We study the linearized time-invariant OD system formed by two pendula mounted on a cart.

See J. W. Polderman, J. C. Willems, *Introduction to Mathematical Systems Theory. A Behavioral Approach*, Springer, 1998, p. 159-160.

```
> with(Ore_algebra):
> with(OreModules):
```

1 Two pendula mounted on a cart: case without frictions in the joints of the rods

We define the Ore algebra *Alg*, where *Dt* acts as differentiation w.r.t. t. Since in this example the system has constant coefficients (the following matrix R is independent of t), we actually are in a commutative context. Note that the constants *m1*, *m2*, M, *L1*, *L2*, g have to be declared in the definition of the Ore algebra.

```
> Alg := DefineOreAlgebra(diff=[Dt,t], polynom=[t], comm=[m1, m2, M, L1, L2, g]):
```

We enter the matrix which defines the system:

```
> R := evalm([[m1*L1*Dt^2, m2*L2*Dt^2,(M+m1+m2)*Dt^2,-1],
> [m1*L1^2*Dt^2-m1*L1*g, 0, m1*L1*Dt^2,0],
> [0, m2*L2^2*Dt^2-m2*L2*g, m2*L2*Dt^2,0]]);
```

$$R := \begin{bmatrix} m1\,L1\,Dt^2 & m2\,L2\,Dt^2 & (M+m1+m2)\,Dt^2 & -1 \\ m1\,L1^2\,Dt^2 - m1\,L1\,g & 0 & m1\,L1\,Dt^2 & 0 \\ 0 & m2\,L2^2\,Dt^2 - m2\,L2\,g & m2\,L2\,Dt^2 & 0 \end{bmatrix}$$

We compute the formal adjoint *R_ adj* of R:

```
> R_adj := Involution(R, Alg);
```

$$R\_adj := \begin{bmatrix} m1\,L1\,Dt^2 & m1\,L1^2\,Dt^2 - m1\,L1\,g & 0 \\ m2\,L2\,Dt^2 & 0 & m2\,L2^2\,Dt^2 - m2\,L2\,g \\ Dt^2\,M + Dt^2\,m1 + Dt^2\,m2 & m1\,L1\,Dt^2 & m2\,L2\,Dt^2 \\ -1 & 0 & 0 \end{bmatrix}$$

In order to check controllability and parametrizability of the system, we compute the first extension module ext^1 with values in *Alg* of the *Alg*-module N which is associated with *R_adj*:

```
> st := time(): Ext1 := Exti(R_adj, Alg, 1): time()-st;
```

$$0.479$$

```
> Ext1[1];
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This shows that ext^1 of N is the zero module. Hence, the system R is generically controllable and, equivalently, generically parametrizable.

We shall see later that these assertions hold when the lengths of the two pendula are different ($L1 \neq L2$). A parametrization of R is given by *Ext1*[3]:

```
> map(collect, Ext1[3], Dt);
```

$$[-L2\,Dt^4 + Dt^2\,g]$$
$$[-Dt^4\,L1 + Dt^2\,g]$$
$$[L2\,Dt^4\,L1 + (-L2\,g - g\,L1)\,Dt^2 + g^2]$$
$$[L2\,Dt^6\,L1\,M + (-g\,L1\,M - L2\,g\,M - L2\,m1\,g - L1\,g\,m2)\,Dt^4 + (g^2\,M + m1\,g^2 + g^2\,m2)\,Dt^2]$$

Or equivalently, we can directly computed the generic parametrization of R by using the command *Parametrization*:

```
>  Parametrization(R, Alg);
```

$$[g\,\%2 - L2\,\%1]$$
$$[g\,\%2 - L1\,\%1]$$
$$[g^2\,\xi_1(t) - \%2\,L2\,g - \%2\,g\,L1 + L1\,L2\,\%1]$$
$$\Big[\%2\,g^2\,M + \%2\,m1\,g^2 + \%2\,g^2\,m2 - \%1\,g\,L1\,M - \%1\,L2\,g\,M - \%1\,L2\,m1\,g - \%1\,L1\,g\,m2 + M\,L2\,L1\,(\tfrac{d^6}{dt^6}\,\xi_1(t))\Big]$$
$$\%1 := \tfrac{d^4}{dt^4}\,\xi_1(t)$$
$$\%2 := \tfrac{d^2}{dt^2}\,\xi_1(t)$$

Since the algebra *Alg* is a principal ideal domain (every ideal of *Alg* is generated by a single element), we know that the module M which is associated with the system R is actually *free* and the system R is *flat*. Hence, we can compute a left inverse of the parametrization and get a *flat output* of the system:

```
>  S := LeftInverse(Ext1[3], Alg);
```

$$S := \left[\begin{array}{cccc} \dfrac{L1^2}{g^2\,(-L2 + L1)} & -\dfrac{L2^2}{g^2\,(-L2 + L1)} & -\dfrac{L2 - L1}{g^2\,(-L2 + L1)} & 0 \end{array}\right]$$

Thus, we obtain that $\xi = S\,(x1 : x2 : x3 : u)^T$ is a flat output of the system which satisfies

$$(x1 : x2 : x3 : u)^T = Ext1[3]\,\xi.$$

We know that R admits a right-inverse if and only if the module M associated with R is projective. Since M is free (as we have seen above), we obtain a right-inverse T of R by means of *RightInverse*(R, *Alg*). Let us compute it:

```
>  T := RightInverse(R, Alg);
```

$$T :=$$
$$\left[0\,,\ -\frac{1}{g\,m1\,(-L2+L1)}+\frac{L2}{g\,L1\,m1\,(-L2+L1)}+\frac{L2\,Dt^2}{g^2\,m1\,(-L2+L1)}\,,\right.$$
$$\left.-\frac{L2\,Dt^2}{g^2\,(-L2+L1)\,m2}\right]$$
$$\left[0\,,\ \frac{L1\,Dt^2}{g^2\,m1\,(-L2+L1)}\,,\right.$$
$$\left.-\frac{Dt^2\,L1}{g^2\,(-L2+L1)\,m2}-\frac{L1}{g\,(-L2+L1)\,m2\,L2}+\frac{1}{g\,(-L2+L1)\,m2}\right]$$
$$\left[0\,,\ \frac{L1}{g\,m1\,(-L2+L1)}-\frac{L1\,L2\,Dt^2}{g^2\,m1\,(-L2+L1)}\,,\ -\frac{L2}{g\,(-L2+L1)\,m2}+\frac{L2\,Dt^2\,L1}{g^2\,(-L2+L1)\,m2}\right]$$
$$\left[-1\,,\ \frac{L2\,Dt^2}{g\,(-L2+L1)}+\frac{L1\,Dt^2\,M}{g\,m1\,(-L2+L1)}+\frac{L1\,Dt^2\,m2}{g\,m1\,(-L2+L1)}-\frac{L1\,L2\,Dt^4\,M}{g^2\,m1\,(-L2+L1)}\,,\right.$$
$$\left.-\frac{L2\,Dt^2\,M}{g\,(-L2+L1)\,m2}-\frac{L2\,Dt^2\,m1}{g\,(-L2+L1)\,m2}+\frac{L2\,Dt^4\,L1\,M}{g^2\,(-L2+L1)\,m2}-\frac{Dt^2\,L1}{g\,(-L2+L1)}\right]$$

```
> Mult(R, T, Alg);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that this holds only for the generic case when the denominators of the entries in T do not vanish. We discover the same obstruction to flatness as above, namely the condition $L1 \neq L2$.

Let us compute the Brunovský canonical form of the system in the case where $L1 \neq L2$.

```
> B := Brunovsky(R, Alg);
```

$$B := \begin{bmatrix} \frac{L1^2}{g^2\,(-L2+L1)} & -\frac{L2^2}{g^2\,(-L2+L1)} & \frac{1}{g^2} & 0 \\ \frac{Dt\,L1^2}{g^2\,(-L2+L1)} & -\frac{Dt\,L2^2}{g^2\,(-L2+L1)} & \frac{Dt}{g^2} & 0 \\ \frac{L1}{g\,(-L2+L1)} & -\frac{L2}{g\,(-L2+L1)} & 0 & 0 \\ \frac{L1\,Dt}{g\,(-L2+L1)} & -\frac{L2\,Dt}{g\,(-L2+L1)} & 0 & 0 \\ \frac{1}{-L2+L1} & -\frac{1}{-L2+L1} & 0 & 0 \\ \frac{Dt}{-L2+L1} & -\frac{Dt}{-L2+L1} & 0 & 0 \\ \frac{g\,(L2\,M+L2\,m1-m1\,L1)}{M\,L1\,L2\,(-L2+L1)} & -\frac{g\,(-m2\,L2+L1\,M+L1\,m2)}{M\,L1\,L2\,(-L2+L1)} & 0 & \frac{1}{L2\,L1\,M} \end{bmatrix}$$

In other words, we have the following transformation between the system variables $x1$, $x2$ and u and the Brunovský variables $z[i]$ and v:

```
> evalm([seq([z[i](t)],i=1..6),[v(t)]])=ApplyMatrix(B, [x1(t),x2(t),x3(t),u(t)],
> Alg);
```

$$\left[\begin{array}{c} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \\ z_6(t) \\ \mathrm{v}(t) \end{array}\right] = \left[\begin{array}{c} \frac{L1^2\,\mathrm{x1}(t)}{g^2\,(-L2+L1)} - \frac{L2^2\,\mathrm{x2}(t)}{g^2\,(-L2+L1)} + \frac{\mathrm{x3}(t)}{g^2} \\ \frac{L1^2\,(\frac{d}{dt}\,\mathrm{x1}(t))}{g^2\,(-L2+L1)} - \frac{L2^2\,(\frac{d}{dt}\,\mathrm{x2}(t))}{g^2\,(-L2+L1)} + \frac{\frac{d}{dt}\,\mathrm{x3}(t)}{g^2} \\ \frac{L1\,\mathrm{x1}(t)}{g\,(-L2+L1)} - \frac{L2\,\mathrm{x2}(t)}{g\,(-L2+L1)} \\ \frac{L1\,(\frac{d}{dt}\,\mathrm{x1}(t))}{g\,(-L2+L1)} - \frac{L2\,(\frac{d}{dt}\,\mathrm{x2}(t))}{g\,(-L2+L1)} \\ \frac{\mathrm{x1}(t)}{-L2+L1} - \frac{\mathrm{x2}(t)}{-L2+L1} \\ \frac{\frac{d}{dt}\,\mathrm{x1}(t)}{-L2+L1} - \frac{\frac{d}{dt}\,\mathrm{x2}(t)}{-L2+L1} \\ \frac{g\,(L2\,M + L2\,m1 - m1\,L1)\,\mathrm{x1}(t)}{M\,L1\,L2\,(-L2+L1)} - \frac{g\,(-m2\,L2 + L1\,M + L1\,m2)\,\mathrm{x2}(t)}{M\,L1\,L2\,(-L2+L1)} + \frac{\mathrm{u}(t)}{L2\,L1\,M} \end{array}\right]$$

Let us check that the new variables $z[i]$ and v satisfy a Brunovský canonical form:

```
> F := Elimination(linalg[stackmatrix](B, R), [x1,x2,x3,u],
> [seq(z[i],i=1..6),v,0,0,0], Alg):
> ApplyMatrix(F[1], [x1(t),x2(t),x3(t),u(t)], Alg)=
> ApplyMatrix(F[2], [seq(z[i](t),i=1..6),v(t)], Alg);
```

$$\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mathrm{u}(t) \\ \mathrm{x3}(t) \\ \mathrm{x2}(t) \\ \mathrm{x1}(t) \end{array}\right] =$$

$$\left[-(\tfrac{d}{dt}\,z_6(t)) + \mathrm{v}(t)\right]$$
$$\left[-(\tfrac{d}{dt}\,z_5(t)) + z_6(t)\right]$$
$$\left[-(\tfrac{d}{dt}\,z_4(t)) + z_5(t)\right]$$
$$\left[-(\tfrac{d}{dt}\,z_3(t)) + z_4(t)\right]$$
$$\left[-(\tfrac{d}{dt}\,z_2(t)) + z_3(t)\right]$$
$$\left[-(\tfrac{d}{dt}\,z_1(t)) + z_2(t)\right]$$
$$\left[(m1\,g^2 + g^2\,m2 + g^2\,M)\,z_3(t) + (-L2\,g\,M - L2\,g\,m1 - g\,L1\,M - g\,L1\,m2)\,z_5(t) + L2\,L1\,M\,\mathrm{v}(t)\right]$$
$$\left[g^2\,z_1(t) + (-g\,L1 - L2\,g)\,z_3(t) + L2\,L1\,z_5(t)\right]$$
$$\left[g\,z_3(t) - L1\,z_5(t)\right]$$
$$\left[g\,z_3(t) - L2\,z_5(t)\right]$$

