```
> with(OreModules):
> with(OreMorphisms):
> with(Stafford):
> with(linalg):
```

Let us consider the second Weyl algebra $A = A_2(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers,

```
> A := DefineOreAlgebra(diff=[dx,x], diff=[dy,y], polynom=[x,y]):
```

the left A-module M finitely presented by the matrix R defined by

```
> R := transpose(evalm([[dx,dy,0,0,0,0],[0,1,-1,0,dx,dy],[0,0,dx,dy,0,0]]));
```

$$R := \begin{bmatrix} dx & 0 & 0 \\ dy & 1 & 0 \\ 0 & -1 & dx \\ 0 & 0 & dy \\ 0 & dx & 0 \\ 0 & dy & 0 \end{bmatrix}$$

and the left A-module M' finitely presented by the matrix R' defined by:

```
> Rp := evalm([[dx,0],[dy/2,dx/2],[0,dy]]);
```

$$Rp := \begin{bmatrix} dx & 0 \\ \frac{dy}{2} & \frac{dx}{2} \\ 0 & dy \end{bmatrix}$$

Let f be the left A-homomorphism from M' to M induced by S, where S is given by

```
> S := evalm([[1,0,0],[0,0,1]]);
```

$$S := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can check that f is a left A-isomorphism:

```
> TestIso(Rp,R,S,A);
```

$$true$$

Then the direct sum of $A^{1\times 6}$ and N_1 is isomorphic to the direct sum of $A^{1\times 8}$ and N_1', where N_1 (resp., N_1') is the left A-module finitely presented by the formal adjoint of R (resp., R'), namely:

```
> Involution(R,A);
```

$$\begin{bmatrix} -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

```
> Involution(Rp,A);
```

$$\begin{bmatrix} -dx & -\frac{dy}{2} & 0 \\ 0 & -\frac{dx}{2} & -dy \end{bmatrix}$$

The corresponding isomorphism can be computed by the command *AuslanderEquivalence*:

```
> unprotect(O);
> O := AuslanderEquivalence(Rp,R,S,A,opt):
```

The left A-module L finitely presented by the first entry P = O[1] of O

```
> P := O[1];
```

$$P := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

is isomorphic to the direct sum of $A^{1\times 6}$ and N_1, and the left A-module L' finitely presented by the second entry P' = O[2] of O, namely,

```
> Pp := O[2];
```

$$Pp := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is isomorphic to the direct sum of $A^{1\times 8}$ and N_1'.

The rank of the left A-module L is:

```
> OreRank(P,A);
```

$$9$$

The rank of the left A-module L' is:

```
> OreRank(Pp,A);
```

$$9$$

The left A-homomorphism f_1 from L to L' induced by O[3], where O[3] is defined by

```
> O[3];
```

$$
\begin{bmatrix}
0 & 0 & -1 & 0 & 0 & 0 & -dy & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & -2\,dy & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & dx & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & dx & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & dx & dy & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & dx & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -\dfrac{dy}{2} & 0 & -1 & -dy & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\dfrac{dx}{2} & -dy & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -dy & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & dx & 0 & 0 & \dfrac{1}{2} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -dy & 0 & \dfrac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & dx & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

is a left A-isomorphism:

```
> TestIso(P,Pp,O[3],A);
```

$$true$$

If P = (0 P2), where P2 is defined by

```
> P2 := submatrix(P,1..5,2..14);
```

$$
P2 := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0
\end{bmatrix}
$$

then L is isomorphic to the direct sum of A and L_2, where the left A-module L_2 is finitely presented by P2. The rank of L_2 is:

```
> OreRank(P2,A);
```

$$8$$

Similarly, if P' = (0 P2'), where P2' is defined by

```
> P2p := submatrix(Pp,1..5,2..14);
```

$$
P2p := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -\dfrac{dy}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\dfrac{dx}{2} & -dy & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

then L' is isomorphic to the direct sum of A and L_2', where the left A-module L_2' is finitely presented by P2'. The rank of L_2' is:

```
> OreRank(P2p,A);
```

$$8$$

Then we have that L_2 is isomorphic to L_2'. Let us compute the corresponding left A-isomorphism f_2 from L_2 onto L_2'.

```
> X := Cancellation(P2p,O[3],A);
```

