, the reader must keep in mind that it is generally not the case and the command *InjectiveParametrization* is really faster than *InjectiveParametrization2*. This is mainly due to the fact that *InjectiveParametrization* avoids as much as possible the time-consuming computations of two generators of ideals, which is not the case of *InjectiveParametrization2*.

Let us consider the example of a time-varying control linear system defined by means of the following matrix of ordinary differential operators with polynomial coefficients in t :

```

> R2:=evalm([[D, 0, -t, 0],[0, D, 0, -t]]); 

$$R2 := \begin{bmatrix} D & 0 & -t & 0 \\ 0 & D & 0 & -t \end{bmatrix}$$


```

The system is then defined by the following equations:

```

> ApplyMatrix(R2,[x[1](t),x[2](t),u[1](t),u[2](t)],Alg)=evalm([[0]$2]);

$$\begin{bmatrix} (\frac{d}{dt}x_1(t)) - tu_1(t) \\ (\frac{d}{dt}x_2(t)) - tu_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$


```

Let us check whether or not the previous time-varying linear control system is flat, i.e., whether or not the left Alg -module $M2=Alg^{\{1*4\}}/(Alg^{\{1*2\}}R2)$ is free. In order to do that, we can first test whether $M2$ is projective, i.e., a stably free left Alg -module.

```

> RightInverse(R2,Alg);

$$\begin{bmatrix} t & 0 \\ 0 & t \\ D & 0 \\ 0 & D \end{bmatrix}$$


```

The fact that $R2$ admits a right-inverse implies that $M2$ is a stably free left Alg -module. We can also compute its rank over Alg :

```
> OreRank(R2,Alg);
```

2

Using a result due to J. T. Stafford, we then obtain that $M2$ is a free left Alg -module, as $M2$ is a stably free left Alg -module of rank 2. Let us compute an injective parametrization of the system $R2(x_1, x_2, u_1, u_2)^T = 0$:

```
> st:=time(): Q2:=InjectiveParametrization(R2,Alg); time()-st;
```

$$\begin{aligned}
Q2 := & \\
& [1 - tD + D^2t - D + t^2D^2 + D^3t^3 + D^3t^2 - D^2t^4 - 3t^3D - t^5D^2 - 4t^4D - t^3D^2 \\
& - 2Dt^2, D^2t - D + D^2 + t^2D^2 - D^3t - D^3t^2 + D^2t^4 + 3t^3D + t^3D^2 + 2Dt^2] \\
& [-t^2D^2 + Dt^2 + t^4D, -D + D^2t + 1 - tD - t^3D] \\
& [4D^3t + 3D^3 - 9t^3D^2 - 16Dt^2 - 7t^2D^2 - 9tD + D^4t^2 + D^4t - D^3t^2 - 5D^2t \\
& - 4D - t^4D^3 - D^3t^3 + D^2, \\
& -D^4t - D^3 - D^4 + D^3t + 2D^2 + D^3t^3 + 7t^2D^2 + 9tD + D^3t^2 + 5D^2t + 4D] \\
& [-2D^2 + 4Dt^2 - D^3t + D^2t + 2D + t^3D^2, D^3 - D^2 - t^2D^2 - 3tD]
\end{aligned}$$

0.640

We then check that $Q2$ is a parametrization by showing that the syzygy module of $Q2$ is generated by the rows of the matrix $R2$:

```

> SyzygyModule(Q2,Alg);

```

$$\left[\begin{array}{cccc} D & 0 & -t & 0 \\ 0 & D & 0 & -t \end{array} \right]$$

Moreover, the parametrization $Q2$ admits a left-inverse $B2$ defined by:

```

> B2:=LeftInverse(Q2,Alg);

```

$$B2 := \left[\begin{array}{cc} 1 & 0 \quad t^2 - D + 1 & t^3D - D^2t + Dt^2 - D^2 + tD + 3t^2 - 2D + t + 2 \\ 0 & 1 \quad t^3 - tD + t + 1 & t^4D - t^2D^2 + t^3D - D^2t + Dt^2 + 2t^3 - tD + D + 2t + 2 \end{array} \right]$$

Hence, we obtain

$$R2(x_1, x_2, u_1, u_2)^T = 0 \iff (x_1, x_2, u_1, u_2)^T = Q2(\xi_1, \xi_2)^T$$

and $(\xi_1, \xi_2)^T = B2(x_1, x_2, u_1, u_2)^T$. Finally, the residue classes of the rows of the matrix $B2$ define a basis of the left Alg -module $M2$.

Now, using *InjectiveParametrization2*, we obtain the following injective parametrization of the system $R2(x_1, x_2, u_1, u_2)^T = 0$

```

> st:=time(): Q2bis:=InjectiveParametrization2(R2,Alg); time()-st;

```

```

Q2bis :=


$$\left[ -8D^2t^3 - t^2D - 4D^3t^3 - 6t^2D^2 - \frac{9}{2}t^4D^3 - \frac{1}{2}D^4t^4 + \frac{7}{2}t^4D^2 + 4t^3D - \frac{1}{2}D^4t^5 \right.$$


$$+ \frac{1}{2}D^3t^5 + t^2,$$


$$\left. -\frac{1}{2}D^5t^2 - \frac{1}{2}D^5t^3 - \frac{1}{2}D^4t^2 + tD^3 - D^2 + D - D^2t + \frac{1}{2}t^3D^4 + \frac{1}{2}D^3t^2 \right]$$


$$\left[ -t^3D - \frac{1}{2}t^4D^2 - t^4D - \frac{1}{2}t^5D^2, -D + D^2t - \frac{1}{2}D^3t^2 + 1 - tD + t^2D^2 - \frac{1}{2}D^3t^3 \right]$$


$$\left[ 2 - 25D^2t - 2D + 13tD - 26D^3t^2 - \frac{1}{2}D^5t^3 - 6D^4t^2 - 18tD^3 - 12D^2 \right.$$


$$- 7t^3D^4 + 6D^3t^3 + 18t^2D^2 - \frac{1}{2}D^5t^4 + \frac{1}{2}D^4t^4,$$


$$\left. -\frac{1}{2}D^6t - D^5 - 2D^5t + 2D^4t - \frac{1}{2}D^6t^2 + \frac{1}{2}D^5t^2 \right]$$


$$\left[ -\frac{1}{2}D^3t^3 - 3t^2D^2 - 3tD - \frac{7}{2}D^2t^3 - 4t^2D - \frac{1}{2}t^4D^3, \right.$$


$$\left. -\frac{1}{2}D^4t - \frac{1}{2}tD^3 + D^2 - \frac{1}{2}D^4t^2 \right]$$

0.709

> LeftInverse(Q2bis,Alg);


$$\left[ 0, 0, -\frac{1}{2}t^2D^2 - \frac{1}{2}D^2t - \frac{3}{2}tD - D + \frac{1}{2}, \right.$$


$$\left. \frac{1}{2}D^4t^2 + \frac{1}{2}D^4t - \frac{1}{2}D^3t^2 + tD^3 + D^3 - 2D^2t - 2D^2 \right]$$


$$\left[ t + 1, 1, \frac{1}{2}t^4D + \frac{1}{2}t^3D - \frac{1}{2}t^3 - \frac{1}{2}t^2, \right.$$


$$\left. -\frac{1}{2}t^4D^3 - \frac{1}{2}D^3t^3 + \frac{1}{2}t^4D^2 + D^2t^3 + \frac{1}{2}t^2D^2 - t^2D - tD + t + 1 \right]$$

> BasisOfModule2(R2,Alg);


$$\left[ 0, 0, -\frac{1}{2}t^2D^2 - \frac{1}{2}D^2t - \frac{3}{2}tD - D + \frac{1}{2}, \right.$$


$$\left. \frac{1}{2}D^4t^2 + \frac{1}{2}D^4t - \frac{1}{2}D^3t^2 + tD^3 + D^3 - 2D^2t - 2D^2 \right]$$


$$\left[ t + 1, 1, \frac{1}{2}t^4D + \frac{1}{2}t^3D - \frac{1}{2}t^3 - \frac{1}{2}t^2, \right.$$


$$\left. -\frac{1}{2}t^4D^3 - \frac{1}{2}D^3t^3 + \frac{1}{2}t^4D^2 + D^2t^3 + \frac{1}{2}t^2D^2 - t^2D - tD + t + 1 \right]$$


```

We point out again that the algorithms which compute bases of free modules over the Weyl algebras are very sensitive with respect the permutation of the rows of the presentation matrix.

