Let us consider the first Weyl algebra $A = A_1(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers,

$$ A := \text{DefineOreAlgebra}(\text{diff}=[d,t], \text{polynom}=[t]) $$

the left $A$-module $M$ finitely presented by the matrix defined by

$$ R := \text{evalm}([[t^2,t],[t*d+2,d]]) $$

and the left $A$-module $M'$ finitely presented by the matrix defined by

$$ R_p := \text{evalm}([[t^2],[t*d+2]]) $$

Let us also consider the left $A$-homomorphism $\chi$ from $M'$ to $M$ induced by the matrix defined by

$$ W := \text{evalm}([[1,0]]) $$

We can check that $\chi$ is a left $A$-isomorphism:

$$ \text{TestIso}(R_p,R,W,A); $$

true

Since $M$ is isomorphic to $M'$, the direct sum of $A^{1\times 4}$ and $N_1$ and the direct sum of $A^{1\times 3}$ and $N_2$, where $N_1$ and $N_2$ are finitely presented by the formal adjoint of $R$ and $R'$, respectively, are isomorphic. The corresponding left $A$-isomorphism can be computed by the command \texttt{AuslanderEquivalence}:

$$ \text{unprotect}(O); $$

$$ O := \text{AuslanderEquivalence}(R_p,R,W,A,\text{opt}); $$

We obtain that the left $A$-module $L$ finitely presented by the matrix defined by

$$ P := O[1]; $$

$$ P := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^2 & 1-td \\ 0 & 0 & 0 & 0 & t & -d \end{bmatrix} $$

is isomorphic to the direct sum of $A^{1\times 4}$ and $N_1$, and the left $A$-module $L'$ finitely presented by the matrix defined by

$$ P_p := O[2]; $$
\[
P_p := \begin{bmatrix}
0 & 0 & 0 & t^2 & 1 - t d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
is isomorphic to the direct sum of \( A^{1 \times 3} \) and \( N_2 \). Moreover, the matrix defined by
\[
X := \begin{bmatrix}
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 \\
1 & -t^2 & -1 + t d & 0 & 0 & 1 & 0 \\
0 & -t & d & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & t^2 & 1 - t d & -1 & 0 \\
0 & 0 & 0 & t d + 2 & -d^2 & 0 & -d \\
0 & 0 & 0 & t^2 & 1 - t d & 0 & -t \\
\end{bmatrix}
\]
induces a left \( A \)-homomorphism \( f \) from \( L \) to \( L' \) which is a left \( A \)-isomorphism:
\[
\text{true}
\]
The inverse \( f^{-1} \) of \( f \) is induced by the matrix defined by:
\[
X_p := \begin{bmatrix}
-t^2 & -1 + t d & 1 & 0 & 1 & 0 & 0 \\
-t d - 2 & d^2 & 0 & d & 0 & 1 & 0 \\
-t^2 & -1 + t d & 0 & t & 0 & 0 & 1 \\
-t d - 1 & d^2 & 0 & d & 0 & 1 & 0 \\
-t^2 & t d & 0 & t & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & t^2 & 1 - t d \\
0 & 0 & 0 & 0 & 0 & t & -d \\
\end{bmatrix}
\]
We can check again that the above left \( A \)-homomorphism is an isomorphism:
\[
\text{true}
\]
If \( P = (0 \ P_2) \), where \( P_2 \) is defined by
\[
P_2 := \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & t^2 & 1 - t d \\
0 & 0 & 0 & t & -d \\
\end{bmatrix}
\]
and \( P' = (0, P_2') \), where \( P_2' \) is defined by
\[
P_2' := \begin{bmatrix}
0 & t^2 & 1 - t d & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
L2 is the left \( A \)-module finitely presented by \( P_2 \) and \( L_2' \) is the left \( A \)-module finitely presented by \( P_2' \), then \( L_2 \) is isomorphic to \( L_2' \). Let us compute the corresponding left \( A \)-isomorphism \( f_2 \).
We obtain that the left $A$-homomorphism $f_2$ from $L_2$ to $L_2'$ induced by $Y$ is a left $A$-isomorphism.

> TestIso(P2,P2p,Y,A);

true

If $P_2 = (0 P_3)$ and $P_2' = (0 P_3')$, where

> P3 := submatrix(P2,1..3,2..6);
P3 :=
\[
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & t^2 & 1-td \\
0 & 0 & t & -d
\end{bmatrix}
\]

> P3p := submatrix(P2p,1..3,2..6);
P3p :=
\[
\begin{bmatrix}
0 & t^2 & 1-td & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

$L_3$ is the left $A$-module finitely presented by $P_3$ and $L_3'$ is the left $A$-module finitely presented by $P_3'$, then $L_3$ is isomorphic to $L_3'$. Let us compute the corresponding left $A$-isomorphism $f_3$.

> Z := Cancellation(P3p,Y,A);

Z :=
\[
\begin{bmatrix}
1 & 1-t^2 & -2+td & 1 & 0 \\
0 & -t & d & 0 & 1 \\
0 & t^2 & 1-td & -1 & 0 \\
0 & td+2 & -d^2 & 0 & -d \\
0 & t^2 & 1-td & 0 & -t
\end{bmatrix}
\]

We obtain that the left $A$-homomorphism $f_3$ from $L_3$ to $L_3'$ induced by $Z$ is an isomorphism. Let us check that $f_3$ is a left $A$-isomorphism.

> TestIso(P3,P3p,Z,A);

true

Finally, if $P_3 = (0 P_4)$, where

> P4 := submatrix(P3p,1..3,2..5);
P4 :=
\[
\begin{bmatrix}
t^2 & 1-td & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

$P_3' = (0 P_4')$, where

> P4p := submatrix(P3,1..3,2..5);
\[ P_4p := \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & t^2 & 1 - td \\
0 & 0 & t & -d \\
\end{bmatrix} \]

L4 is the left \( A \)-module finitely presented by \( P_4 \), and \( L_4' \) is the left \( A \)-module finitely presented by \( P_4' \), then we cannot deduce a left \( A \)-isomorphism \( f_4 \) from \( L_4 \) to \( L_4' \) from \( f_3 \) since \( \text{rank}_A(L_4') = 1 \):

\[
\text{> U := Cancellation(P4p,Z,A);}
\]

Error, (in Stafford/ReductionOfUnimodularElement) expecting that the rank of the left module presented by the first matrix is at least 2.