We now turn to the special case, where the two pendula have the same length: $L1 = L2$.

```
> R2 := subs(L2=L1, evalm(R));
```

$$R2 := \begin{bmatrix} m1\,L1\,Dt^2 & Dt^2\,L1\,m2 & (M+m1+m2)\,Dt^2 & -1 \\ m1\,L1^2\,Dt^2 - m1\,L1\,g & 0 & m1\,L1\,Dt^2 & 0 \\ 0 & m2\,L1^2\,Dt^2 - g\,L1\,m2 & Dt^2\,L1\,m2 & 0 \end{bmatrix}$$

```
> Ext1 := Exti(Involution(R2, Alg), Alg, 1):  Ext1[1];
```

$$\begin{bmatrix} L1\,Dt^2 - g & 0 & 0 \\ 0 & L1\,Dt^2 - g & 0 \\ 0 & 0 & L1\,Dt^2 - g \end{bmatrix}$$

We see that the module M which is associated with *R2* is not torsion-free. The torsion submodule is generated by the rows of *Ext1*[2]:

```
> Ext1[2];
```

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & g\,m1 + g\,m2 & Dt^2\,M & -1 \\ 0 & Dt^2\,L1\,M - g\,M - g\,m1 - g\,m2 & 0 & 1 \end{bmatrix}$$

These elements are killed by the diagonal entry in *Ext1*[1]: ($L1\ Dt^2$ - g). We can also obtain this result by invoking *TorsionElements*:

```
> TorsionElements(R2, [x1(t),x2(t),x3(t),u(t)], Alg);
```

$$\left[\begin{bmatrix} -g\,\theta_1(t) + L1\,(\frac{d^2}{dt^2}\,\theta_1(t)) = 0 \\ -g\,\theta_2(t) + L1\,(\frac{d^2}{dt^2}\,\theta_2(t)) = 0 \\ -g\,\theta_3(t) + L1\,(\frac{d^2}{dt^2}\,\theta_3(t)) = 0 \end{bmatrix}, \begin{bmatrix} \theta_1(t) = \mathrm{x1}(t) - \mathrm{x2}(t) \\ \theta_2(t) = (g\,m1 + g\,m2)\,\mathrm{x2}(t) + M\,(\frac{d^2}{dt^2}\,\mathrm{x3}(t)) - \mathrm{u}(t) \\ \theta_3(t) = (-g\,m2 - g\,M - g\,m1)\,\mathrm{x2}(t) + L1\,M\,(\frac{d^2}{dt^2}\,\mathrm{x2}(t)) + \mathrm{u}(t) \end{bmatrix}\right]$$

```
> AutonomousElements(R2, [x1(t),x2(t),x3(t),u(t)], Alg);
```

$$\left[\begin{bmatrix} m2\,m1\,L1\,g\,\theta_1(t) - L1\,m2\,\theta_3(t) = 0 \\ L1\,m2\,\theta_2(t) + L1\,m2\,\theta_3(t) = 0 \\ -g\,L1\,m2\,\theta_3(t) + L1^2\,m2\,(\frac{d^2}{dt^2}\,\theta_3(t)) = 0 \end{bmatrix}, \begin{bmatrix} \theta_1 = \dfrac{\_C1\,e^{(\frac{\sqrt{g}\,t}{\sqrt{L1}})} + \_C2\,e^{(-\frac{\sqrt{g}\,t}{\sqrt{L1}})}}{m1\,g} \\ \theta_2 = -\_C1\,e^{(\frac{\sqrt{g}\,t}{\sqrt{L1}})} - \_C2\,e^{(-\frac{\sqrt{g}\,t}{\sqrt{L1}})} \\ \theta_3 = \_C1\,e^{(\frac{\sqrt{g}\,t}{\sqrt{L1}})} + \_C2\,e^{(-\frac{\sqrt{g}\,t}{\sqrt{L1}})} \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} \theta_1 = \mathrm{x1}(t) - \mathrm{x2}(t) \\ \theta_2 = (g\,m1 + g\,m2)\,\mathrm{x2}(t) + M\,(\frac{d^2}{dt^2}\,\mathrm{x3}(t)) - \mathrm{u}(t) \\ \theta_3 = (-g\,m2 - g\,M - g\,m1)\,\mathrm{x2}(t) + L1\,M\,(\frac{d^2}{dt^2}\,\mathrm{x2}(t)) + \mathrm{u}(t) \end{bmatrix}\right]$$

The second matrix gives the explicit autonomous elements of the system in function of two arbitrary constants which can be computed using the knowledge of the initial conditions.

```
> V := FirstIntegral(R2, [x1(t),x2(t),x3(t),u(t)], Alg);
```

$$\begin{aligned} V := m1\,L1(&L1\,(\tfrac{d}{dt}\,\mathrm{x1}(t))\,\_C1\,e^{(\frac{2\sqrt{g}\,t}{\sqrt{L1}})} + L1\,(\tfrac{d}{dt}\,\mathrm{x1}(t))\,\_C2 - \sqrt{L1}\,\mathrm{x1}(t)\,\_C1\,\sqrt{g}\,e^{(\frac{2\sqrt{g}\,t}{\sqrt{L1}})} \\ &+ \sqrt{L1}\,\mathrm{x1}(t)\,\_C2\,\sqrt{g} - L1\,(\tfrac{d}{dt}\,\mathrm{x2}(t))\,\_C1\,e^{(\frac{2\sqrt{g}\,t}{\sqrt{L1}})} - L1\,(\tfrac{d}{dt}\,\mathrm{x2}(t))\,\_C2 \\ &+ \sqrt{L1}\,\mathrm{x2}(t)\,\_C1\,\sqrt{g}\,e^{(\frac{2\sqrt{g}\,t}{\sqrt{L1}})} - \sqrt{L1}\,\mathrm{x2}(t)\,\_C2\,\sqrt{g})e^{(-\frac{\sqrt{g}\,t}{\sqrt{L1}})} \end{aligned}$$

Let us check that the time derivative of V is 0.

```
> Vdot := simplify(diff(V, t));
```

$$Vdot := -m1\, L1 (-L1\, e^{(\frac{\sqrt{g}\, t}{\sqrt{L1}})} (\frac{d^2}{dt^2}\, \mathrm{x1}(t))\, \_C1 - L1\, e^{(-\frac{\sqrt{g}\, t}{\sqrt{L1}})} (\frac{d^2}{dt^2}\, \mathrm{x1}(t))\, \_C2$$
$$+ e^{(\frac{\sqrt{g}\, t}{\sqrt{L1}})}\, \mathrm{x1}(t)\, \_C1\, g + L1\, e^{(\frac{\sqrt{g}\, t}{\sqrt{L1}})} (\frac{d^2}{dt^2}\, \mathrm{x2}(t))\, \_C1 + L1\, e^{(-\frac{\sqrt{g}\, t}{\sqrt{L1}})} (\frac{d^2}{dt^2}\, \mathrm{x2}(t))\, \_C2$$
$$- e^{(\frac{\sqrt{g}\, t}{\sqrt{L1}})}\, \mathrm{x2}(t)\, \_C1\, g + g\, e^{(-\frac{\sqrt{g}\, t}{\sqrt{L1}})}\, \mathrm{x1}(t)\, \_C2 - g\, e^{(-\frac{\sqrt{g}\, t}{\sqrt{L1}})}\, \mathrm{x2}(t)\, \_C2)$$

The equations of the system are defined by:

```
> Sys_mod := ApplyMatrix(R2, [x1(t),x2(t),x3(t),u(t)], Alg) = evalm([[0],[0]]);
```

$$Sys\_mod := \begin{bmatrix} m1\, L1\, (\frac{d^2}{dt^2}\, \mathrm{x1}(t)) + L1\, m2\, (\frac{d^2}{dt^2}\, \mathrm{x2}(t)) + (M + m1 + m2)\, (\frac{d^2}{dt^2}\, \mathrm{x3}(t)) - \mathrm{u}(t) \\ -m1\, L1\, g\, \mathrm{x1}(t) + m1\, L1^2\, (\frac{d^2}{dt^2}\, \mathrm{x1}(t)) + m1\, L1\, (\frac{d^2}{dt^2}\, \mathrm{x3}(t)) \\ -g\, L1\, m2\, \mathrm{x2}(t) + m2\, L1^2\, (\frac{d^2}{dt^2}\, \mathrm{x2}(t)) + L1\, m2\, (\frac{d^2}{dt^2}\, \mathrm{x3}(t)) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From the second and third equations, let us extract the second derivatives of *x1* and *x2*.

```
> lhs1 := solve(lhs(Sys_mod)[2,1], diff(x1(t),t$2));
```

$$lhs1 := \frac{g\, \mathrm{x1}(t) - (\frac{d^2}{dt^2}\, \mathrm{x3}(t))}{L1}$$

```
> lhs2 := solve(lhs(Sys_mod)[3,1], diff(x2(t),t$2));
```

$$lhs2 := \frac{g\, \mathrm{x2}(t) - (\frac{d^2}{dt^2}\, \mathrm{x3}(t))}{L1}$$

Finally, let us substitute the second derivatives of *x1* and *x2* in *Vdot*. We obtain:

```
> simplify(subs({diff(x1(t),t$2)=lhs1, diff(x2(t),t$2)=lhs2}, Vdot));
```

$$0$$

Finally, let us compute the parametrization of the uncontrollable system defined by *R2* by means of an arbitrary function ξ_1 and two arbitrary constants *C1* and *C2*:

```
> P := Parametrization(R2, Alg);
```

$$P := \begin{bmatrix} \frac{\_C1\, e^{(\frac{\sqrt{g}\, t}{\sqrt{L1}})} + \_C2\, e^{(-\frac{\sqrt{g}\, t}{\sqrt{L1}})} - g\, \%1\, m1}{g\, m1} \\ -\%1 \\ -g\, \xi_1(t) + L1\, \%1 \\ \_C1\, e^{(\frac{\sqrt{g}\, t}{\sqrt{L1}})} + \_C2\, e^{(-\frac{\sqrt{g}\, t}{\sqrt{L1}})} + L1\, M\, (\frac{d^4}{dt^4}\, \xi_1(t)) - g\, \%1\, M - g\, \%1\, m1 - g\, \%1\, m2 \end{bmatrix}$$
$$\%1 := \frac{d^2}{dt^2}\, \xi_1(t)$$

We can check that P parametrizes some solutions of the system defined by R:

```
> simplify(ApplyMatrix(R2, P, Alg));
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In fact, we can prove that P parametrizes all the C^∞solutions. For more details, we refer the reader to A. Quadrat, D. Robertz, "*On Monge problem for uncontrollable linear systems*", to appear.

2 Two pendula mounted on a cart: case with frictions in the joints of the rods

In the previous model, we have neglected the effect of friction in the joints of the rods.

As in J. W. Polderman, J. C. Willems, *Introduction to Mathematical Systems Theory. A Behavioral Approach*, Springer, 1998, p. 171-172, let us incorporate these effects into the model.