Let us illustrate this fact by computing a basis of the previous system but where we have permuted the two rows of $R2$, i.e.:

```

> R3:=evalm([[0, D, 0, -t],[D, 0, -t, 0]]);

R3 :=  $\begin{bmatrix} 0 & D & 0 & -t \\ D & 0 & -t & 0 \end{bmatrix}$ 

```

```

> st:=time(): Q3:=InjectiveParametrization(R3,Alg); time()-st;
Q3 := 
[-t^2 D^2 + D t^2 + t^4 D, -D + D^2 t + 1 - t D - t^3 D]
[1 - t D + D^2 t - D + t^2 D^2 + D^3 t^3 + D^3 t^2 - D^2 t^4 - 3 t^3 D - t^5 D^2 - 4 t^4 D - t^3 D^2
- 2 D t^2, D^2 t - D + D^2 + t^2 D^2 - D^3 t - D^3 t^2 + D^2 t^4 + 3 t^3 D + t^3 D^2 + 2 D t^2]
[-2 D^2 + 4 D t^2 - D^3 t + D^2 t + 2 D + t^3 D^2, D^3 - D^2 - t^2 D^2 - 3 t D]
[4 D^3 t + 3 D^3 - 9 t^3 D^2 - 16 D t^2 - 7 t^2 D^2 - 9 t D + D^4 t^2 + D^4 t - D^3 t^2 - 5 D^2 t
- 4 D - t^4 D^3 - D^3 t^3 + D^2,
-D^4 t - D^3 - D^4 + D^3 t + 2 D^2 + D^3 t^3 + 7 t^2 D^2 + 9 t D + D^3 t^2 + 5 D^2 t + 4 D]
0.691

```

We check that $Q3$ is an injective parametrization of the system $R3(x_1, x_2, u_1, u_2)^T = 0$, i.e., of the system $R2(x_1, x_2, u_1, u_2)^T = 0$:

```

> SyzygyModule(Q3,Alg);
      ⎡ D   0   -t   0 ⎤
      ⎢ 0   D   0   -t ⎥
> B3:=LeftInverse(Q3,Alg);
B3 := ⎣ 0   1   t^3 D - D^2 t + D t^2 - D^2 + t D + 3 t^2 - 2 D + t + 2   t^2 - D + 1
        1   0   t^4 D - t^2 D^2 + t^3 D - D^2 t + D t^2 + 2 t^3 - t D + D + 2 t + 2   t^3 - t D + t + 1 ⎦

```

We obtain that the residue classes of the rows of $B3$ define a basis of the left Alg -module M .

We can also compute an injective parametrization and a basis by means of the commands *InjectiveParametrization2* and *BasisOfModule2*:

```

> st:=time(): Q3bis:=InjectiveParametrization2(R3,Alg); time()-st;

```

Q3bis :=

$$\begin{aligned}
& \left[-\frac{31}{2}tD - 81D^2t + 81D + 84D^2 - 23t^2D^2 - 84D^3t - \frac{1}{2}D^3t^3 - \frac{157}{2}D^3t^2 \right. \\
& \quad - 48D^4t^2 - 42D^4t - 4D^5t^3 + \frac{7}{2}t^2 + \frac{1}{2}D^2t^4 - \frac{19}{2}D^4t^3 + \frac{1}{2}D^4t^4 - \frac{9}{2}D^6t^2 \\
& \quad - \frac{21}{2}D^5t^2 + \frac{17}{2}t^3D^2 + 42D^3 + t^4D^3 + \frac{45}{2}D^4 - \frac{45}{2}D^5t + \frac{31}{2} + 14Dt^2 + \frac{7}{2}t^3D, \\
& \quad D^2t - D - 4D^2 + 4D^3t + \frac{5}{2}D^3t^2 + 8D^4t^2 - \frac{25}{2}D^4t + \frac{3}{2}D^5t^3 + \frac{1}{2}D^4t^3 - 12D^6t^2 \\
& \quad + 3D^5t^2 + \frac{137}{2}D^5 + \frac{25}{2}D^3 + \frac{77}{2}D^6 + \frac{1}{2}D^7t^3 + \frac{3}{2}D^6t^3 - \frac{137}{2}D^6t + 15D^7 + 56D^4 \\
& \quad - 56D^5t - 15D^8t - 4D^8t^2 - \frac{77}{2}D^7t - \frac{27}{2}D^7t^2 + \frac{9}{2}D^8 - \frac{9}{2}D^9t \Big] \\
& \left[\frac{81}{2}tD + 27D^2t - 27D - \frac{81}{2}D^2 + \frac{21}{2}t^2D^2 + \frac{81}{2}D^3t - 26D^3t^3 - \frac{21}{2}D^3t^2 - 9D^4t^2 \right. \\
& \quad - \frac{9}{2}D^5t^3 - 3t^2 + \frac{5}{2}D^2t^4 - \frac{81}{2} + 2t^3 - 6D^4t^3 - 4D^4t^4 + \frac{1}{2}t^5D^2 - \frac{53}{2}t^3D^2 \\
& \quad - \frac{11}{2}t^4D^3 + \frac{1}{2}t^5D^3 + 3t^4D + 6Dt^2 - t^3D, 1 - tD - 26D^2 + 26D^3t + 2D^3t^3 \\
& \quad - 3D^3t^2 - 13D^4t^2 + \frac{133}{2}D^4t - \frac{9}{2}D^5t^3 + 3D^4t^3 + \frac{1}{2}D^4t^4 + D^5t^4 - 16D^6t^2 \\
& \quad - 20D^5t^2 - 27D^5 - \frac{133}{2}D^3 - \frac{27}{2}D^6 - 4D^7t^3 - \frac{19}{2}D^6t^3 + \frac{1}{2}D^6t^4 + 27D^6t \\
& \quad - 54D^4 + 54D^5t - \frac{9}{2}D^8t^2 + \frac{27}{2}D^7t - \frac{21}{2}D^7t^2 \Big] \\
& \left[7 + \frac{11}{2}t^2D^2 + 14tD + \frac{79}{2}D^2t + 28D + \frac{25}{2}D^3t^2 + \frac{1}{2}D^3t^3 + D^4t^3 + \frac{3}{2}D^4t^2 \right. \\
& \quad - \frac{49}{2}D^3t - \frac{123}{2}D^2 + \frac{1}{2}D^5t^3 - 60D^5t - 180D^4 - \frac{21}{2}D^6t - 63D^5 - 4D^6t^2 \\
& \quad - \frac{9}{2}D^7t - \frac{63}{2}D^6 - 107D^4t - 238D^3 - \frac{19}{2}D^5t^2, 4D^4t + 6D^3 + \frac{3}{2}D^6t^2 + \frac{25}{2}D^5t \\
& \quad + 20D^4 + \frac{1}{2}D^5t^2 - \frac{13}{2}D^5 - 80D^6 + \frac{15}{2}D^6t - \frac{191}{2}D^7 - \frac{27}{2}D^8t + \frac{1}{2}D^8t^2 - \frac{21}{2}D^7t \\
& \quad + \frac{3}{2}D^7t^2 - \frac{93}{2}D^8 - 15D^9 - \frac{9}{2}D^{10} - 4D^9t \Big] \\
& \left[-6 - 6tD - \frac{147}{2}D^2t + 12D + \frac{123}{2}D^2 + 9t^2D^2 - \frac{135}{2}D^3t + 5D^3t^3 - \frac{97}{2}D^3t^2 \right. \\
& \quad - 42D^4t^2 - \frac{57}{2}D^4t - 4D^5t^3 - \frac{11}{2}D^4t^3 + \frac{1}{2}D^4t^4 - \frac{9}{2}D^6t^2 - 6D^5t^2 + 6t \\
& \quad + \frac{11}{2}t^3D^2 + 6D^3 + \frac{1}{2}t^4D^3 + \frac{45}{2}D^4 - \frac{45}{2}D^5t + 14Dt^2, -D^2 + 6D^3t + 4D^4t^2 \\
& \quad + 6D^4t - 6D^3 + \frac{1}{2}D^5t^3 - \frac{5}{2}D^6t^2 - \frac{53}{2}D^5t + 7D^5t^2 + \frac{53}{2}D^5 + 22D^6 + \frac{1}{2}D^7t^3 \\
& \quad + D^6t^3 - \frac{97}{2}D^6t + 6D^7 - \frac{21}{2}D^8t - 4D^8t^2 - 28D^7t - \frac{19}{2}D^7t^2 + \frac{9}{2}D^8 - \frac{9}{2}D^9t \Big]
\end{aligned}$$