$$X := \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & -2\,dy & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & dx & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & dy & 0 & 0 & -(1+dx)\,dy & 0 & 1+dx & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & dx & dy & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & dx & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & -1 & -dy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -dy & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & dx & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -dy & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & dx & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain that f_2 is induced by X. We can check again that f_2 is a left A-isomorphism:

```
> TestIso(P2,P2p,X,A);
```

$$true$$

If P2 = (0 P3), where P3 is defined by

```
> P3 := submatrix(P2,1..5,2..13);
```

$$P3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

then L_2 is isomorphic to the direct sum of A and L_3, where the left A-module L_3 is finitely presented by P3. The rank of L_3 is:

```
> OreRank(P3,A);
```

$$7$$

If P2' = (0 P3'), where P3' is defined by

```
> P3p := submatrix(P2p,1..5,2..13);
```

$$P3p := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then L_2' is isomorphic to the direct sum of A and L_3', where the left A-module L_3' is finitely presented by P3'. The rank of L_3' is:

```
> OreRank(P3p,A);
```

$$7$$

Then we have that L_3 and L_3' are isomorphic. Let us compute the corresponding left A-isomorphism f_3 from L_3 onto L_3'.

```
> Y := Cancellation(P3p,X,A);
```

$$Y := \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & dx & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & dy & 0 & 0 & -(1+dx)\,dy & 0 & 1+dx & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & dx & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & dy & 0 & -(2+dx)\,dy & 0 & 1+\frac{dx}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & -1 & -dy & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -dy & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dx & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -dy & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & dx & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain that f_3 is induced by Y. We can check again that f_3 is a left A-isomorphism:

```
> TestIso(P3,P3p,Y,A);
```

true

If P3 = (0 P4), where P4 is defined by

```
> P4 := submatrix(P3,1..5,2..12);
```

$$P4 := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

then L_3 is isomorphic to the direct sum of A and L_4, where the left A-module L_4 is finitely presented by P4. The rank of L_4 is:

```
> OreRank(P4,A);
```

$$6$$

If P3' = (0 P4'), where P4' is defined by

```
> P4p := submatrix(P3p,1..5,2..12);
```

$$P4p := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then L_3' is isomorphic to the direct sum of A and L_4', where the left A-module L_4' is finitely presented by P4'. The rank of L_4' is:

```
> OreRank(P4p,A);
```

$$6$$

Then we have that L_4 and L_4' are isomorphic. Let us compute the corresponding left A-isomorphism f_4:

```
> Z := Cancellation(P4p,Y,A);
```

$$Z := \begin{bmatrix} dy\,,\,0\,,\,1\,,\,-(1+dx)\,dy\,,\,-dx\,,\,1+dx\,,\,0\,,\,-1\,,\,1\,,\,0\,,\,0 \\ -1\,,\,0\,,\,0\,,\,dx\,,\,0\,,\,0\,,\,\frac{1}{2}\,,\,0\,,\,0\,,\,1\,,\,0 \\ 0\,,\,-2\,,\,0\,,\,0\,,\,-(2+dx)\,dy+dy\,dx\,,\,0\,,\,1+\frac{dx}{2}\,,\,dy\,,\,0\,,\,0\,,\,1 \\ 0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,-dx\,,\,-\frac{dy}{2}\,,\,0\,,\,-1\,,\,-dy\,,\,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,-\frac{dx}{2}\,,\,-dy\,,\,0\,,\,0\,,\,-1 \\ 0\,,\,0\,,\,0\,,\,-dy\,,\,0\,,\,1\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0 \\ 0\,,\,0\,,\,0\,,\,dx\,,\,0\,,\,0\,,\,\frac{1}{2}\,,\,0\,,\,0\,,\,1\,,\,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,-dy\,,\,0\,,\,\frac{1}{2}\,,\,0\,,\,0\,,\,0\,,\,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,dx\,,\,0\,,\,0\,,\,1\,,\,0\,,\,0\,,\,0 \\ 0\,,\,0\,,\,0\,,\,1\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0 \\ 0\,,\,0\,,\,0\,,\,0\,,\,1\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0 \end{bmatrix}$$