```
> Alg2 := DefineOreAlgebra(diff=[Dt,t], polynom=[t],
> comm=[m1,m2,M,L1,L2,g,d1,d2,k1,k2]):
```

The new model is defined by the following matrix:

```
> R2 := evalm([[m1*L1*Dt^2, m2*L2*Dt^2,(M+m1+m2)*Dt^2,-1],
> [m1*L1^2*Dt^2+d1*Dt+k1-m1*L1*g, 0, m1*L1*Dt^2,0],
> [0, m2*L2^2*Dt^2+d2*Dt+k2-m2*L2*g, m1*L1*Dt^2,0]]);
```

$$R2 := \begin{bmatrix} m1\ L1\ Dt^2\,,\ m2\ L2\ Dt^2\,,\ (M + m1 + m2)\ Dt^2\,,\ -1 \\ m1\ L1^2\ Dt^2 + d1\ Dt + k1 - m1\ L1\ g\,,\ 0\,,\ m1\ L1\ Dt^2\,,\ 0 \\ 0\,,\ m2\ L2^2\ Dt^2 + d2\ Dt + k2 - m2\ L2\ g\,,\ m1\ L1\ Dt^2\,,\ 0 \end{bmatrix}$$

Let us compute the formal adjoint of *R2*:

```
> R2_adj := Involution(R2, Alg2);
```

$$R2\_adj := \begin{bmatrix} m1\ L1\ Dt^2\,,\ m1\ L1^2\ Dt^2 - d1\ Dt + k1 - m1\ L1\ g\,,\ 0 \\ m2\ L2\ Dt^2\,,\ 0\,,\ m2\ L2^2\ Dt^2 - d2\ Dt + k2 - m2\ L2\ g \\ Dt^2\ M + Dt^2\ m1 + Dt^2\ m2\,,\ m1\ L1\ Dt^2\,,\ m1\ L1\ Dt^2 \\ -1\,,\ 0\,,\ 0 \end{bmatrix}$$

Let us check whether or not the new system is controllable.

```
> st := time():  Ext2 := Exti(R2_adj, Alg2, 1):  time()-st; Ext2[1];
```

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We obtain that the new system is generically controllable. A parametrization of the system is then defined by:

```
> P2 := map(collect, Ext2[3], Dt):
> ApplyMatrix(P2, [xi(t)], Alg2);
```

$$\left[(-m2\,L2\,m1\,L1\,g + m1\,k2\,L1)\,(\tfrac{d^2}{dt^2}\,\xi(t)) + L2^2\,m2\,m1\,L1\,(\tfrac{d^4}{dt^4}\,\xi(t)) + L1\,m1\,d2\,(\tfrac{d^3}{dt^3}\,\xi(t))\right]$$

$$\left[(-m1^2\,L1^2\,g + m1\,k1\,L1)\,(\tfrac{d^2}{dt^2}\,\xi(t)) + m1^2\,L1^3\,(\tfrac{d^4}{dt^4}\,\xi(t)) + L1\,m1\,d1\,(\tfrac{d^3}{dt^3}\,\xi(t))\right]$$

$$\left[(-k2\,k1 - m2\,m1\,L2\,L1\,g^2 + g\,m1\,k2\,L1 + g\,m2\,L2\,k1)\,\xi(t) + (g\,d1\,m2\,L2 - k1\,d2 + g\,d2\,m1\,L1 - d1\,k2)\,(\tfrac{d}{dt}\,\xi(t)) + (m2\,L2^2\,m1\,L1\,g - d1\,d2 - m1\,L1^2\,k2 - m2\,L2^2\,k1 + m1\,L1^2\,m2\,L2\,g)\,(\tfrac{d^2}{dt^2}\,\xi(t)) - m1\,L1^2\,m2\,L2^2\,(\tfrac{d^4}{dt^4}\,\xi(t)) + (-L2^2\,d1\,m2 - m1\,L1^2\,d2)\,(\tfrac{d^3}{dt^3}\,\xi(t))\right]$$

$$\left[(g\,L1\,k2\,m1^2 + g\,L1\,m1\,m2\,k2 - k2\,k1\,m1 - k2\,k1\,m2 + M\,g\,m1\,k2\,L1 - M\,k2\,k1 - g^2\,L1\,L2\,m1\,m2^2 + g\,L2\,m2\,k1\,m1 - M\,m2\,m1\,L2\,L1\,g^2 + g\,L2\,k1\,m2^2 + M\,g\,m2\,L2\,k1 - g^2\,L1\,L2\,m2\,m1^2)(\tfrac{d^2}{dt^2}\,\xi(t)) + (-g\,L1^2\,L2\,m2\,m1^2 + g\,L1^2\,L2\,m1\,m2^2 - M\,d1\,d2 + g\,L1\,L2^2\,m2\,m1^2 + g\,L1\,L2^2\,m1\,m2^2 - L1^2\,m1\,m2\,k2 - L2^2\,m2\,k1\,m1 + L2\,m2\,k1\,m1\,L1 + M\,m1\,L1^2\,m2\,L2\,g + M\,m2\,L2^2\,m1\,L1\,g - M\,m2\,L2^2\,k1 - L2^2\,k1\,m2^2 - d1\,d2\,m1 - d1\,d2\,m2 - M\,m1\,L1^2\,k2)(\tfrac{d^4}{dt^4}\,\xi(t)) + (L2\,d1\,m2\,m1\,L1 - M\,m1\,L1^2\,d2 - L2^2\,d1\,m2\,m1 - L2^2\,d1\,m2^2 - M\,m2\,L2^2\,d1 - L1^2\,d2\,m2\,m1)(\tfrac{d^5}{dt^5}\,\xi(t)) + (-M\,k1\,d2 - M\,d1\,k2 + M\,g\,d2\,m1\,L1 + g\,L1\,d2\,m1^2 + g\,L1\,d2\,m2\,m1 - m1\,k1\,d2 - m1\,d1\,k2 - m2\,k1\,d2 - k2\,d1\,m2 + g\,L2\,d1\,m2\,m1 + M\,g\,d1\,m2\,L2 + g\,L2\,d1\,m2^2)(\tfrac{d^3}{dt^3}\,\xi(t)) + (-L1^2\,L2^2\,m1\,m2^2 - M\,m1\,L1^2\,L2^2\,m2 + m2\,m1^2\,L1^3\,L2)\,(\tfrac{d^6}{dt^6}\,\xi(t))\right]$$

The fact that the *Alg2*-module *M2* associated with *R2* is generically torsion-free implies that *M2* is also generically a projective *Alg2*-module as *Alg2* is a hereditary ring. This result can be directly verified by checking whether or not a right-inverse exists for *R2*.

```
> st := time():  T2 := RightInverse(R2, Alg2):  time()-st;
```

$$9.570$$

```
> Mult(R2, T2, Alg2);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, a right-inverse *T2* of *R2* is defined by:

```
> map(collect, simplify(T2), Dt);
```

$$[0\,, -(m1^2\, L1^4\, L2^2\, m2\, d1\, k2 + 2\, m1^2\, L1^3\, g\, L2^4\, m2^2\, d1 + m1^2\, L1^4\, L2^2\, m2\, k1\, d2$$
$$- 2\, m1\, L1^2\, L2^4\, m2^2\, d1\, k1 - m1\, L1^2\, L2^2\, m2\, d1^2\, d2 + L2^4\, m2^2\, d1^3$$
$$- m1^2\, L1^4\, g\, L2^3\, m2^2\, d1 - m1^3\, L1^5\, g\, L2^2\, m2\, d2)Dt^3/(\%2) - (m1\, L1^2\, L2^4\, m2^2\, k1^2$$
$$- L2^4\, m2^2\, d1^2\, k1 - m1\, L1^2\, d1^2\, d2^2 + m1^2\, L1^4\, k1\, d2^2 - m1^2\, L1^4\, L2^2\, m2\, k1\, k2$$
$$+ L2^2\, m2\, d1^3\, d2 - m1^3\, L1^5\, g\, d2^2 + m1^3\, L1^4\, g^2\, L2^4\, m2^2 - 2\, m1^2\, L1^3\, g\, L2^4\, m2^2\, k1$$
$$+ m1\, L1\, g\, L2^4\, m2^2\, d1^2 + m1^3\, L1^5\, g\, L2^2\, m2\, k2 + m1^2\, L1^3\, g\, L2^2\, m2\, d1\, d2$$
$$+ m1^2\, L1^4\, d1\, d2\, k2 - m1^2\, L1^4\, g\, L2\, m2\, d1\, d2 + m1^2\, L1^4\, g\, L2^3\, m2^2\, k1$$
$$- m1\, L1^2\, L2^2\, m2\, d1\, d2\, k1 - m1^3\, L1^5\, g^2\, L2^3\, m2^2)Dt^2/(\%2)$$
$$-(m1^2\, L1^4\, g^2\, L2^2\, m2^2\, d1 - m1^2\, L1^3\, g\, d2^2\, d1 + 2\, m1\, L1^2\, g\, L2^3\, m2^2\, k1\, d1$$
$$- 2\, m1^2\, L1^3\, g^2\, L2^3\, m2^2\, d1 + m1^2\, L1^4\, k2^2\, d1 + m1\, L1^2\, d2^2\, k1\, d1$$
$$- 2\, m1^2\, L1^4\, g\, L2\, m2\, k2\, d1 - m1\, L1^2\, d1^2\, k2\, d2 - 2\, m1\, L1^2\, L2^2\, m2\, k2\, k1\, d1$$
$$+ L2^2\, m2\, d1^3\, k2 + m1\, L1^2\, g\, L2\, m2\, d1^2\, d2 - L2^2\, m2\, d1^2\, d2\, k1$$
$$+ 2\, m1^2\, L1^3\, g\, L2^2\, m2\, k2\, d1 + m1\, L1\, g\, L2^2\, m2\, d1^2\, d2 + L2^4\, m2^2\, k1^2\, d1$$
$$- g\, L2^3\, m2^2\, d1^3 + m1^2\, L1^2\, g^2\, L2^4\, m2^2\, d1 - 2\, m1\, L1\, g\, L2^4\, m2^2\, k1\, d1)Dt/(\%2)$$
$$-(L2^2\, m2\, k1^2\, d1\, d2 + m1\, L1^2\, d1\, d2\, k1\, k2 - m1\, L1^2\, k1^2\, d2^2 - m1^2\, L1^4\, k2^2\, k1$$
$$- L2^2\, m2\, d1^2\, k2\, k1 + m1^3\, L1^5\, g\, k2^2 - m1^3\, L1^4\, g^2\, d2^2 + 2\, m1\, L1^2\, L2^2\, m2\, k1^2\, k2$$
$$- 2\, m1^3\, L1^4\, g^3\, m2^2\, L2^3 + m1^3\, L1^3\, g^3\, m2^2\, L2^4 - 3\, m1^2\, L1^2\, g^2\, k1\, m2^2\, L2^4$$
$$+ m1^3\, L1^5\, g^3\, m2^2\, L2^2 + m1^2\, L1^2\, g^2\, L2^2\, m2\, d1\, d2 - k1^3\, m2^2\, L2^4$$
$$- m1^2\, L1^4\, g^2\, L2^2\, m2^2\, k1 + 4\, m1^2\, L1^3\, g^2\, L2^3\, m2^2\, k1 - m1\, L1\, g^2\, L2^3\, m2^2\, d1^2$$
$$+ 3\, m1\, L1\, g\, k1^2\, m2^2\, L2^4 + m1^2\, L1^3\, g^2\, L2\, m2\, d1\, d2 - 2\, m1^3\, L1^5\, g^2\, L2\, m2\, k2$$
$$+ 2\, m1^3\, L1^4\, g^2\, L2^2\, m2\, k2 + g\, L2^3\, m2^2\, k1\, d1^2 - 2\, m1\, L1^2\, g\, L2^3\, m2^2\, k1^2$$
$$- 2\, m1\, L1\, g\, L2^2\, m2\, d1\, d2\, k1 + m1\, L1\, g\, L2^2\, m2\, d1^2\, k2$$
$$- 4\, m1^2\, L1^3\, g\, L2^2\, m2\, k1\, k2 + 2\, m1^2\, L1^4\, g\, L2\, m2\, k1\, k2$$
$$- m1\, L1^2\, g\, L2\, m2\, d1\, d2\, k1 + 2\, m1^2\, L1^3\, g\, k1\, d2^2 - m1^2\, L1^3\, g\, d1\, d2\, k2)/(\%2),$$
$$-(-2\, m1\, L1^2\, L2^4\, m2^2\, d2\, k2 - L2^4\, m2^2\, d1\, d2^2 - L2^7\, m2^4\, d1\, g + L2^6\, m2^3\, k2\, d1$$
$$- m1\, L1\, L2^6\, m2^3\, g\, d2 + L2^6\, m2^3\, k1\, d2 + 2\, m1\, L1^2\, L2^5\, m2^3\, d2\, g$$
$$+ m1\, L1^2\, L2^2\, m2\, d2^3)Dt^3/(\%1) - (-3\, m1\, L1^2\, L2^2\, m2\, d2^2\, k2 - L2^2\, m2\, d1\, d2^3$$
$$+ 3\, m1\, L1^2\, L2^3\, m2^2\, d2^2\, g + 2\, L2^4\, m2^2\, k2\, d1\, d2 + m1\, L1^2\, L2^4\, m2^2\, k2^2$$
$$+ L2^4\, m2^2\, k1\, d2^2 - m1\, L1\, L2^4\, m2^2\, g\, d2^2 + m1\, L1^2\, d2^4 + L2^7\, m2^4\, k1\, g$$
$$- m1\, L1\, L2^7\, m2^4\, g^2 + m1\, L1^2\, L2^6\, m2^4\, g^2 - L2^6\, m2^3\, k1\, k2 + m1\, L1\, L2^6\, m2^3\, g\, k2$$
$$- 2\, L2^5\, m2^3\, d1\, g\, d2 - 2\, m1\, L1^2\, L2^5\, m2^3\, k2\, g)Dt^2/(\%1)]$$
$$[0\,, (m1^3\, L1^6\, g\, L2\, m2\, d1 + m1^4\, L1^7\, g\, d2 - m1\, L1^2\, L2^2\, m2\, d1^3$$
$$+ 2\, m1^2\, L1^4\, L2^2\, m2\, d1\, k1 - m1^3\, L1^6\, d1\, k2 + m1^2\, L1^4\, d1^2\, d2 - m1^3\, L1^6\, k1\, d2$$
$$- 2\, m1^3\, L1^5\, g\, L2^2\, m2\, d1)Dt^3/(\%2) + (m1^2\, L1^4\, g\, L2\, m2\, d1^2 - m1^3\, L1^6\, g\, L2\, m2\, k1$$
$$+ 2\, m1^3\, L1^5\, g\, d1\, d2 - m1^4\, L1^7\, g\, k2 - m1^4\, L1^6\, g^2\, L2^2\, m2 + m1^4\, L1^7\, g^2\, L2\, m2$$
$$- m1^2\, L1^4\, L2^2\, m2\, k1^2 + 3\, m1\, L1^2\, L2^2\, m2\, k1\, d1^2 - L2^2\, m2\, d1^4 + m1\, L1^2\, d1^3\, d2$$
$$+ m1^3\, L1^6\, k1\, k2 - 2\, m1^2\, L1^4\, d1\, k1\, d2 - m1^2\, L1^4\, d1^2\, k2 - 3\, m1^2\, L1^3\, g\, L2^2\, m2\, d1^2$$
$$+ 2\, m1^3\, L1^5\, g\, L2^2\, m2\, k1)Dt^2/(\%2), -(m1^2\, L1^4\, d2^3 - m1\, L1^2\, L2^2\, m2\, d1\, d2^2$$
$$+ m1\, L1^2\, L2^4\, m2^2\, k2\, d1 - m1^2\, L1^3\, L2^4\, m2^2\, g\, d2 - 2\, m1^2\, L1^4\, L2^2\, m2\, d2\, k2$$
$$+ 2\, m1^2\, L1^4\, L2^3\, m2^2\, d2\, g + m1\, L1^2\, L2^4\, m2^2\, k1\, d2 - m1\, L1^2\, L2^5\, m2^3\, d1\, g)Dt^3/(\%1)$$