1.701

> SyzygyModule(Q3bis,Alg);

$$\left[\begin{array}{cccc} D & 0 & -t & 0 \\ 0 & D & 0 & -t \end{array} \right]$$

```

> B3bis:=LeftInverse(Q3bis,Alg);

B3bis :=


$$\left[ 0, 0, 180D^4 + \frac{49}{2}D^3t - \frac{3}{2}D^4t^2 + \frac{123}{2}D^2 - 28D - 14tD - \frac{1}{2}D^5t^3 - \frac{79}{2}D^2t \right.$$


$$- \frac{25}{2}D^3t^2 - \frac{11}{2}t^2D^2 - \frac{1}{2}D^3t^3 + 238D^3 + \frac{9}{2}D^7t + \frac{63}{2}D^6 - D^4t^3 + \frac{21}{2}D^6t$$


$$+ 60D^5t + 4D^6t^2 + 63D^5 + 107D^4t + \frac{19}{2}D^5t^2 - 5, \frac{15}{2}D^4t - \frac{13}{2}D^3 + \frac{25}{2}D^3t$$


$$- \frac{27}{2}D^6t - \frac{191}{2}D^5 + \frac{3}{2}D^4t^2 - \frac{93}{2}D^6 + \frac{3}{2}D^5t^2 + \frac{1}{2}D^3t^2 + 4D^2t + \frac{1}{2}D^6t^2 - \frac{9}{2}D^8$$


$$- 4D^7t - \frac{21}{2}D^5t - 80D^4 + 20D^2 - 15D^7 + 6D \Big]$$


$$\left[ -t, 1, -\frac{9}{2}D^3t - \frac{9}{2}D^4t^2 + \frac{81}{2}D^2 + \frac{3}{2}t^3 + 27D + \frac{57}{2}tD - \frac{9}{2}D^2t - 6D^3t^2 + \frac{43}{2}t \right.$$


$$- 18t^2D^2 - 4D^3t^3 + \frac{1}{2}t^4D - 3t^2 + t^3D + \frac{1}{2}D^2t^4 - \frac{31}{2}D^2t^2 - \frac{11}{2}t^3D^2 + \frac{81}{2},$$


$$\frac{21}{2}D^4t - 27D^3 + 12D^3t - t^3D^2 - Dt^2 + \frac{15}{2}tD + 4D^4t^2 - \frac{1}{2}t^3D + \frac{19}{2}D^3t^2$$


$$+ \frac{21}{2}D^2t + 4t + \frac{9}{2}D^5t - \frac{1}{2}D^3t^3 + \frac{11}{2}t^2D^2 - \frac{27}{2}D^4 - t^2 - 54D^2 - \frac{133}{2}D - 26 \Big]$$

> BasisOfModule2(R3,Alg);


$$\left[ 0, 0, 238D^3 + 180D^4 + 60D^5t + 4D^6t^2 - \frac{79}{2}D^2t - \frac{25}{2}D^3t^2 + \frac{123}{2}D^2 - \frac{1}{2}D^5t^3 \right.$$


$$+ 63D^5 - t^3D^4 - \frac{11}{2}t^2D^2 - 5 + \frac{21}{2}D^6t - \frac{1}{2}D^3t^3 - 14tD - 28D - \frac{3}{2}D^4t^2$$


$$+ \frac{49}{2}tD^3 + \frac{9}{2}D^7t + 107D^4t + \frac{63}{2}D^6 + \frac{19}{2}D^5t^2, 6D - \frac{27}{2}D^6t + 20D^2 + \frac{1}{2}D^3t^2$$


$$+ 4D^2t - \frac{9}{2}D^8 - \frac{191}{2}D^5 - \frac{93}{2}D^6 + \frac{3}{2}D^5t^2 - \frac{21}{2}D^5t - 80D^4 + \frac{3}{2}D^4t^2 + \frac{25}{2}tD^3$$


$$- 4D^7t - 15D^7 + \frac{1}{2}D^6t^2 + \frac{15}{2}D^4t - \frac{13}{2}D^3 \Big]$$


$$\left[ -t, 1, \frac{43}{2}t - \frac{9}{2}D^2t - 6D^3t^2 + \frac{81}{2}D^2 - 3t^2 - \frac{31}{2}t^2D - \frac{11}{2}D^2t^3 - 18t^2D^2 + \frac{81}{2} \right.$$


$$+ t^3D - 4D^3t^3 + \frac{1}{2}t^4D^2 + \frac{1}{2}t^4D + \frac{3}{2}t^3 + \frac{57}{2}tD + 27D - \frac{9}{2}D^4t^2 - \frac{9}{2}tD^3,$$


$$- \frac{133}{2}D - 54D^2 + \frac{19}{2}D^3t^2 + \frac{21}{2}D^2t - \frac{1}{2}D^3t^3 + \frac{11}{2}t^2D^2 - D^2t^3 - t^2D - \frac{1}{2}t^3D$$


$$- t^2 + \frac{9}{2}D^5t + \frac{15}{2}tD - \frac{27}{2}D^4 + 4D^4t^2 + 12tD^3 - 26 + \frac{21}{2}D^4t + 4t - 27D^3 \Big]$$