We obtain that f_4 is induced by Z. We can check again that f_4 is a left A-isomorphism:

```
> TestIso(P4,P4p,Z,A);
```

$$true$$

If P4 = (0 P5), where P5 is defined by

```
> P5 := submatrix(P4,1..5,2..11);
```

$$P5 := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

then L_4 is isomorphic to the direct sum of A and L_5, where the left A-module L_5 is finitely presented by P5. The rank of L_5 is:

```
> OreRank(P5,A);
```

$$5$$

If P4' = (0 P5'), where P5' is defined by

```
> P5p := submatrix(P4p,1..5,2..11);
```

$$P5p := \begin{bmatrix} 0 & 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then L_4' is isomorphic to the direct sum of A and L_5', where the left A-module L_5' is finitely presented by P5'. The rank of L_5' is:

```
> OreRank(P5p,A);
```

$$5$$

Then we have that L_5 and L_5' are isomorphic. Let us compute the corresponding left A-isomorphism f_5.

```
> U := Cancellation(P5p,Z,A);
```

$$U := \begin{bmatrix} 0\,,\ -1\,,\ dx + (1+dx)\,dy\,,\ dx\,,\ -1-dx\,,\ \frac{1}{2}\,,\ 1\,,\ -1\,,\ 1\,,\ 0 \\ -2\,,\ 0\,,\ 0\,,\ -(2+dx)\,dy + dy\,dx\,,\ 0\,,\ 1+\frac{dx}{2}\,,\ dy\,,\ 0\,,\ 0\,,\ 1 \\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -dx\,,\ -\frac{dy}{2}\,,\ 0\,,\ -1\,,\ -dy\,,\ 0 \\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ -\frac{dx}{2}\,,\ -dy\,,\ 0\,,\ 0\,,\ -1 \\ 0\,,\ 0\,,\ -dy\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0 \\ 0\,,\ 0\,,\ dx\,,\ 0\,,\ 0\,,\ \frac{1}{2}\,,\ 0\,,\ 0\,,\ 1\,,\ 0 \\ 0\,,\ 0\,,\ 0\,,\ -dy\,,\ 0\,,\ \frac{1}{2}\,,\ 0\,,\ 0\,,\ 0\,,\ 0 \\ 0\,,\ 0\,,\ 0\,,\ dx\,,\ 0\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0 \\ 0\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0 \\ 0\,,\ 0\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0 \end{bmatrix}$$

We obtain that f_5 is induced by U. We can check again that f_5 is a left A-isomorphism:

```
> TestIso(P5,P5p,U,A);
```

$$true$$

If P5 = (0 P6), where P6 is defined by

```
> P6 := submatrix(P5,1..5,2..10);
```

$$P6 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

then L_5 is isomorphic to the direct sum of A and L_6, where the left A-module L_6 is finitely presented by P6. The rank of L_6 is:

```
> OreRank(P6,A);
```

$$4$$

If P5' = (0 P6'), where P6' is defined by

```
> P6p := submatrix(P5p,1..5,2..10);
```

$$P6p := \begin{bmatrix} 0 & 0 & 0 & -dx & -\frac{dy}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then L_5' is isomorphic to the direct sum of A and L_6', where the left A-module L_6' is finitely presented by P6'. The rank of L_6' is:

```
> OreRank(P6p,A);
```

$$4$$

Then we have that L_6 and L_6' are isomorphic. Let us compute the corresponding left A-isomorphism f_6.

```
> V := Cancellation(P6p,U,A);
```

$V :=$

$\left[2\,,\ -2\,dx - 2\,dy - 2\,dy\,dx\,,\ -(2+dx)\,dy + dy\,dx - 2\,dx\,,\ 2 + 2\,dx\,,\ \frac{dx}{2}\,,\ dy - 2\,,\ 2\,,\right.$
$\left.-2\,,\ 1\right]$

$\left[0\,,\ 0\,,\ 0\,,\ -dx\,,\ -\frac{dy}{2}\,,\ 0\,,\ -1\,,\ -dy\,,\ 0\right]$