$$-(m1^2\,L1^4\,L2^4\,m2^3\,g^2 - m1^2\,L1^3\,L2^5\,m2^3\,g^2 - L2^2\,m2\,d1^2\,d2^2$$
$$- L2^5\,m2^3\,d1^2\,g - m1^2\,L1^4\,d2^2\,k2 + L2^4\,m2^2\,k2\,d1^2 - m1\,L1^2\,L2^2\,m2\,k2\,d1\,d2$$
$$- m1\,L1\,L2^4\,m2^2\,g\,d2\,d1 + m1^2\,L1^3\,L2^4\,m2^2\,g\,k2 + m1\,L1^2\,L2^5\,m2^3\,k1\,g$$
$$+ m1^2\,L1^4\,L2\,m2\,d2^2\,g + m1\,L1^2\,L2^3\,m2^2\,d1\,g\,d2 + L2^4\,m2^2\,k1\,d2\,d1$$
$$+ m1\,L1^2\,d1\,d2^3 - 2\,m1^2\,L1^4\,L2^3\,m2^2\,k2\,g - m1\,L1^2\,L2^4\,m2^2\,k1\,k2$$
$$+ m1^2\,L1^4\,L2^2\,m2\,k2^2)Dt^2/(\%1) - (m1\,L1\,L2^2\,m2\,g\,d2^2\,d1$$
$$+ m1^2\,L1^4\,L2^2\,m2^2\,d2\,g^2 - L2^2\,m2\,k1\,d2^2\,d1 + 2\,m1\,L1^2\,L2^3\,m2^2\,k1\,d2\,g$$
$$+ 2\,m1^2\,L1^3\,L2^2\,m2\,g\,d2\,k2 - 2\,m1\,L1^2\,L2^2\,m2\,k1\,d2\,k2 + m1\,L1^2\,k1\,d2^3$$
$$- m1\,L1^2\,k2\,d1\,d2^2 - 2\,m1\,L1\,L2^4\,m2^2\,g\,d2\,k1 + L2^2\,m2\,k2\,d1^2\,d2$$
$$+ m1^2\,L1^2\,L2^4\,m2^2\,g^2\,d2 + L2^4\,m2^2\,k1^2\,d2 - 2\,m1^2\,L1^4\,L2\,m2\,d2\,g\,k2$$
$$- 2\,m1^2\,L1^3\,L2^3\,m2^2\,g^2\,d2 - L2^3\,m2^2\,d1^2\,g\,d2 + m1\,L1^2\,L2\,m2\,d1\,g\,d2^2$$
$$+ m1^2\,L1^4\,d2\,k2^2 - m1^2\,L1^3\,g\,d2^3)Dt/(\%1) - (L2^5\,m2^3\,k1^2\,g$$
$$- 2\,m1\,L1\,L2^5\,m2^3\,k1\,g^2 - m2\,L2^2\,d1^2\,k2^2 + 3\,m1^2\,L1^4\,m2\,L2\,g\,k2^2$$
$$- m1^2\,L1^3\,L2\,m2\,g^2\,d2^2 - 2\,m1^2\,L1^3\,m2\,L2^2\,k2^2\,g + 2\,m1\,L1^2\,m2\,L2^2\,k2^2\,k1$$
$$- m1\,L1\,L2^2\,m2\,k2\,d1\,g\,d2 - 2\,m1\,L1^2\,L2\,m2\,d1\,g\,d2\,k2 + m1^2\,L1^2\,L2^5\,m2^3\,g^3$$
$$+ m1^2\,L1^3\,g\,d2^2\,k2 - 4\,m1\,L1^2\,m2^2\,L2^3\,g\,k1\,k2 + 2\,m2^2\,L2^3\,d1^2\,g\,k2$$
$$+ 4\,m1^2\,L1^3\,m2^2\,L2^3\,g^2\,k2 + m1\,L1\,L2^3\,m2^2\,d1\,g^2\,d2 + m1\,L1^2\,L2\,m2\,k1\,d2^2\,g$$
$$- m1\,L1^2\,k1\,d2^2\,k2 - 2\,m1^2\,L1^3\,m2^3\,L2^4\,g^3 - m1^2\,L1^4\,k2^3 + L2^2\,m2\,k2\,d1\,k1\,d2$$
$$+ m1^2\,L1^4\,m2^3\,L2^3\,g^3 + m1\,L1^2\,L2^2\,m2^2\,d1\,g^2\,d2 - 3\,m1^2\,L1^4\,m2^2\,L2^2\,g^2\,k2$$
$$- m1^2\,L1^2\,L2^4\,m2^2\,g^2\,k2 + 2\,m1\,L1^2\,m2^3\,L2^4\,g^2\,k1 - m2^3\,L2^4\,d1^2\,g^2$$
$$- L2^3\,m2^2\,d1\,g\,k1\,d2 + 2\,m1\,L1\,L2^4\,m2^2\,g\,k2\,k1 + m1\,L1^2\,k2^2\,d1\,d2$$
$$- L2^4\,m2^2\,k1^2\,k2)/(\%1)]$$
$$[0\,,(-m1^4\,L1^7\,g\,L2^2\,m2\,d2 + m1^3\,L1^6\,L2^2\,m2\,k1\,d2 + 2\,m1^3\,L1^5\,g\,L2^4\,m2^2\,d1$$
$$- 2\,m1^2\,L1^4\,L2^4\,m2^2\,d1\,k1 - m1^2\,L1^4\,L2^2\,m2\,d1^2\,d2 + L1^2\,L2^4\,m2^2\,d1^3\,m1$$
$$- m1^3\,L1^6\,g\,L2^3\,m2^2\,d1 + m1^3\,L1^6\,L2^2\,m2\,d1\,k2)Dt^3/(m1\,L1\,\%2)$$
$$+(m1^2\,L1^4\,L2^4\,m2^2\,k1^2 - m1^2\,L1^4\,g\,L2^3\,m2^2\,d1^2 + 3\,m1^2\,L1^3\,g\,L2^4\,m2^2\,d1^2$$
$$- m1^4\,L1^7\,g\,d2^2 + L2^4\,m2^2\,d1^4 + m1^3\,L1^6\,k1\,d2^2 + m1^4\,L1^6\,g^2\,L2^4\,m2^2$$
$$- m1^3\,L1^6\,L2^2\,m2\,k1\,k2 - m1^4\,L1^7\,g^2\,L2^3\,m2^2 + m1^3\,L1^6\,d1\,d2\,k2$$
$$- m1^3\,L1^6\,g\,L2\,m2\,d1\,d2 - 3\,L1^2\,L2^4\,m2^2\,k1\,d1^2\,m1 - 2\,m1^3\,L1^5\,g\,L2^4\,m2^2\,k1$$
$$+ m1^2\,L1^4\,L2^2\,m2\,d1^2\,k2 + m1^3\,L1^6\,g\,L2^3\,m2^2\,k1 - m1^2\,L1^4\,d1^2\,d2^2$$
$$+ m1^4\,L1^7\,g\,L2^2\,m2\,k2)Dt^2/(m1\,L1\,\%2) + (m1^3\,L1^6\,g^2\,L2^2\,m2^2\,d1$$
$$+ m1^4\,L1^6\,g^2\,L2^2\,m2\,d2 - 2\,m1^3\,L1^5\,g\,L2^2\,m2\,k1\,d2 + 2\,m1^2\,L1^4\,g\,L2^3\,m2^2\,k1\,d1$$
$$- 2\,m1^3\,L1^5\,g\,d2^2\,d1 + L1^2\,L2^2\,m2\,d1^3\,k2\,m1 - 2\,m1^3\,L1^6\,g\,L2\,m2\,k2\,d1$$
$$+ 2\,m1^3\,L1^5\,g\,L2^2\,m2\,k2\,d1 + m1^3\,L1^6\,k2^2\,d1 - L1^2\,g\,L2^3\,m2^2\,d1^3\,m1$$
$$- L1^2\,d1^3\,d2^2\,m1 + 2\,m1^2\,L1^4\,d2^2\,k1\,d1 + L2^2\,m2\,d1^4\,d2$$
$$- 2\,m1^2\,L1^4\,L2^2\,m2\,k2\,k1\,d1 - 3\,L1^2\,L2^2\,m2\,d1^2\,d2\,k1\,m1$$
$$+ m1^2\,L1^4\,L2^2\,m2\,k1^2\,d2 - 2\,m1^3\,L1^5\,g^2\,L2^3\,m2^2\,d1 + 3\,m1^2\,L1^3\,g\,L2^2\,m2\,d1^2\,d2)$$
$$Dt/(m1\,L1\,\%2)$$