```

Finally, let us consider an example taken from F. Malrait, *Problèmes d'identification et d'observation du moteur à induction pour la variation de vitesse industrielle "sans capteur"*, PhD thesis, Ecole Nationale des Mines de Paris, 07/02/01, Example 2, p. 105. The system is defined by the following matrix of ordinary differential operators with polynomial coefficients in t :

```
> R4:=evalm([[D, -t, 0, -t^4, 0, 0], [0, D, -1, 0, -t, 0], [0, 0, D, 0, 0, -t^2]]);
```

$$R4 := \begin{bmatrix} D & -t & 0 & -t^4 & 0 & 0 \\ 0 & D & -1 & 0 & -t & 0 \\ 0 & 0 & D & 0 & 0 & -t^2 \end{bmatrix}$$

We then have the following time-varying linear control system:

$$> \text{ApplyMatrix}(\text{R4}, [\text{seq}(x[i](t), i=1..3), \text{seq}(u[i](t), i=1..3)], \text{Alg}) = \text{evalm}([[0]\$3]);$$

$$\begin{bmatrix} \left(\frac{d}{dt}x_1(t)\right) - t x_2(t) - t^4 u_1(t) \\ \left(\frac{d}{dt}x_2(t)\right) - x_3(t) - t u_2(t) \\ \left(\frac{d}{dt}x_3(t)\right) - t^2 u_3(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let us check whether or not this system is flat. In order to do that, we need to check whether or not the left Alg -module $M_4 = \text{Alg}^{\wedge}\{1^*6\}/(\text{Alg}^{\wedge}\{1^*3\} \text{R4})$ is free.

Let us first understand if the left Alg -module M_4 is projective.

> `RightInverse(R4,Alg);`

$$\begin{bmatrix} -\frac{1}{3}D t^2 + t & \frac{t^3}{3} & 0 \\ -\frac{1}{3}D^2 t + \frac{1}{3}D & \frac{1}{3}D t^2 + t & 0 \\ 0 & 0 & t - \frac{1}{2}D t^2 \\ 0 & 0 & 0 \\ -\frac{D^3}{3} & \frac{1}{3}D^2 t + \frac{5}{3}D & -1 + \frac{tD}{2} \\ 0 & 0 & -\frac{D^2}{2} \end{bmatrix}$$

As a right-inverse of R4 exists, we then obtain that M_4 is a projective, and thus, a stably free left Alg -module as R4 has full row rank. Let us compute the rank of M_4 over Alg .

> `OreRank(R4,Alg);`

3

As M_4 is a stably free left Alg -module of rank 3, from a classical result due to J. T. Stafford, we obtain that M_4 is a free left Alg -module. Let us compute a basis.

> `st:=time(): B4:=BasisOfModule(R4,Alg); time()-st;`

$B_4 :=$

$$\begin{aligned}
& \left[-8tD - 7D^2t - 2D - 8D^2 - 4t^2D^2 - \frac{35}{2}D^3t + 2D^3t^3 - 13D^3t^2 - \frac{21}{2}D^4t^2 \right. \\
& \quad - \frac{3}{2}D^4t - \frac{3}{2}D^5t^3 + 6D^2t^4 - \frac{1}{2}D^2t^6 - 3t^5D - \frac{11}{2}D^4t^3 + \frac{1}{2}D^4t^4 - \frac{1}{2}D^5t^4 \\
& \quad - \frac{3}{2}D^5t^2 + \frac{1}{2}t^5D^2 + \frac{7}{2}D^5 + \frac{5}{2}t^3D^2 - \frac{11}{2}D^3 + \frac{5}{2}D^6 - \frac{1}{2}D^6t^3 + 3D^6t + 3t^4D^3 \\
& \quad + \frac{1}{2}D^7 + 3D^4 + 3D^5t + \frac{1}{2}t^5D^3 + D^7t + \frac{1}{2}D^7t^2 + \frac{5}{2}t^4D + \frac{1}{2}D^4t^5 + \frac{3}{2}Dt^2 \\
& \quad + 14t^3D, \frac{7}{2}t^6 + 9 + 9t - 4t^3 - 3t^5 - 20t^4 + 12t^2, 11 + 10tD - 15D^2t - 8D \right. \\
& \quad - 14D^2 + \frac{15}{2}t^2D^2 - \frac{37}{2}D^3t + 4D^3t^3 - 3D^3t^2 - 6D^4t^2 - \frac{17}{2}D^4t - \frac{1}{2}D^5t^3 \\
& \quad + 33t^2 + \frac{15}{2}D^2t^4 - \frac{1}{2}D^2t^6 - \frac{9}{2}t^5D + \frac{1}{2}D^4t^4 - D^5t^2 - \frac{1}{2}t^5D^2 + 43t - \frac{1}{2}t^6D - 7t^5 \\
& \quad - \frac{17}{2}t^4 + \frac{33}{2}t^3D^2 - \frac{25}{2}D^3 + \frac{3}{2}t^4D^3 - 3D^4 - \frac{1}{2}D^5t + \frac{1}{2}t^5D^3 - \frac{7}{2}t^4D + 49Dt^2 \\
& \quad - \frac{1}{2}t^6 + \frac{1}{2}t^7 + \frac{59}{2}t^3D, D^4t^2 + 7D^3t + 8D^2 + \frac{5}{2}D^3 + 9D + 6 + 5D^3t^2 + \frac{1}{2}D^4t^3 \\
& \quad + \frac{1}{2}D^4t + \frac{7}{2}t^3D^2 + 12tD + \frac{37}{2}Dt^2 + 16t + 3t^2D^2 + 13D^2t, -9 - 39tD \\
& \quad - \frac{139}{2}D^2t - 28D - \frac{75}{2}D^2 - 24t^2D^2 - \frac{93}{2}D^3t - 3D^3t^3 - \frac{85}{2}D^3t^2 - \frac{27}{2}D^4t^2 \\
& \quad - \frac{11}{2}D^4t - D^5t^3 + 92t^2 + 21D^2t^4 - \frac{1}{2}D^2t^6 - \frac{9}{2}t^5D + \frac{125}{2}t^3 - \frac{17}{2}D^4t^3 - \frac{1}{2}D^5t^4 \\
& \quad - \frac{1}{2}D^5t^2 + 9t^5D^2 + 21t - \frac{1}{2}t^7D^2 - \frac{11}{2}t^6D - 13t^5 - \frac{7}{2}t^4 + \frac{1}{2}t^8 + \frac{39}{2}t^3D^2 - 12D^3 \\
& \quad + 6t^4D^3 + \frac{3}{2}t^5D^3 + \frac{89}{2}t^4D + \frac{1}{2}D^4t^5 + \frac{1}{2}D^3t^6 + 25Dt^2 - \frac{15}{2}t^6 - \frac{1}{2}t^7 - \frac{1}{2}t^7D \\
& \quad + 82t^3D, -1 - D - tD \Big] \\
& \left[1 - \frac{5}{2}tD - 4D^2t + \frac{7}{2}D^2 - \frac{11}{2}t^2D^2 - 6D^3t^3 - \frac{3}{2}D^3t^2 + 3D^4t^2 + \frac{3}{2}D^4t + \frac{1}{2}D^2t^6 \right. \\
& \quad + \frac{7}{2}t^5D - \frac{3}{2}D^4t^4 - \frac{1}{2}D^5t^4 + D^6t^2 + 2D^5t^2 + \frac{1}{2}t^5D^2 + \frac{1}{2}t^6D - \frac{5}{2}t^3D^2 + \frac{1}{2}D^6t^3 \\
& \quad + \frac{1}{2}D^6t - \frac{7}{2}t^4D^3 + \frac{1}{2}D^4 + \frac{3}{2}D^5t + \frac{1}{2}t^5D^3 + \frac{1}{2}t^4D - \frac{1}{2}D^4t^5 + \frac{1}{2}D^3t^6 - 2Dt^2 \\
& \quad - \frac{1}{2}t^7D - 4t^3D, -4t^6 - \frac{7}{2} + 4t + \frac{1}{2}t^8 + \frac{9}{2}t^3 - t^5 - \frac{1}{2}t^7 + 4t^4 + 8t^2, -\frac{9}{2}tD \\
& \quad - \frac{13}{2}D^2t - \frac{5}{2}D - \frac{19}{2}t^2D^2 - \frac{5}{2}D^3t - \frac{9}{2}D^3t^3 - \frac{13}{2}D^3t^2 - \frac{1}{2}D^4t^2 + 7t^2 + 2D^2t^4 \\
& \quad + \frac{1}{2}D^2t^6 + 5t^5D + \frac{35}{2}t^3 - D^4t^3 - \frac{1}{2}D^4t^4 + \frac{3}{2}t^5D^2 - \frac{5}{2}t - \frac{1}{2}t^6D - t^5 + \frac{19}{2}t^4 \\
& \quad - 3t^3D^2 + \frac{1}{2}t^5D^3 + \frac{21}{2}t^4D - 9Dt^2 - \frac{3}{2}t^6 - \frac{1}{2}t^7D + \frac{3}{2}t^3D, \frac{1}{2}D^3t^2 \\
& \quad - \frac{5}{2}t^3D^2 + 3tD - 12Dt^2 + 3t^3D - 9t + 5t^2D^2 + 2D^2t - 6t^3 + 6t^2 + D^3t^3, \\
& \quad - \frac{29}{2}tD - \frac{15}{2}D^2t - \frac{1}{2}t^8D - 26t^2D^2 - \frac{21}{2}D^3t^3 - \frac{9}{2}D^3t^2 - \frac{25}{2}t^2 - 3D^2t^4 + \frac{3}{2}D^2t^6 \\
& \quad + \frac{27}{2}t^5D + \frac{17}{2}t^3 - \frac{1}{2}D^4t^3 - D^4t^4 + \frac{7}{2}t^5D^2 - \frac{9}{2}t + \frac{1}{2}t^7D^2 + 6t^6D + \frac{29}{2}t^5 - \frac{3}{2} \\
& \quad + 28t^4 - \frac{1}{2}t^8 - 23t^3D^2 - \frac{13}{2}t^4D^3 + \frac{11}{2}t^4D - \frac{1}{2}D^4t^5 + \frac{1}{2}D^3t^6 - \frac{47}{2}Dt^2 - \frac{3}{2}t^6 \\
& \quad - 2t^7 - \frac{1}{2}t^7D - 15t^3D, -t \Big] \quad 12
\end{aligned}$$