$\left[0\,,\ 0\,,\ 0\,,\ 0\,,\ -\frac{dx}{2}\,,\ -dy\,,\ 0\,,\ 0\,,\ -1\right]$

$[0\,,\ -dy\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0]$

$\left[0\,,\ dx\,,\ 0\,,\ 0\,,\ \frac{1}{2}\,,\ 0\,,\ 0\,,\ 1\,,\ 0\right]$

$\left[0\,,\ 0\,,\ -dy\,,\ 0\,,\ \frac{1}{2}\,,\ 0\,,\ 0\,,\ 0\,,\ 0\right]$

$[0\,,\ 0\,,\ dx\,,\ 0\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0]$

$[0\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0]$

$[0\,,\ 0\,,\ 1\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0\,,\ 0]$

We obtain that f_6 is induced by V. We can check again that f_6 is a left A-isomorphism:

```
> TestIso(P6,P6p,V,A);
```

$$true$$

If P6 = (0 P7), where P7 is defined by

```
> P7 := submatrix(P6,1..5,2..9);
```

$$P7 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

then L_6 is isomorphic to the direct sum of A and L_7, where the left A-module L_7 is finitely presented by P7. The rank of L_7 is:

```
> OreRank(P7,A);
```

$$3$$

If P6' = (0 P7'), where P7' is defined by

```
> P7p := submatrix(P6p,1..5,2..9);
```

$$P7p := \begin{bmatrix} 0 & 0 & -dx & -\frac{dy}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{dx}{2} & -dy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

then L_6' is isomorphic to the direct sum of A and L_7', where the left A-module L_7' is finitely presented by P7'. The rank of L_7' is:

```
> OreRank(P7p,A);
```

$$3$$

Then we have that L_7 and L_7' are isomorphic. Let us compute the corresponding left A-isomorphism f_7.

```
> W := map(collect,Cancellation(P7p,V,A),[y,dx,dy]):
```

We obtain that f_7 is induced by W. We can check again that f_7 is a left A-isomorphism:

```
> TestIso(P7,P7p,W,A);
```

$$\mathit{true}$$

Let us compute a matrix Q satisfying P7 W = Q P7':

```
> Q := Factorize(Mult(P7,W,A),P7p,A);
```

$$Q := \begin{bmatrix} 1 & 0 & -1 & -dy & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -dy & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

If P7 = (0 P8), where P8 is defined by

```
> P8 := submatrix(P7,3..5,3..8);
```

$$P8 := \begin{bmatrix} -dx & -dy & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -dx & -dy \\ 0 & 0 & -dx & -dy & 0 & 0 \end{bmatrix}$$

then L_7 is isomorphic to the direct sum of A and L_8, where the left A-module L_8 is finitely presented by P8. The rank of L_8 is:

```
> OreRank(P8,A);
```

$$3$$

If P7' = (0 P8'), where P8' is defined by

```
> P8p := submatrix(P7p,1..2,1..5);
```

$$P8p := \begin{bmatrix} 0 & 0 & -dx & -\frac{dy}{2} & 0 \\ 0 & 0 & 0 & -\frac{dx}{2} & -dy \end{bmatrix}$$

then L_7' is isomorphic to the direct sum of A and L_8', where the left A-module L_8' is finitely presented by P8'. The rank of L_8' is:

```
> OreRank(P8p,A);
```

$$3$$

Let us consider the matrix W' obtained by taking the last five rows of W

```
> Wp := submatrix(W,3..8,1..5):
```

i.e., the rows of W' are respectively defined by:

```
> for i from 1 to 5 do submatrix(Wp,1..2,i..i); od;
```

$$[((3\,dy^3+6\,dy^2+3\,dy)\,dx+3\,dy^3+3\,dy^2)\,y+(-3\,dy^3-6\,dy^2-3\,dy)\,dx^2 + (3-3\,dy^4-9\,dy^3-6\,dy^2+3\,dy)\,dx-3\,dy^4-3\,dy^3+2\,dy]$$
$$[((-6\,dy-3-3\,dy^2)\,dx^2+(-3\,dy^2-3\,dy)\,dx)\,y+(3\,dy^2+6\,dy+3)\,dx^3 + (3+3\,dy^3+9\,dy^2+9\,dy)\,dx^2+(1+3\,dy^3+3\,dy^2+3\,dy)\,dx]$$