$$+(m1^{2}\,L1^{4}\,L2^{2}\,m2\,k2\,k1^{2}-3\,L1^{2}\,L2^{2}\,m2\,k2\,k1\,d1^{2}\,m1$$

$$+\,2\,m1^{2}\,L1^{4}\,d1\,k2\,d2\,k1-L1^{2}\,d1^{3}\,k2\,d2\,m1-2\,m1^{2}\,L1^{4}\,g\,L2\,m2\,d1\,d2\,k1$$

$$-\,2\,m1^{3}\,L1^{5}\,g\,d1\,d2\,k2-2\,m1^{4}\,L1^{7}\,m2\,L2\,g^{2}\,k2+m1^{4}\,L1^{7}\,m2^{2}\,L2^{2}\,g^{3}$$

$$+\,m1^{4}\,L1^{7}\,g\,k2^{2}+m1^{2}\,L1^{4}\,d1^{2}\,L2^{2}\,m2^{2}\,g^{2}+m1^{2}\,L1^{4}\,d1^{2}\,k2^{2}$$

$$-\,m1^{3}\,L1^{6}\,g^{2}\,k1\,m2^{2}\,L2^{2}-m1^{3}\,L1^{6}\,k2^{2}\,k1+2\,m1^{3}\,L1^{6}\,g\,k1\,m2\,L2\,k2$$

$$-\,2\,m1^{2}\,L1^{4}\,d1^{2}\,L2\,m2\,k2\,g+2\,m1^{3}\,L1^{5}\,g^{2}\,m2^{2}\,k1\,L2^{3}$$

$$-\,3\,m1^{2}\,L1^{3}\,g^{2}\,L2^{3}\,m2^{2}\,d1^{2}+m1^{4}\,L1^{6}\,g^{2}\,L2^{2}\,m2\,k2+2\,m1^{3}\,L1^{5}\,g^{2}\,d1\,d2\,m2\,L2$$

$$+\,3\,L1^{2}\,g\,L2^{3}\,m2^{2}\,k1\,d1^{2}\,m1-m1^{2}\,L1^{4}\,g\,L2^{3}\,k1^{2}\,m2^{2}-2\,m1^{3}\,L1^{5}\,g\,L2^{2}\,m2\,k1\,k2$$

$$+\,3\,m1^{2}\,L1^{3}\,g\,L2^{2}\,m2\,d1^{2}\,k2+L1^{2}\,g\,L2\,m2\,d1^{3}\,d2\,m1-g\,L2^{3}\,m2^{2}\,d1^{4}$$

$$-\,m1^{4}\,L1^{6}\,g^{3}\,m2^{2}\,L2^{3}+L2^{2}\,m2\,d1^{4}\,k2)/(m1\,L1\,\%2),-(2\,m1^{2}\,L1^{4}\,L2^{4}\,m2^{2}\,d2\,k2$$

$$+\,L1^{2}\,L2^{7}\,m2^{4}\,d1\,g\,m1-2\,m1^{2}\,L1^{4}\,L2^{5}\,m2^{3}\,d2\,g+L1^{2}\,L2^{4}\,m2^{2}\,d1\,d2^{2}\,m1$$

$$-\,m1^{2}\,L1^{4}\,L2^{2}\,m2\,d2^{3}-L1^{2}\,L2^{6}\,m2^{3}\,k1\,d2\,m1+m1^{2}\,L1^{3}\,L2^{6}\,m2^{3}\,g\,d2$$

$$-\,L1^{2}\,L2^{6}\,m2^{3}\,k2\,d1\,m1)Dt^{3}/(m1\,L1\,\%1)-(-L2^{6}\,m2^{3}\,k1\,d2\,d1+L2^{7}\,m2^{4}\,d1^{2}\,g$$

$$+\,L1^{2}\,L2^{6}\,m2^{3}\,k1\,k2\,m1+L2^{6}\,m2^{3}\,g\,d2\,d1\,m1\,L1+2\,m1^{2}\,L1^{4}\,L2^{5}\,m2^{3}\,k2\,g$$

$$+\,m1^{2}\,L1^{3}\,L2^{4}\,m2^{2}\,g\,d2^{2}-L1^{2}\,L2^{4}\,m2^{2}\,k1\,d2^{2}\,m1-m1^{2}\,L1^{3}\,L2^{6}\,m2^{3}\,g\,k2$$

$$-\,m1^{2}\,L1^{4}\,d2^{4}+3\,m1^{2}\,L1^{4}\,L2^{2}\,m2\,d2^{2}\,k2+L2^{4}\,m2^{2}\,d1^{2}\,d2^{2}-m1^{2}\,L1^{4}\,L2^{6}\,m2^{4}\,g^{2}$$

$$-\,3\,m1^{2}\,L1^{4}\,L2^{3}\,m2^{2}\,d2^{2}\,g-L1^{2}\,L2^{7}\,m2^{4}\,k1\,g\,m1-m1^{2}\,L1^{4}\,L2^{4}\,m2^{2}\,k2^{2}$$

$$+\,m1^{2}\,L1^{3}\,L2^{7}\,m2^{4}\,g^{2}-L2^{6}\,m2^{3}\,k2\,d1^{2})Dt^{2}/(m1\,L1\,\%1)-(-L1^{2}\,d2^{4}\,d1\,m1$$

$$-\,L1^{2}\,L2^{6}\,m2^{4}\,g^{2}\,d1\,m1-L2^{6}\,m2^{3}\,k1^{2}\,d2-3\,L1^{2}\,L2^{3}\,m2^{2}\,d2^{2}\,g\,d1\,m1$$

$$+\,2\,m1^{2}\,L1^{3}\,L2^{5}\,m2^{3}\,d2\,g^{2}+2\,L2^{6}\,m2^{3}\,g\,d2\,k1\,m1\,L1$$

$$+\,3\,L1^{2}\,L2^{2}\,m2\,d2^{2}\,k2\,d1\,m1-2\,L1^{2}\,L2^{5}\,m2^{3}\,d2\,g\,k1\,m1$$

$$-\,2\,m1^{2}\,L1^{3}\,L2^{4}\,m2^{2}\,d2\,k2\,g-2\,L2^{4}\,m2^{2}\,k2\,d1^{2}\,d2-m1^{2}\,L1^{2}\,L2^{6}\,m2^{3}\,g^{2}\,d2$$

$$+\,2\,L2^{5}\,m2^{3}\,d1^{2}\,g\,d2+2\,L1^{2}\,L2^{4}\,m2^{2}\,d2\,k2\,k1\,m1+m1^{2}\,L1^{3}\,L2^{2}\,m2\,d2^{3}\,g$$

$$-\,L1^{2}\,L2^{4}\,m2^{2}\,k2^{2}\,d1\,m1+L2^{2}\,m2\,d1^{2}\,d2^{3}+2\,L1^{2}\,L2^{5}\,m2^{3}\,k2\,g\,d1\,m1$$

$$-\,L1^{2}\,L2^{2}\,m2\,d2^{3}\,k1\,m1)Dt/(m1\,L1\,\%1)-(-L1^{2}\,L2^{6}\,m2^{4}\,g^{2}\,k1\,m1$$

$$-\,L2^{7}\,m2^{4}\,k1^{2}\,g-L2^{4}\,m2^{2}\,k1^{2}\,d2^{2}+L2^{6}\,m2^{3}\,k1^{2}\,k2+2\,L2^{5}\,m2^{3}\,d1\,g\,d2\,k1$$

$$+\,L2^{2}\,m2\,d1\,d2^{3}\,k1-2\,L2^{4}\,m2^{2}\,k2\,d1\,d2\,k1+m1^{2}\,L1^{3}\,L2^{6}\,m2^{4}\,g^{3}$$

$$+\,m1^{2}\,L1^{2}\,L2^{6}\,m2^{3}\,g^{2}\,k2-2\,L2^{5}\,m2^{3}\,d1\,g^{2}\,d2\,m1\,L1+2\,L1^{2}\,L2^{5}\,m2^{3}\,k2\,g\,k1\,m1$$

$$-\,2\,m1^{2}\,L1^{3}\,L2^{5}\,m2^{3}\,k2\,g^{2}-L2^{2}\,m2\,d1\,d2^{3}\,g\,m1\,L1-3\,m1^{2}\,L1^{3}\,L2^{2}\,m2\,d2^{2}\,k2\,g$$

$$+\,3\,L1^{2}\,L2^{2}\,m2\,d2^{2}\,k2\,k1\,m1-m1^{2}\,L1^{2}\,L2^{7}\,m2^{4}\,g^{3}-3\,L1^{2}\,L2^{3}\,m2^{2}\,d2^{2}\,g\,k1\,m1$$

$$+\,3\,m1^{2}\,L1^{3}\,L2^{3}\,m2^{2}\,d2^{2}\,g^{2}-L1^{2}\,L2^{4}\,m2^{2}\,k2^{2}\,k1\,m1+2\,L2^{4}\,m2^{2}\,k1\,d2^{2}\,g\,m1\,L1$$

$$+\,2\,L2^{4}\,m2^{2}\,k2\,d1\,d2\,g\,m1\,L1-m1^{2}\,L1^{2}\,L2^{4}\,m2^{2}\,g^{2}\,d2^{2}+m1^{2}\,L1^{3}\,d2^{4}\,g$$

$$+\,m1^{2}\,L1^{3}\,L2^{4}\,m2^{2}\,k2^{2}\,g-L1^{2}\,d2^{4}\,k1\,m1-2\,L2^{6}\,m2^{3}\,g\,k2\,k1\,m1\,L1$$