$$\begin{aligned}
& \left[-\frac{1}{2}tD - \frac{7}{2}D^2t - \frac{3}{2}D - \frac{1}{2}D^2 - 3t^2D^2 + \frac{3}{2}D^3t - 4D^3t^3 - \frac{5}{2}D^3t^2 + D^4t^2 + \frac{5}{2}D^4t \right. \\
& - \frac{1}{2}D^5t^3 + t^5D - D^4t^3 - \frac{1}{2}D^4t^4 + \frac{1}{2}D^6t^2 + \frac{1}{2}D^5t^2 + \frac{1}{2}t^5D^2 - \frac{1}{2}t^6D - \frac{3}{2}D^3 \\
& + \frac{1}{2}D^6t - D^4 + 2D^5t + \frac{1}{2}t^5D^3 + 3t^4D - \frac{3}{2}Dt^2 - 3t^3D, \\
& -t^6 + \frac{1}{2} + 5t + \frac{3}{2}t^3 - \frac{7}{2}t^5 + \frac{1}{2}t^7 + 3t^4 + \frac{7}{2}t^2, D^2t^4 - \frac{5}{2}t - \frac{9}{2}t^2D^2 - 7D^2t + 3D \\
& - \frac{1}{2}t^6 + 4 - \frac{1}{2}D^4t^3 + \frac{17}{2}t^2 - t^5 + \frac{11}{2}t^3D - \frac{1}{2}D^3t^3 - \frac{5}{2}D^3t - \frac{9}{2}Dt^2 - 6tD \\
& - \frac{1}{2}t^6D - \frac{1}{2}D^4t^2 + 11t^3 + \frac{1}{2}t^5D^2 + \frac{11}{2}t^4D - \frac{9}{2}D^3t^2 + \frac{1}{2}t^4D^3 + t^3D^2, -1 \\
& + \frac{1}{2}D^3t^2 + \frac{1}{2}t^3D^2 + \frac{7}{2}tD + \frac{7}{2}Dt^2 + 4t^3D + 4t + \frac{7}{2}t^2D^2 + 2D^2t + \frac{3}{2}t^3 + 9t^2 \\
& + \frac{1}{2}D^3t^3, 4 - 11tD - \frac{15}{2}D^2t - \frac{41}{2}t^2D^2 - \frac{13}{2}D^3t^3 - \frac{9}{2}D^3t^2 - 7t^2 + \frac{5}{2}D^2t^4 \\
& + \frac{1}{2}D^2t^6 + \frac{13}{2}t^5D + 14t^3 - \frac{1}{2}D^4t^3 - \frac{1}{2}D^4t^4 + t^5D^2 - 3t + \frac{33}{2}t^4 - 6t^3D^2 \\
& \left. - \frac{1}{2}t^4D^3 + \frac{1}{2}t^5D^3 + \frac{15}{2}t^4D - 15Dt^2 - \frac{3}{2}t^6 - \frac{1}{2}t^7 - \frac{1}{2}t^7D - \frac{5}{2}t^3D, -t \right]
\end{aligned}$$

17.489

In particular, we obtain that $(\xi_1, \xi_2, \xi_3)^T = B4(x_1, x_2, x_3, u_1, u_2, u_3)^T$ is a flat output of the system $R4(x_1, x_2, x_3, u_1, u_2, u_3)^T = 0$ or, equivalently, the residue classes of the rows of $B4$ define a basis of $M4$. Let us compute an injective parametrization of the system $R4(x_1, x_2, x_3, u_1, u_2, u_3)^T = 0$.

```
> st:=time(): Q4:=InjectiveParametrization(R4,Alg); time()-st;
```