$$[(-3\,dy^2-3\,dy)\,y^2+((6\,dy+6\,dy^2)\,dx+6\,dy^3-6\,dy-6+6\,dy^2)\,y + (-3\,dy^2-3\,dy)\,dx^2+(6\,dy+6-6\,dy^3-6\,dy^2)\,dx+9\,dy-3\,dy^4-3\,dy^3+9\,dy^2]$$
$$[(3\,dy+3)\,dx\,y^2+((-6-6\,dy)\,dx^2+(-6\,dy^2-6\,dy)\,dx)\,y+(3\,dy+3)\,dx^3 + (6\,dy+6\,dy^2)\,dx^2+(-3\,dy-3+3\,dy^3+3\,dy^2)\,dx]$$

$$[((-3\,dy^2-3\,dy)\,dx-3\,dy^2-3\,dy)\,y+(3\,dy+3\,dy^2)\,dx^2 + (3\,dy^3+6\,dy^2-3+3\,dy)\,dx-2+3\,dy^3+3\,dy^2]$$
$$[((3\,dy+3)\,dx^2+(3\,dy+3)\,dx)\,y+(-3-3\,dy)\,dx^3+(-6-3\,dy^2-6\,dy)\,dx^2 + (-3-3\,dy^2-3\,dy)\,dx]$$

$$\begin{bmatrix} (-\frac{3}{4}\,dy^2-\frac{3}{4}\,dy)\,dx\,y+(\frac{3}{4}\,dy+\frac{3}{4}\,dy^2)\,dx^2+(\frac{3}{4}\,dy^3+\frac{3}{4}\,dy^2-\frac{3}{4})\,dx \\ (\frac{3\,dy}{4}+\frac{3}{4})\,dx^2\,y+\frac{1}{2}+(-\frac{3}{4}-\frac{3\,dy}{4})\,dx^3+(-\frac{3}{4}-\frac{3}{4}\,dy^2-\frac{3}{4}\,dy)\,dx^2 \end{bmatrix}$$

$$\begin{bmatrix} (-\frac{3}{2}\,dy^3+\frac{3}{2}\,dy^2+3\,dy)\,y+(\frac{3}{2}\,dy^3-\frac{3}{2}\,dy^2-3\,dy)\,dx+3+\frac{3\,dy^4}{2}-\frac{3\,dy^3}{2}-3\,dy^2-\frac{3\,dy}{2} \\ (-3+\frac{3}{2}\,dy^2-\frac{3}{2}\,dy)\,dx\,y+(3-\frac{3}{2}\,dy^2+\frac{3}{2}\,dy)\,dx^2+(3-\frac{3}{2}\,dy^3+\frac{3}{2}\,dy^2+\frac{3}{2}\,dy)\,dx \end{bmatrix}$$

Let us consider the matrix Q' obtained by taking the last three rows of Q:

```
> Qp := submatrix(Q,3..5,1..2);
```

$$Qp := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We can easily check that P8 W' = Q' P8':

```
> simplify(evalm(Mult(P8,Wp,A)-Mult(Qp,P8p,A)));
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can also check that the left A-homomorphism f_8 from L_8 to L_8' induced by W' is a left A-isomorphism:

```
> TestIso(P8,P8p,Wp,A);
```

$$true$$

We note that the left A-module L_8 corresponds to Cosserat's equations. Moreover, from the structure of the matrix P8', it is clear that L_8' is isomorphic to the direct sum of $A^{1\times 2}$ and L_9', where the left A-module L_9' is finitely presented by P9' and P9' is defined by:

```
> P9p := submatrix(P8p,1..2,3..5);
```

$$P9p := \begin{bmatrix} -dx & -\frac{dy}{2} & 0 \\ 0 & -\frac{dx}{2} & -dy \end{bmatrix}$$

We also note that L_9' corresponds to the linear PD system defined by the equilibrium of the symmetric stress tensor. Hence, we find again that L_8 is isomorphic to the direct sum of $A^{1\times 2}$ and L_9'.