$$+\,2\,L2^{7}\,m2^{4}\,k1\,g^{2}\,m1\,L1)/(m1\,L1\,\%1)]$$

$$[-1\,,(-m1^4\,L1^7\,L2\,m2\,d1\,k2-m1^3\,L1^6\,g\,L2^3\,m2^3\,d1+m1^3\,L1^6\,L2^2\,m2\,d1\,k2\,M$$
$$-m1^2\,L1^4\,L2^2\,m2^2\,d1^2\,d2-2\,m1^2\,L1^4\,L2^4\,m2^3\,d1\,k1-2\,m1^4\,L1^6\,g\,L2^3\,m2^2\,d1$$
$$+L1^2\,L2^4\,m2^3\,d1^3\,m1-2\,m1^2\,L1^4\,L2^4\,m2^2\,d1\,k1\,M+2\,m1^3\,L1^5\,L2^3\,m2^2\,d1\,k1$$
$$+m1^5\,L1^8\,g\,L2\,m2\,d2+m1^4\,L1^7\,g\,L2^2\,m2^2\,d1+m1^3\,L1^6\,L2^2\,m2\,k1\,d2\,M$$
$$+2\,m1^3\,L1^5\,g\,L2^4\,m2^3\,d1+L1^2\,L2^4\,m2^2\,d1^3\,M\,m1-m1^4\,L1^7\,L2\,m2\,k1\,d2$$
$$-m1^3\,L1^6\,g\,L2^3\,m2^2\,d1\,M-m1^4\,L1^7\,g\,L2^2\,m2^2\,d2+m1^3\,L1^6\,L2^2\,m2^2\,k1\,d2$$
$$+m1^3\,L1^6\,L2^2\,m2^2\,d1\,k2+2\,m1^3\,L1^5\,g\,L2^4\,m2^2\,d1\,M-m1^2\,L1^3\,L2^3\,m2^2\,d1^3$$
$$-m1^2\,L1^4\,L2^2\,m2\,d1^2\,d2\,M-m1^4\,L1^7\,g\,L2^2\,m2\,d2\,M+m1^3\,L1^5\,L2\,m2\,d1^2\,d2)$$
$$Dt^5/(m1\,L1\,\%2)+(L2^4\,m2^3\,d1^4+L2^4\,m2^2\,d1^4\,M+d1^4\,L2^4\,m2^2\,m1$$
$$+m1^3\,L1^6\,m2\,k1\,d2^2+m1^3\,L1^4\,L2^2\,m2\,k1\,d2\,d1-m1^3\,L1^6\,L2^2\,m2\,k1\,k2\,M$$
$$-2\,m1^3\,L1^5\,L2\,m2\,d1\,k1\,d2+m1^4\,L1^7\,L2\,m2\,k1\,k2-3\,L1^2\,L2^4\,m2^2\,k1\,d1^2\,M\,m1$$
$$-m1^3\,L1^5\,L2^3\,m2^2\,k1^2-L2^3\,m2^2\,d1^4\,m1\,L1+3\,m1^2\,L1^3\,L2^3\,m2^2\,k1\,d1^2$$
$$-m1^3\,L1^5\,L2\,m2\,d1^2\,k2+m1^2\,L1^3\,L2\,m2\,d1^3\,d2-m1^2\,L1^4\,m2\,d1^2\,d2^2$$
$$+m1^2\,L1^4\,L2^4\,m2^3\,k1^2-3\,L1^2\,L2^4\,m2^3\,k1\,d1^2\,m1-2\,m1^2\,L1^2\,L2^4\,m2^2\,k1\,d1^2$$
$$+m1^2\,L1^4\,L2^4\,m2^2\,k1^2\,M-m1^3\,L1^6\,L2^2\,m2^2\,k1\,k2+m1^3\,L1^6\,k1\,d2^2\,M$$
$$-m1^2\,L1^4\,d1^2\,d2^2\,M-m1^3\,L1^6\,g\,L2\,m2^2\,d1\,d2-m1^4\,L1^7\,g\,m2\,d2^2$$
$$-2\,m1^3\,L1^5\,g\,L2^4\,m2^3\,k1+2\,m1^3\,L1^3\,g\,L2^4\,m2^2\,d1^2-2\,m1^3\,L1^5\,g\,L2^4\,m2^2\,k1\,M$$
$$+3\,m1^2\,L1^3\,g\,L2^4\,m2^2\,d1^2\,M-m1^2\,L1^4\,g\,L2^3\,m2^3\,d1^2+m1^4\,L1^6\,g^2\,L2^4\,m2^3$$
$$-m1^4\,L1^7\,g^2\,L2^3\,m2^3+3\,m1^2\,L1^3\,g\,L2^4\,m2^3\,d1^2+m1^2\,L1^4\,L2^2\,m2^2\,d1^2\,k2$$
$$+m1^5\,L1^8\,g^2\,L2^2\,m2^2-m1^2\,L1^4\,g\,L2^3\,m2^2\,d1^2\,M-4\,m1^3\,L1^4\,g\,L2^3\,m2^2\,d1^2$$
$$+2\,m1^4\,L1^6\,g\,L2^3\,m2^2\,k1+m1^4\,L1^6\,g^2\,L2^4\,m2^2\,M-m1^5\,L1^7\,g^2\,L2^3\,m2^2$$
$$-m1^4\,L1^7\,g^2\,L2^3\,m2^2\,M+m1^4\,L1^7\,g\,L2^2\,m2\,k2\,M+2\,m1^4\,L1^6\,g\,L2\,m2\,d1\,d2$$
$$-m1^5\,L1^8\,g\,L2\,m2\,k2-m1^3\,L1^6\,g\,L2\,m2\,d1\,d2\,M-m1^4\,L1^7\,g\,d2^2\,M$$
$$+m1^3\,L1^6\,g\,L2^3\,m2^3\,k1+m1^3\,L1^6\,g\,L2^3\,m2^2\,k1\,M-m1^4\,L1^7\,g\,L2^2\,m2^2\,k1$$
$$+m1^4\,L1^7\,g\,L2^2\,m2^2\,k2+m1^3\,L1^5\,g\,L2^2\,m2^2\,d1^2-m1^4\,L1^5\,g\,L2^2\,m2\,d2\,d1$$
$$-m1^2\,L1^2\,L2^2\,m2\,d1^3\,d2+m1^2\,L1^4\,L2^2\,m2\,d1^2\,k2\,M+m1^3\,L1^6\,d1\,d2\,k2\,M$$
$$+m1^3\,L1^4\,L2^2\,m2\,d1^2\,k2+m1^3\,L1^6\,m2\,d1\,d2\,k2)Dt^4/(m1\,L1\,\%2)$$
$$+(L2^2\,m2^2\,d1^4\,d2-m1^2\,L1^2\,d1^3\,d2^2+m1^4\,L1^6\,g^2\,L2^2\,m2^2\,d2$$
$$+m1^3\,L1^4\,L2^2\,m2\,k1^2\,d2-3\,L1^2\,L2^2\,m2^2\,d1^2\,d2\,k1\,m1$$
$$+L1^2\,L2^2\,m2\,d1^3\,k2\,M\,m1-m1^2\,L1^2\,L2^4\,m2^2\,d1\,k1^2-L1^2\,g\,L2^3\,m2^3\,d1^3\,m1$$
$$+m1^2\,L1^4\,L2^2\,m2^2\,k1^2\,d2+m1^3\,L1^4\,k1\,d2^2\,d1+m1^3\,L1^4\,d1^2\,k2\,d2$$
$$+m1^3\,L1^6\,k2^2\,d1\,M-2\,m1^3\,L1^6\,g\,L2\,m2^2\,k2\,d1-2\,m1^3\,L1^6\,g\,L2\,m2\,k2\,d1\,M$$
$$+m1^3\,L1^6\,m2\,k2^2\,d1+2\,m1^3\,L1^3\,g\,L2^2\,m2\,d1^2\,d2+3\,m1^2\,L1^3\,g\,L2^2\,m2\,d1^2\,d2\,M$$
$$+2\,m1^3\,L1^5\,g\,L2^2\,m2\,k2\,d1\,M+2\,m1^3\,L1^3\,g\,L2^4\,m2^2\,d1\,k1$$
$$+m1^3\,L1^6\,g^2\,L2^2\,m2^2\,d1\,M-2\,m1^3\,L1^5\,g\,m2\,d2^2\,d1+L2^2\,m2\,d1^4\,d2\,M$$
$$+m1^3\,L1^6\,g^2\,L2^2\,m2^3\,d1-L1^2\,g\,L2^3\,m2^2\,d1^3\,M\,m1-m1^4\,L1^4\,g^2\,L2^4\,m2^2\,d1$$
$$+m1^4\,L1^6\,g^2\,L2^2\,m2\,d2\,M+m1^5\,L1^6\,g^2\,L2^2\,m2\,d2-2\,m1^3\,L1^5\,g^2\,L2^3\,m2^3\,d1$$
$$-2\,m1^3\,L1^5\,g^2\,L2^3\,m2^2\,d1\,M-m1^4\,L1^5\,g\,d2^2\,d1-m1^3\,L1^4\,g\,L2\,m2\,d1^2\,d2$$
$$-2\,m1^3\,L1^5\,g\,d2^2\,d1\,M-2\,m1^2\,L1^4\,L2^2\,m2\,k2\,k1\,d1\,M+d1^4\,L2^2\,m2\,d2\,m1$$
$$-3\,L1^2\,L2^2\,m2\,d1^2\,d2\,k1\,M\,m1+L1^2\,L2^2\,m2^2\,d1^3\,k2\,m1$$
$$-2\,m1^2\,L1^4\,L2^2\,m2^2\,k2\,k1\,d1+2\,m1^2\,L1^4\,g\,L2^3\,m2^3\,k1\,d1$$