$$\begin{aligned}
Q4 := & \\
& [-t^8 D - 3t^7 + t^2 + 5D^2 t^6 + 3t^5 D + 4t^5 D^2 - t^8 D^2 - 3t^7 D - t^8 + D^3 t^7 + D^3 t^6 + t^9 \\
& , 1 - t D - t^8 D + t^7 D^2 + 2t^6 D - D^4 t^6 - t^5 D^3 + D^3 t^7 - D^4 t^5 + t^7 D , \\
& -t^6 D - t^5 + t^8 - 2t^6 - t^7 D] \\
& [2 + t D + t^8 D + 8t^7 + 28D^2 t^4 + 12t^3 D - 10D^2 t^6 - 18t^5 D - t^7 D^2 - 10t^6 D \\
& - 18t^5 + 16t^3 D^2 + 9t^4 D^3 + D^4 t^6 + 11t^5 D^3 - D^3 t^7 + D^4 t^5 - t^7 D - 7t^6 , -D^2 \\
& - 4D^3 t^3 + D^2 t^6 + 6t^5 D - 4D^4 t^3 - 6D^4 t^4 - D^5 t^4 + 8t^5 D^2 - t^7 D^2 - 7t^6 D \\
& + D^4 t^6 - D^5 t^5 + 6t^5 D^3 + 10t^4 D + D^3 t^6 , \\
& -D^2 t^6 - 8t^5 D - 10t^4 - t^5 D^2 - 6t^4 D - 4t^3 + t^7 D + 7t^6] \\
& \left[-3D^2 t + 3D + \frac{1267}{2} t^8 D + 115D^3 t^3 + \frac{3}{2} D^3 t^2 + 492 D^2 t^4 - 1657 D^2 t^6 \right. \\
& - \frac{2569}{2} t^5 D - 4t^3 + \frac{7}{2} D^4 t^3 + 179 D^4 t^4 + \frac{1}{2} D^5 t^4 - \frac{1055}{2} t^5 D^2 - \frac{375}{2} t^8 D^2 \\
& + 194 t^9 D^2 + \frac{129}{2} t^{10} D^2 + \frac{1}{2} t^{14} - \frac{2323}{2} t^7 D^2 - \frac{97}{2} D^4 t^7 - \frac{697}{2} t^8 D^3 + \frac{51}{2} t^7 D^6 \\
& - 10D^4 t^{10} + 15D^6 t^8 - 16D^5 t^9 + \frac{1}{2} D^7 t^7 - \frac{3329}{2} t^6 D - 797 t^5 - \frac{565}{2} t^4 - \frac{21}{2} t^{10} \\
& + 212 t^8 + 220 t^3 D^2 - t^{13} - \frac{3}{2} t^{12} + 785 t^4 D^3 + 11D^6 t^6 + 618 D^4 t^6 + \frac{1}{2} t^{11} D \\
& + 13D^2 t^{11} + \frac{1}{2} D^2 t^{12} + 63 D t^{10} + 212 D^5 t^6 + 75 D^5 t^5 - 824 D^3 t^7 + 179 D t^9 \\
& + t^{11} D^4 - 2t^{10} D^5 + t^8 D^7 + 10t^{11} D^3 - 30t^{11} + 990 t^5 D^3 - \frac{505}{2} t^4 D - \frac{19}{2} D t^{12} \\
& + 683 D^4 t^5 + \frac{1}{2} D^4 t^{12} - 45 t^9 D^4 - \frac{557}{2} D^3 t^6 - t^{13} D^2 - \frac{5}{2} t^8 D^5 - \frac{1}{2} t^{13} D - 34 t^6 \\
& + \frac{1385}{2} t^7 - 205 t^7 D - \frac{1}{2} D^5 t^{11} + \frac{1}{2} D^7 t^9 - \frac{1}{2} D^6 t^{10} + D^3 t^{12} + 151 D^5 t^7 - 67 D^3 t^9 \\
& + 36 t^3 D + \frac{221}{2} t^9 + \frac{47}{2} t^{10} D^3 - 176 t^8 D^4 , -\frac{135}{2} t^8 D - 12 D^3 t^3 + D^4 t - 23 D^5 t^3 \\
& + \frac{479}{2} D^2 t^4 + 49 D^2 t^6 - 15 t^5 D - 55 D^4 t^3 - 120 D^4 t^4 - 146 D^5 t^4 - \frac{1}{2} D^5 t^2 \\
& + \frac{1169}{2} t^5 D^2 - \frac{181}{2} t^8 D^2 - 43 t^9 D^2 + t^{10} D^2 - \frac{701}{2} t^7 D^2 + \frac{389}{2} D^4 t^7 - 131 t^8 D^3 \\
& + \frac{5}{2} t^7 D^6 - \frac{15}{2} D^4 t^{10} + \frac{25}{2} D^6 t^8 + 7 D^5 t^9 - 11 D^7 t^7 - 342 t^6 D + 4 t^3 D^2 - D^3 \\
& - \frac{1}{2} D^6 t^3 - 26 D^6 t^4 + 84 t^4 D^3 - 74 D^6 t^6 - \frac{179}{2} D^6 t^5 + \frac{657}{2} D^4 t^6 + t^{11} D + 7 D^2 t^{11} \\
& - \frac{1}{2} D^8 t^6 + \frac{1}{2} D^2 t^{12} + 23 D t^{10} - \frac{35}{2} D^7 t^6 + \frac{67}{2} D^5 t^6 - 174 D^5 t^5 - 7 D^7 t^5 \\
& + \frac{133}{2} D^3 t^7 + \frac{19}{2} D t^9 - t^{11} D^4 + 2 t^9 D^6 - t^{10} D^5 - \frac{1}{2} t^{11} D^3 + \frac{869}{2} t^5 D^3 + \frac{425}{2} t^4 D \\
& + D t^{12} + 111 D^4 t^5 - 19 t^9 D^4 + \frac{1009}{2} D^3 t^6 + \frac{67}{2} t^8 D^5 - \frac{1}{2} t^{13} D - \frac{243}{2} t^7 D \\
& - \frac{1}{2} D^5 t^{11} + \frac{1}{2} D^7 t^9 + \frac{1}{2} D^6 t^{10} + D^3 t^{12} - \frac{1}{2} D^8 t^8 - D^8 t^7 + \frac{203}{2} D^5 t^7 - \frac{71}{2} D^3 t^9 \\
& + 43 t^3 D - \frac{19}{2} t^{10} D^3 + \frac{65}{2} t^8 D^4 , \frac{227}{2} t^8 D - 101 D^2 t^4 - 396 D^2 t^6 - 468 t^5 D \\
& - 55 t^3 - \frac{829}{2} t^5 D^2 + \frac{251}{2} t^8 D^2 + \frac{49}{2} t^9 D^2 + 8 t^{10} D^2 + 41 t^7 D^2 - \frac{45}{2} D^4 t^7 + \frac{5}{2} t^8 D^3 \\
& + \frac{1}{2} D^4 t^{10} - \frac{1}{2} D^5 t^9 + \frac{355}{2} t^6 D + 191 t^5 - 132 t^4 - 24 t^{10} + \frac{93}{2} t^8 - 2 t^3 D^2 + \frac{1}{2} t^{13} \\
& - \frac{1}{2} t^{12} - \frac{1}{2} t^4 D^3 - \frac{19}{2} D^4 t^6 - \frac{15}{2} t^{11} D - \frac{1}{2} D^2 t^{11} - \frac{1}{2} D^2 t^{12} - 7 D t^{10} - 118 D^3 t^7 \\
& + \frac{73}{2} D t^9 + \frac{1}{2} t^{11} D^3 - \frac{3}{2} t^{11} - 54 t^5 D^3 - 347 t^4 D - \frac{1}{2} D t^{12} - \frac{319}{2} D^3 t^6 - t^8 D^5 \\
& + 413 t^6 + \frac{289}{2} t^7 + 416 t^7 D - \frac{1}{2} D^5 t^7 + 14 D^3 t^9 - 46 t^3 D - 27 t^9 + \frac{3}{2} t^{10} D^3 \\
& \left. - \frac{27}{2} t^8 D^4 \right]
\end{aligned}$$

$$\begin{aligned}
& [5D^2t + 3D + 4D^2 + D^3t + D^3t^2 - t^3 + t^4 - t^3D^2 - 3Dt^2 - t^3D - 3t^2, \\
& 2tD + t^2D^2 + D^3t^2 - D^4t - D^3 - D^4 + Dt^2 - t^3D, t^3 - Dt^2 - 2t - 1 - tD] \\
& \left[36tD + 48D^2t + 4D^2 - 179t^8D - 96t^2D^2 - \frac{3}{2}D^3t - 702D^3t^3 - 63D^3t^2 \right. \\
& - \frac{7}{2}D^4t^2 - \frac{1}{2}D^5t^3 + 4t^2 + \frac{899}{2}D^2t^4 + \frac{2321}{2}D^2t^6 + \frac{3301}{2}t^5D + \frac{385}{2}t^3 - 165D^4t^3 \\
& - 666D^4t^4 - 74D^5t^4 + 1640t^5D^2 - 194t^8D^2 - \frac{129}{2}t^9D^2 - 13t^{10}D^2 + \frac{377}{2}t^7D^2 \\
& + 176D^4t^7 + 67t^8D^3 - 15t^7D^6 - D^4t^{10} + 2D^5t^9 - D^7t^7 + 221t^6D + 90t^5 \\
& + 755t^4 + 30t^{10} - \frac{221}{2}t^8 - 492t^3D^2 - \frac{1}{2}t^{13} + t^{12} - 990t^4D^3 - \frac{51}{2}D^6t^6 - 11D^6t^5 \\
& + \frac{95}{2}D^4t^6 + \frac{19}{2}t^{11}D - \frac{1}{2}D^2t^{11} + D^2t^{12} - \frac{1}{2}Dt^{10} - \frac{1}{2}D^7t^6 - 151D^5t^6 - 211D^5t^5 \\
& + \frac{697}{2}D^3t^7 - 63Dt^9 - \frac{1}{2}t^{11}D^4 + \frac{1}{2}t^9D^6 + \frac{1}{2}t^{10}D^5 - \frac{1}{2}t^8D^7 - t^{11}D^3 + \frac{3}{2}t^{11} \\
& + \frac{523}{2}t^5D^3 + \frac{2413}{2}t^4D + \frac{1}{2}Dt^{12} - 618D^4t^5 + 10t^9D^4 + 823D^3t^6 + 16t^8D^5 \\
& - 36Dt^2 - \frac{1385}{2}t^6 - 212t^7 - \frac{1267}{2}t^7D + \frac{5}{2}D^5t^7 - \frac{47}{2}D^3t^9 + \frac{325}{2}t^3D + \frac{21}{2}t^9 \\
& - 10t^{10}D^3 + 45t^8D^4, -\frac{19}{2}t^8D - 4t^2D^2 - 12D^3t - 54D^3t^3 + 12D^3t^2 \\
& + 27D^4t^2 - 12D^4t + 135D^5t^3 - \frac{1145}{2}D^2t^4 + \frac{701}{2}D^2t^6 + 342t^5D + 120D^4t^3 \\
& - 99D^4t^4 + 174D^5t^4 + \frac{1}{2}D^6t^2 + 15D^5t^2 - 63t^5D^2 + 7t^4D^7 + 43t^8D^2 - t^9D^2 \\
& - 7t^{10}D^2 + \frac{181}{2}t^7D^2 - \frac{65}{2}D^4t^7 + \frac{71}{2}t^8D^3 - \frac{25}{2}t^7D^6 + D^4t^{10} - 2D^6t^8 + D^5t^5 \\
& + \frac{243}{2}t^6D - \frac{379}{2}t^3D^2 + 25D^6t^3 + \frac{177}{2}D^6t^4 - \frac{841}{2}t^4D^3 - \frac{5}{2}D^6t^6 + 74D^6t^5 \\
& - \frac{389}{2}D^4t^6 - t^{11}D - \frac{1}{2}D^2t^{11} + D^8t^6 + \frac{1}{2}D^8t^5 - Dt^{10} + 11D^7t^6 - \frac{203}{2}D^5t^6 \\
& - \frac{65}{2}D^5t^5 + \frac{35}{2}D^7t^5 + 131D^3t^7 - D^4 + \frac{1}{2}D^5t - 23Dt^9 - \frac{1}{2}t^9D^6 + \frac{1}{2}t^{10}D^5 \\
& - \frac{1}{2}t^8D^7 - t^{11}D^3 - \frac{1007}{2}t^5D^3 - 27t^4D + \frac{1}{2}Dt^{12} - \frac{655}{2}D^4t^5 + \frac{15}{2}t^9D^4 \\
& - \frac{135}{2}D^3t^6 - 7t^8D^5 - 3Dt^2 + \frac{135}{2}t^7D + \frac{1}{2}D^8t^7 - \frac{67}{2}D^5t^7 + \frac{19}{2}D^3t^9 - \frac{365}{2}t^3D \\
& + \frac{1}{2}t^{10}D^3 + 19t^8D^4, -\frac{73}{2}t^8D + 2t^2D^2 + \frac{1}{2}D^3t^3 + 15t^2 + \frac{801}{2}D^2t^4 - 40D^2t^6 \\
& - \frac{327}{2}t^5D + 132t^3 + 396t^5D^2 - \frac{49}{2}t^8D^2 - 8t^9D^2 + \frac{1}{2}t^{10}D^2 - 12t - \frac{251}{2}t^7D^2 \\
& + \frac{27}{2}D^4t^7 - 14t^8D^3 - 416t^6D - 413t^5 - 149t^4 + \frac{3}{2}t^{10} + 27t^8 + 90t^3D^2 - \frac{1}{2}t^{12} \\
& + 53t^4D^3 + \frac{45}{2}D^4t^6 + \frac{1}{2}t^{11}D + \frac{1}{2}D^2t^{11} + \frac{15}{2}Dt^{10} + \frac{1}{2}D^5t^6 - \frac{5}{2}D^3t^7 + 7Dt^9 \\
& + \frac{1}{2}t^{11} + \frac{317}{2}t^5D^3 + 468t^4D + \frac{19}{2}D^4t^5 - \frac{1}{2}t^9D^4 + 118D^3t^6 + \frac{1}{2}t^8D^5 + 18Dt^2 \\
& \left. - \frac{289}{2}t^6 - \frac{93}{2}t^7 - \frac{227}{2}t^7D + D^5t^7 - \frac{3}{2}D^3t^9 + 297t^3D + 24t^9 - \frac{1}{2}t^{10}D^3 \right]
\end{aligned}$$