$$+2\,m1^2\,L1^4\,g\,L2^3\,m2^2\,k1\,d1\,M-2\,m1^3\,L1^5\,g\,L2^2\,m2^2\,k1\,d2$$
$$+\,3\,m1^2\,L1^3\,g\,L2^2\,m2^2\,d1^2\,d2+2\,m1^3\,L1^5\,g\,L2^2\,m2^2\,k2\,d1$$
$$-\,2\,m1^3\,L1^5\,g\,L2^2\,m2\,k1\,d2\,M-2\,m1^4\,L1^5\,g\,L2^2\,m2\,k1\,d2$$
$$-\,2\,m1^2\,L1^2\,L2^2\,m2\,d1^2\,d2\,k1+2\,m1^2\,L1^4\,m2\,d2^2\,k1\,d1+2\,m1^2\,L1^4\,d2^2\,k1\,d1\,M$$
$$-\,L1^2\,d1^3\,d2^2\,M\,m1-L1^2\,m2\,d1^3\,d2^2\,m1+m1^2\,L1^4\,L2^2\,m2\,k1^2\,d2\,M)Dt^3/(m1\,L1$$
$$\%2)+(-2\,m1^4\,L1^7\,g^2\,L2\,m2\,k2\,M+m1^3\,L1^4\,g^2\,L2^2\,m2^2\,d1^2-g\,L2^3\,m2^3\,d1^4$$
$$-\,g\,L2^3\,m2^2\,d1^4\,M-L1^2\,m2\,d1^3\,k2\,d2\,m1+3\,L1^2\,g\,L2^3\,m2^3\,k1\,d1^2\,m1$$
$$+\,L2^2\,m2^2\,d1^4\,k2+L2^2\,m2\,d1^4\,k2\,M-m1^3\,L1^6\,k2^2\,k1\,M-L1^2\,d1^3\,k2\,d2\,M\,m1$$
$$-\,m1^3\,L1^6\,m2\,k2^2\,k1+m1^2\,L1^4\,k2^2\,d1^2\,M-m1^2\,L1^2\,d1^3\,k2\,d2$$
$$+\,m1^2\,L1^4\,m2\,k2^2\,d1^2+m1^2\,L1^4\,L2^2\,m2\,k2\,k1^2\,M-m1^2\,L1^2\,L2^2\,m2\,d1\,d2\,k1^2$$
$$-\,m1^3\,L1^4\,L2^2\,m2\,k1^2\,k2-2\,m1^3\,L1^3\,g^2\,L2^3\,m2^2\,d1^2$$
$$-\,3\,m1^2\,L1^3\,g^2\,L2^3\,m2^2\,d1^2\,M+2\,m1^3\,L1^5\,g^2\,L2\,m2^2\,d1\,d2$$
$$+\,m1^4\,L1^5\,g^2\,L2\,m2\,d1\,d2-m1^3\,L1^6\,g^2\,L2^2\,m2^2\,k1\,M-2\,m1^4\,L1^7\,g^2\,L2\,m2^2\,k2$$
$$-\,2\,m1^2\,L1^2\,L2^2\,m2\,k1\,k2\,d1^2+d1^4\,L2^2\,m2\,k2\,m1$$
$$-\,3\,L1^2\,L2^2\,m2\,k2\,k1\,d1^2\,M\,m1+m1^2\,L1^4\,L2^2\,m2^2\,k2\,k1^2+m1^2\,L1^2\,L2^4\,m2^2\,k1^3$$
$$+\,2\,m1^3\,L1^3\,g\,L2^2\,m2\,d1\,d2\,k1-2\,m1^3\,L1^5\,g\,L2^2\,m2\,k2\,k1\,M$$
$$+\,2\,m1^4\,L1^5\,g\,L2^2\,m2\,k2\,k1+3\,m1^2\,L1^3\,g\,L2^2\,m2\,d1^2\,k2\,M$$
$$-\,3\,m1^3\,L1^3\,g\,L2^4\,m2^2\,k1^2-m1^2\,L1^4\,g\,L2^3\,m2^3\,k1^2+m1^3\,L1^4\,k1^2\,d2^2$$
$$+\,m1^3\,L1^4\,k2^2\,d1^2+m1^3\,L1^4\,k1\,k2\,d1\,d2+2\,m1^2\,L1^4\,d1\,k2\,d2\,k1\,M$$
$$+\,L1^2\,g\,L2\,m2\,d1^3\,d2\,M\,m1-2\,m1^2\,L1^4\,g\,L2\,m2^2\,k2\,d1^2$$
$$+\,2\,m1^3\,L1^6\,g\,L2\,m2^2\,k2\,k1+L1^2\,g\,L2\,m2^2\,d1^3\,d2\,m1$$
$$-\,2\,m1^2\,L1^4\,g\,L2\,m2^2\,d1\,d2\,k1-2\,m1^2\,L1^4\,g\,L2\,m2\,k2\,d1^2\,M$$
$$-\,2\,m1^2\,L1^4\,g\,L2\,m2\,d1\,d2\,k1\,M-2\,m1^3\,L1^5\,g\,m2\,d1\,k2\,d2+m1^4\,L1^7\,g\,m2\,k2^2$$
$$+\,2\,m1^2\,L1^4\,m2\,d1\,k2\,d2\,k1+2\,m1^3\,L1^5\,g^2\,L2\,m2\,d1\,d2\,M$$
$$+\,m1^4\,L1^6\,g^2\,L2^2\,m2\,k2\,M+m1^2\,L1^4\,g^2\,L2^2\,m2^3\,d1^2-m1^3\,L1^6\,g^2\,L2^2\,m2^3\,k1$$
$$+\,m1^4\,L1^6\,g^2\,L2^2\,m2^2\,k2-m1^4\,L1^5\,g\,k2\,d1\,d2+3\,m1^4\,L1^4\,g^2\,L2^4\,m2^2\,k1$$
$$-\,3\,m1^2\,L1^3\,g^2\,L2^3\,m2^3\,d1^2+m1^5\,L1^6\,g^3\,L2^3\,m2^2-m1^4\,L1^6\,g^3\,L2^3\,m2^2\,M$$
$$+\,m1^4\,L1^7\,g^3\,L2^2\,m2^3+m1^4\,L1^7\,g^3\,L2^2\,m2^2\,M-m1^4\,L1^4\,g^2\,L2^2\,m2\,d1\,d2$$
$$-\,m1^5\,L1^6\,g^2\,L2^2\,m2\,k2+2\,m1^3\,L1^5\,g^2\,L2^3\,m2^3\,k1+2\,m1^3\,L1^5\,g^2\,L2^3\,m2^2\,k1\,M$$
$$-\,3\,L1^2\,L2^2\,m2^2\,k2\,k1\,d1^2\,m1+2\,m1^3\,L1^6\,g\,L2\,m2\,k2\,k1\,M+m1^5\,L1^6\,g^2\,d2^2$$
$$-\,2\,m1^3\,L1^4\,g\,L2\,m2\,k2\,d1^2-m1^3\,L1^4\,g\,L2\,m2\,k1\,d1\,d2$$
$$+\,m1^2\,L1^2\,g\,L2\,m2\,d1^3\,d2+m1^4\,L1^7\,g\,k2^2\,M-2\,m1^3\,L1^5\,g\,d1\,k2\,d2\,M$$
$$-\,m1^5\,L1^5\,g^3\,L2^4\,m2^2-m1^4\,L1^6\,g^3\,L2^3\,m2^3-2\,m1^4\,L1^5\,g\,d2^2\,k1$$
$$-\,2\,m1^4\,L1^5\,g^2\,L2^3\,m2^2\,k1+m1^2\,L1^4\,g^2\,L2^2\,m2^2\,d1^2\,M-d1^4\,g\,L2^3\,m2^2\,m1$$
$$-\,m1^2\,L1^4\,g\,L2^3\,m2^2\,k1^2\,M+2\,m1^2\,L1^2\,g\,L2^3\,m2^2\,k1\,d1^2+m1^3\,L1^4\,g\,L2^3\,m2^2\,k1^2$$
$$+\,3\,L1^2\,g\,L2^3\,m2^2\,k1\,d1^2\,M\,m1-2\,m1^3\,L1^5\,g\,L2^2\,m2^2\,k2\,k1$$
$$+\,3\,m1^2\,L1^3\,g\,L2^2\,m2^2\,d1^2\,k2+2\,m1^3\,L1^3\,g\,L2^2\,m2\,k2\,d1^2)Dt^2/(m1\,L1\,\%2),$$
$$-(2\,m1^2\,L1^4\,L2^4\,m2^3\,d2\,k2-m1^3\,L1^4\,L2^5\,m2^3\,g\,d2-L1^2\,L2^6\,m2^4\,k1\,d2\,m1$$
$$+\,L1^2\,L2^4\,m2^2\,d1\,d2^2\,M\,m1+L1^2\,L2^7\,m2^4\,d1\,g\,M\,m1+L1^2\,L2^4\,m2^3\,d1\,d2^2\,m1$$
$$+\,L1^2\,L2^7\,m2^5\,d1\,g\,m1+2\,m1^2\,L1^4\,L2^4\,m2^2\,d2\,k2\,M-m1^2\,L1^3\,L2^6\,m2^4\,d1\,g$$
$$-\,L1^2\,L2^6\,m2^4\,k2\,d1\,m1-2\,m1^3\,L1^5\,L2^3\,m2^2\,d2\,k2-2\,m1^2\,L1^4\,L2^5\,m2^4\,d2\,g$$

$$- m1^2\,L1^4\,L2^2\,m2^2\,d2^3 + m1^3\,L1^5\,L2\,m2\,d2^3 + m1^2\,L1^3\,L2^6\,m2^4\,g\,d2$$
$$+ m1^2\,L1^3\,L2^5\,m2^3\,k2\,d1 - 2\,m1^2\,L1^4\,L2^5\,m2^3\,d2\,g\,M + m1^2\,L1^3\,L2^5\,m2^3\,k1\,d2$$
$$- m1^2\,L1^4\,L2^2\,m2\,d2^3\,M + m1^2\,L1^3\,L2^6\,m2^3\,g\,d2\,M - L1^2\,L2^6\,m2^3\,k2\,d1\,M\,m1$$
$$+ 2\,m1^3\,L1^5\,L2^4\,m2^3\,d2\,g - L1^2\,L2^6\,m2^3\,k1\,d2\,M\,m1 - m1^2\,L1^3\,L2^3\,m2^2\,d1\,d2^2)$$
$$Dt^5/(m1\,L1\,\%1) - (L2^7\,m2^5\,d1^2\,g + L2^4\,m2^3\,d1^2\,d2^2 - L2^6\,m2^4\,k2\,d1^2$$
$$- m1^2\,L1^4\,d2^4\,M - m1^2\,L1^4\,m2\,d2^4 + m1^3\,L1^4\,L2^5\,m2^3\,g\,k2 + m1^3\,L1^5\,L2^5\,m2^4\,g^2$$
$$+ m1^2\,L1^3\,L2^4\,m2^3\,g\,d2^2 - L2^6\,m2^3\,k1\,d2\,d1\,M - m1^3\,L1^4\,L2^6\,m2^4\,g^2$$
$$- 3\,m1^2\,L1^4\,L2^3\,m2^3\,d2^2\,g - m1^2\,L1^4\,L2^6\,m2^4\,g^2\,M + m1^2\,L1^3\,L2^6\,m2^4\,k1\,g$$
$$+ m1^3\,L1^5\,L2^3\,m2^2\,k2^2 - m1^2\,L1^4\,L2^4\,m2^3\,k2^2 + L2^4\,m2^2\,d1^2\,d2^2\,m1$$
$$- 3\,m1^2\,L1^4\,L2^3\,m2^2\,d2^2\,g\,M - 2\,m1^3\,L1^5\,L2^4\,m2^3\,k2\,g$$
$$+ 2\,m1^2\,L1^2\,L2^4\,m2^2\,d2\,k2\,d1 + 3\,m1^2\,L1^4\,L2^2\,m2^2\,d2^2\,k2$$
$$+ m1^3\,L1^5\,L2^2\,m2^2\,d2^2\,g - m1^2\,L1^4\,L2^4\,m2^2\,k2^2\,M + m1^2\,L1^3\,L2^4\,m2^2\,g\,d2^2\,M$$
$$- L1^2\,L2^4\,m2^2\,k1\,d2^2\,M\,m1 - L2^3\,m2^2\,d1^2\,d2^2\,m1\,L1 - L2^6\,m2^3\,k1\,d2\,d1\,m1$$
$$- m1^2\,L1^3\,L2^6\,m2^3\,g\,k2\,M + 3\,m1^2\,L1^4\,L2^2\,m2\,d2^2\,k2\,M + L2^7\,m2^4\,d1^2\,g\,M$$
$$- L2^6\,m2^3\,k2\,d1^2\,M + L2^4\,m2^2\,d1^2\,d2^2\,M - L2^6\,m2^4\,k1\,d2\,d1$$
$$+ m1^2\,L1^3\,L2\,m2\,d1\,d2^3 - m1^3\,L1^5\,L2\,m2\,d2^2\,k2 - m1^2\,L1^3\,L2^5\,m2^3\,k1\,k2$$
$$+ m1^2\,L1^3\,L2^7\,m2^5\,g^2 + 2\,m1^2\,L1^4\,L2^5\,m2^4\,k2\,g - m1^2\,L1^2\,L2^2\,m2\,d2^3\,d1$$
$$- L1^2\,L2^4\,m2^3\,k1\,d2^2\,m1 + m1^2\,L1^3\,L2^4\,m2^3\,d1\,g\,d2 - m1^2\,L1^3\,L2^3\,m2^2\,k2\,d1\,d2$$
$$- m1^2\,L1^3\,L2^6\,m2^4\,g\,k2 + L1^2\,L2^6\,m2^3\,k1\,k2\,M\,m1 - L2^6\,m2^3\,k2\,d1^2\,m1$$
$$+ L2^6\,m2^3\,g\,d2\,d1\,M\,m1\,L1 + m1^2\,L2^6\,m2^3\,g\,d2\,d1\,L1 - L1^2\,L2^7\,m2^5\,k1\,g\,m1$$
$$- L1^2\,L2^7\,m2^4\,k1\,g\,M\,m1 + L2^5\,m2^3\,k2\,d1^2\,m1\,L1 + L2^7\,m2^4\,d1^2\,g\,m1$$
$$+ m1^2\,L1^3\,L2^7\,m2^4\,g^2\,M - m1^2\,L1^4\,L2^6\,m2^5\,g^2 + L1^2\,L2^6\,m2^4\,k1\,k2\,m1$$
$$- L2^6\,m2^4\,d1^2\,g\,m1\,L1 + L2^6\,m2^4\,g\,d2\,d1\,m1\,L1 - 3\,m1^2\,L1^2\,L2^5\,m2^3\,g\,d2\,d1$$
$$+ L2^5\,m2^3\,k1\,d2\,d1\,m1\,L1 + 2\,m1^2\,L1^4\,L2^5\,m2^3\,k2\,g\,M)Dt^4/(m1\,L1\,\%1)$$
$$-(L2^2\,m2^2\,d1^2\,d2^3 - L2^6\,m2^4\,k1^2\,d2 - m1^2\,L1^2\,d2^4\,d1 + 2\,m1^2\,L1^3\,L2^5\,m2^4\,d2\,g^2$$
$$+ L2^2\,m2\,d1^2\,d2^3\,M + 2\,L1^2\,L2^5\,m2^3\,k2\,g\,d1\,M\,m1 + 2\,m1^2\,L1^3\,L2^4\,m2^3\,k1\,d2\,g$$
$$- L2^6\,m2^3\,k1^2\,d2\,M - L2^4\,m2^3\,d1^2\,g\,d2\,m1\,L1 + 2\,L2^5\,m2^4\,d1^2\,g\,d2$$
$$+ 2\,L2^6\,m2^4\,g\,d2\,k1\,m1\,L1 - 3\,L1^2\,L2^3\,m2^3\,d2^2\,g\,d1\,m1$$
$$+ 2\,m1^2\,L1^2\,L2^5\,m2^3\,k2\,d1\,g + m1^2\,L1^3\,L2^2\,m2^2\,d2^3\,g - 2\,m1^3\,L1^4\,L2^4\,m2^3\,g^2\,d2$$
$$- 2\,m1^2\,L1^3\,L2^4\,m2^2\,d2\,k2\,g\,M + 2\,L1^2\,L2^4\,m2^3\,d2\,k2\,k1\,m1$$
$$- 2\,m1^3\,L1^3\,L2^4\,m2^2\,d2\,k2\,g - m1^2\,L1^2\,L2^4\,m2^2\,k2^2\,d1$$
$$- 2\,m1^3\,L1^5\,L2^2\,m2^2\,d2\,k2\,g - 2\,L2^4\,m2^2\,k2\,d1^2\,d2\,m1$$
$$- 2\,m1^2\,L1^2\,L2^3\,m2^2\,g\,d2^2\,d1 + 2\,m1^2\,L1^2\,L2^4\,m2^2\,d2\,k2\,k1$$
$$+ L2^5\,m2^3\,k1^2\,d2\,m1\,L1 - 2\,m1^2\,L1^3\,L2^3\,m2^2\,k1\,d2\,k2 - L2^3\,m2^2\,k1\,d2^2\,d1\,m1\,L1$$
$$+ m1^3\,L1^5\,L2^3\,m2^3\,d2\,g^2 + m1^2\,L1^3\,L2^2\,m2^2\,d1\,g\,d2^2 - m1^2\,L1^2\,L2^6\,m2^3\,g^2\,d2\,M$$
$$+ 2\,m1^2\,L2^6\,m2^3\,g\,d2\,k1\,L1 + 2\,L2^6\,m2^3\,g\,d2\,k1\,M\,m1\,L1$$
$$+ 2\,L1^2\,L2^5\,m2^4\,k2\,g\,d1\,m1 + 3\,L1^2\,L2^2\,m2^2\,d2^2\,k2\,d1\,m1$$
$$- L1^2\,L2^2\,m2^2\,d2^3\,k1\,m1 + 2\,m1^3\,L1^4\,L2^3\,m2^2\,g\,d2\,k2$$
$$+ 3\,L1^2\,L2^2\,m2\,d2^2\,k2\,d1\,M\,m1 + L2^2\,m2\,d1^2\,d2^3\,m1$$
$$+ 3\,m1^2\,L1^2\,L2^2\,m2\,d2^2\,k2\,d1 + m1^2\,L1^3\,L2^2\,m2\,d2^3\,g\,M - 2\,L2^4\,m2^3\,k2\,d1^2\,d2$$