$$\begin{aligned}
& \left[-12 - 1014tD + 2004D^2t + 108D + 660D^2 - 5t^8D - 2890t^2D^2 + 3360D^3t \right. \\
& - \frac{4397}{2}D^3t^3 + 5442D^3t^2 + 4200D^4t^2 + 831D^4t + 1955D^5t^3 - 3985t^2 \\
& - 9795D^2t^4 + \frac{4759}{2}D^2t^6 + \frac{11521}{2}t^5D - 204t^3 + 4698D^4t^3 - 618D^4t^4 \\
& + 1675D^5t^4 + \frac{1}{2}D^6t^2 + 554D^5t^2 - 1705t^5D^2 + \frac{29}{2}t^4D^7 + 206t^8D^2 + \frac{13}{2}t^9D^2 \\
& - \frac{45}{2}t^{10}D^2 - 1130t + 824t^7D^2 - 167D^4t^7 + \frac{349}{2}t^8D^3 - 21t^7D^6 + D^4t^{10} \\
& - 2D^6t^8 + D^5t^9 + 1823t^6D + 1696t^5 + \frac{9695}{2}t^4 - 13t^{10} - 330t^8 - \frac{22453}{2}t^3D^2 \\
& + 345D^3 + 141D^6t^3 + \frac{781}{2}D^6t^4 - 7425t^4D^3 - \frac{5}{2}D^6t^6 + 271D^6t^5 - \frac{1507}{2}D^4t^6 \\
& - t^{11}D - \frac{1}{2}D^2t^{11} + D^8t^6 + \frac{1}{2}D^8t^5 - 8Dt^{10} + \frac{39}{2}D^7t^6 - 320D^5t^6 - \frac{137}{2}D^5t^5 \\
& + \frac{67}{2}D^7t^5 + 429D^3t^7 + 12D^4 + \frac{11}{2}D^5t - 144Dt^9 - \frac{1}{2}t^9D^6 + \frac{1}{2}t^{10}D^5 - \frac{1}{2}t^8D^7 \\
& - t^{11}D^3 + 7t^{11} - \frac{7899}{2}t^5D^3 - 1469t^4D + \frac{1}{2}Dt^{12} - 2232D^4t^5 + 16t^9D^4 \\
& - \frac{1581}{2}D^3t^6 - \frac{31}{2}t^8D^5 - 6705Dt^2 + \frac{1989}{2}t^6 - 105t^7 + \frac{1481}{2}t^7D + \frac{1}{2}D^8t^7 \\
& - 65D^5t^7 + 25D^3t^9 - 10784t^3D - 18t^9 + \frac{1}{2}t^{10}D^3 + \frac{69}{2}t^8D^4, 850tD \right. \\
& + 1001D^2t + 129D + 12D^2 + 11t^8D + 3135t^2D^2 + 340D^3t + \frac{7223}{2}D^3t^3 \\
& + 2412D^3t^2 + 639D^4t^2 - 492D^4t + 312D^5t^3 - \frac{5591}{2}D^2t^4 - \frac{909}{2}D^2t^6 \\
& - 540t^5D + \frac{4811}{2}D^4t^3 + 1866D^4t^4 + 1039D^5t^4 - \frac{1187}{2}D^6t^2 - 990D^5t^2 \\
& - \frac{1691}{2}t^5D^2 - 10t^3D^8 - \frac{49}{2}t^4D^8 - 151t^4D^7 + 100t^8D^2 + 7t^9D^2 + t^{10}D^2 \\
& - 69D^5 + \frac{39}{2}t^7D^2 - \frac{221}{2}D^4t^7 - \frac{9}{2}t^8D^3 + 12t^7D^6 + D^4t^{10} - D^6t^8 - D^5t^9 \\
& + 2D^7t^7 + \frac{171}{2}t^6D - \frac{1}{2}D^9t^4 + 279t^3D^2 - 36D^3 - 2D^6 - \frac{389}{2}D^7t^3 - 618D^6t^3 \\
& + 51D^6t^4 - 127D^6t + \frac{1029}{2}t^4D^3 + \frac{103}{2}D^6t^6 + \frac{403}{2}D^6t^5 - 302D^4t^6 - \frac{1}{2}D^2t^{11} \\
& - 15D^8t^5 - \frac{13}{2}Dt^{10} + 17D^7t^6 + \frac{191}{2}D^5t^6 + \frac{925}{2}D^5t^5 + \frac{5}{2}D^7t^5 - 138D^3t^7 \\
& - 165D^4 - 639D^5t + 12Dt^9 - \frac{1}{2}t^9D^6 + \frac{1}{2}t^8D^7 - \frac{2797}{2}t^5D^3 - \frac{1}{2}D^7t - 61D^7t^2 \\
& - \frac{1701}{2}t^4D - D^9t^5 + \frac{653}{2}D^4t^5 - \frac{1}{2}t^9D^4 - 410D^3t^6 - 13t^8D^5 - 75Dt^2 \\
& + 230t^7D + \frac{1}{2}D^8t^7 - 29D^5t^7 + 19D^3t^9 - 2052t^3D + \frac{1}{2}t^{10}D^3 - \frac{1}{2}D^9t^6 \\
& - \frac{41}{2}t^8D^4, -165 - 1443tD - 450D^2t - 138D - 6D^2 - \frac{213}{2}t^8D - \frac{4839}{2}t^2D^2 \\
& - 4D^3t - \frac{2743}{2}D^3t^3 - 371D^3t^2 - \frac{1}{2}D^4t^2 + 955t^2 + \frac{929}{2}D^2t^4 + 334D^2t^6 \\
& + \frac{2105}{2}t^5D + 2478t^3 - 111D^4t^3 - 317D^4t^4 - 13D^5t^4 + 1420t^5D^2 - \frac{25}{2}t^8D^2 \\
& - \frac{27}{2}t^9D^2 - \frac{1}{2}t^{10}D^2 - 528t + \frac{233}{2}t^7D^2 + 19D^4t^7 + \frac{27}{2}t^8D^3 - \frac{1}{2}t^7D^6 + 375t^6D \\
& + 372t^5 + \frac{2023}{2}t^4 + \frac{13}{2}t^{10} - \frac{33}{2}t^8 - 2844t^3D^2 - 1222t^4D^3 - D^6t^6 - \frac{1}{2}D^6t^5 \\
& + \frac{5}{2}D^4t^6 + \frac{1}{2}t^{11}D - \frac{1}{2}Dt^{10} - 18D^5t^6 - \frac{61}{2}D^5t^5 + \frac{79}{2}D^3t^7 - \frac{15}{2}Dt^9 + 61t^5D^3 \\
& + 3325t^4D - 226D^4t^5 + \frac{1}{2}t^9D^4 + \frac{503}{2}D^3t^6 + \frac{1}{2}t^8D^5 - 2472Dt^2 - 243t^6 \\
& \left. - 240t^7 - 97t^7D - \frac{1}{2}D^3t^9 - 461256t^3D - 6t^9 - \frac{1}{2}t^{10}D^3 + \frac{3}{2}t^8D^4 \right]
\end{aligned}$$

10.810

We check that Q_4 is a parametrization of the system $R4(x_1, x_2, x_3, u_1, u_2, u_3)^T = 0$ by computing the syzygy module of the left Alg -module generated by the rows of $R4$. We obtain:

```
> SyzygyModule(Q4,Alg);

$$\begin{bmatrix} -D & t & 0 & t^4 & 0 & 0 \\ 0 & D & -1 & 0 & -t & 0 \\ 0 & 0 & -D & 0 & 0 & t^2 \end{bmatrix}$$

```

Finally, the parametrization $(x_1, x_2, x_3, u_1, u_2, u_3)^T = Q4(\xi_1, \xi_2, \xi_3)^T$ is injective, i.e., we obtain

$$(\xi_1, \xi_2, \xi_3)^T = B4(x_1, x_2, x_3, u_1, u_2, u_3)^T$$

as we have:

```
> simplify(evalm(B4-LeftInverse(Q4,Alg)));

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```

More generally, it was proved in A. Quadrat, D. Robertz, *On the blowing-up of stably free behaviours*, Proceedings of CDC-ECC05, Seville (Spain), 12-15/12/05, that any controllable linear system with polynomial coefficients and at least 2 inputs is flat. The three previous examples satisfy these conditions and we have shown how to obtain flat outputs for these systems or, equivalently, bases of the corresponding free modules over the Weyl algebra A_{-1} . Finally, we point out that we do not need to use a dynamical compensator for last example in order to obtain a flat linear system as it was done in F. Malrait, *Problèmes d'identification et d'observation du moteur à induction pour la variation de vitesse industrielle "sans capteur"*, PhD thesis, Ecole Nationale des Mines de Paris, 07/02/01 as this systems is already flat. A similar comment holds for any controllable linear system with polynomial coefficients and at least 2 inputs.

Finally, we do not advice to use *InjectiveParametrization2* and *BasisOfModule2* for computing an injective parametrization and a basis of M_4 as it can take half a day of computations and return quite large injective parametrizations and bases!