$$
\begin{aligned}
&+ 2\,L2^5\,m2^3\,d1^2\,g\,d2\,M - 2\,L2^4\,m2^2\,k2\,d1^2\,d2\,M + 2\,L2^5\,m2^3\,d1^2\,g\,d2\,m1\\
&+ m1^2\,L1^3\,L2\,m2\,k1\,d2^3 - m1^3\,L1^4\,L2\,m2\,g\,d2^3 + m1^3\,L1^5\,L2\,m2\,d2\,k2^2\\
&- 3\,L1^2\,L2^3\,m2^2\,d2^2\,g\,d1\,M\,m1 - L1^2\,d2^4\,d1\,M\,m1 - L1^2\,m2\,d2^4\,d1\,m1\\
&- 2\,L1^2\,L2^5\,m2^4\,d2\,g\,k1\,m1 - m1^2\,L1^2\,L2^2\,m2\,d2^3\,k1 - L1^2\,L2^2\,m2\,d2^3\,k1\,M\,m1\\
&+ m1^3\,L1^3\,L2^2\,m2\,d2^3\,g - m1^2\,L1^3\,L2\,m2\,k2\,d1\,d2^2 + L2^3\,m2^2\,k2\,d1^2\,d2\,m1\,L1\\
&- L1^2\,L2^6\,m2^4\,g^2\,d1\,M\,m1 - m1^3\,L1^2\,L2^6\,m2^3\,g^2\,d2 - L2^6\,m2^3\,k1^2\,d2\,m1\\
&+ 2\,L1^2\,L2^4\,m2^2\,d2\,k2\,k1\,M\,m1 - L1^2\,L2^4\,m2^2\,k2^2\,d1\,M\,m1\\
&- L1^2\,L2^4\,m2^3\,k2^2\,d1\,m1 - 2\,m1^2\,L1^3\,L2^4\,m2^3\,d2\,k2\,g - L1^2\,L2^6\,m2^5\,g^2\,d1\,m1\\
&- m1^2\,L1^2\,L2^6\,m2^4\,d1\,g^2 - m1^2\,L1^2\,L2^6\,m2^4\,g^2\,d2 + 3\,m1^3\,L1^3\,L2^5\,m2^3\,g^2\,d2\\
&- 4\,m1^2\,L1^2\,L2^5\,m2^3\,g\,d2\,k1 + 2\,m1^2\,L1^3\,L2^5\,m2^3\,d2\,g^2\,M\\
&- 2\,L1^2\,L2^5\,m2^3\,d2\,g\,k1\,M\,m1)Dt^3/(m1\,L1\,\%1) - (2\,L1^2\,L2^5\,m2^4\,k2\,g\,k1\,m1\\
&+ 2\,L2^4\,m2^3\,k1\,d2^2\,g\,m1\,L1 + L2^2\,m2\,d1\,d2^3\,k1\,M + L2^2\,m2^2\,d1\,d2^3\,k1\\
&+ L2^6\,m2^3\,k1^2\,k2\,M + 2\,L2^5\,m2^4\,d1\,g\,d2\,k1 + m1^2\,L1^3\,L2^6\,m2^4\,g^3\,M\\
&- 2\,L2^6\,m2^4\,k1\,k2\,g\,m1\,L1 - 3\,L1^2\,L2^3\,m2^3\,d2^2\,g\,k1\,m1 + 2\,m1^3\,L1^3\,L2^6\,m2^4\,g^3\\
&+ 3\,m1^2\,L1^3\,L2^3\,m2^2\,d2^2\,g^2\,M - 2\,m1^3\,L1^4\,L2^3\,m2^2\,k2^2\,g\\
&- L2^2\,m2^2\,d1\,d2^3\,g\,m1\,L1 + 3\,m1^3\,L1^5\,L2^2\,m2^2\,k2^2\,g\\
&- 3\,m1^2\,L1^3\,L2^2\,m2^2\,d2^2\,k2\,g + 3\,L1^2\,L2^2\,m2^2\,d2^2\,k2\,k1\,m1\\
&+ 4\,m1^3\,L1^4\,L2^4\,m2^3\,k2\,g^2 + 2\,m1^2\,L2^4\,m2^2\,k1\,d2^2\,g\,L1\\
&+ 2\,L2^4\,m2^2\,k1\,d2^2\,g\,M\,m1\,L1 - L2^4\,m2^2\,k1^2\,d2^2\,m1 - m1^3\,L1^2\,L2^4\,m2^2\,g^2\,d2^2\\
&- m1^2\,L1^2\,L2^4\,m2^2\,g^2\,d2^2\,M - L1^2\,L2^4\,m2^2\,k2^2\,k1\,M\,m1\\
&+ 2\,m1^2\,L2^4\,m2^2\,k2\,d1\,d2\,g\,L1 + 2\,L2^4\,m2^2\,k2\,d1\,d2\,g\,M\,m1\,L1\\
&- 2\,L2^4\,m2^2\,k2\,d1\,d2\,k1\,m1 + m1^3\,L1^3\,L2^4\,m2^2\,k2^2\,g - L1^2\,d2^4\,k1\,M\,m1\\
&+ m1^2\,L1^3\,L2^4\,m2^2\,k2^2\,g\,M - m1^2\,L1^2\,L2^4\,m2^3\,g^2\,d2^2 + m1^2\,L1^3\,L2^2\,m2^2\,k1\,d2^2\,g\\
&+ 3\,m1^2\,L1^2\,L2^2\,m2\,d2^2\,k2\,k1 - 3\,m1^3\,L1^3\,L2^2\,m2\,d2^2\,k2\,g\\
&- 3\,m1^2\,L1^3\,L2^2\,m2\,d2^2\,k2\,g\,M + L2^2\,m2\,d1\,d2^3\,k1\,m1 - m1^2\,L2^2\,m2\,d1\,d2^3\,g\,L1\\
&- L2^2\,m2\,d1\,d2^3\,g\,M\,m1\,L1 + 3\,L1^2\,L2^2\,m2\,d2^2\,k2\,k1\,M\,m1\\
&+ m1^2\,L1^3\,L2^3\,m2^3\,d1\,g^2\,d2 + 3\,m1^2\,L1^3\,L2^3\,m2^3\,d2^2\,g^2\\
&- 3\,m1^3\,L1^5\,L2^3\,m2^3\,k2\,g^2 + 3\,m1^3\,L1^3\,L2^3\,m2^2\,g^2\,d2^2 + 2\,m1^2\,L1^3\,L2^3\,m2^2\,k1\,k2^2\\
&- 2\,m1^2\,L1^3\,L2^2\,m2^2\,d1\,g\,d2\,k2 - m1^3\,L1^4\,L2^2\,m2^2\,g^2\,d2^2\\
&- m1^2\,L1^2\,L2^4\,m2^2\,k2^2\,k1 - 2\,L2^6\,m2^3\,k1\,k2\,g\,M\,m1\,L1 + L2^6\,m2^3\,k1^2\,k2\,m1\\
&- 2\,m1^2\,L2^6\,m2^3\,g\,k2\,k1\,L1 + m1^2\,L1^2\,L2^6\,m2^3\,g^2\,k2\,M + m1^3\,L1^2\,L2^6\,m2^3\,g^2\,k2\\
&- L2^5\,m2^4\,d1^2\,g^2\,m1\,L1 - 2\,L2^5\,m2^4\,d1\,g^2\,d2\,m1\,L1 + 2\,m1^2\,L1^3\,L2^5\,m2^4\,k1\,g^2\\
&- 2\,m1^3\,L1^4\,L2^5\,m2^4\,g^3 - 2\,m1^2\,L1^3\,L2^5\,m2^4\,k2\,g^2 - L2^5\,m2^3\,k1^2\,k2\,m1\,L1\\
&+ m1^3\,L1^4\,L2\,m2\,g\,d2^2\,k2 - L2^7\,m2^5\,k1^2\,g - L2^7\,m2^4\,k1^2\,g\,M - L2^4\,m2^3\,k1^2\,d2^2\\
&+ L2^6\,m2^4\,k1^2\,k2 + 2\,L2^5\,m2^3\,d1\,g\,d2\,k1\,M - 2\,L2^4\,m2^2\,k2\,d1\,d2\,k1\,M\\
&- L2^4\,m2^2\,k1^2\,d2^2\,M - 2\,L2^4\,m2^3\,k2\,d1\,d2\,k1 + 2\,L1^2\,L2^5\,m2^3\,k2\,g\,k1\,M\,m1\\
&- 2\,L2^5\,m2^3\,d1\,g^2\,d2\,M\,m1\,L1 - 2\,m1^2\,L2^5\,m2^3\,d1\,g^2\,d2\,L1\\
&+ m1^2\,L1^3\,L2\,m2\,k2^2\,d1\,d2 - m1^2\,L1^3\,L2\,m2\,k1\,d2^2\,k2 - m1^3\,L1^5\,L2\,m2\,k2^3\\
&- 3\,m1^2\,L1^2\,L2^3\,m2^2\,g\,d2^2\,k1 - m1^2\,L1^2\,d2^4\,k1 + m1^3\,L1^3\,d2^4\,g\\
&+ m1^2\,L1^3\,d2^4\,g\,M - L1^2\,m2\,d2^4\,k1\,m1 + m1^2\,L1^3\,m2\,d2^4\,g + m1^3\,L1^5\,L2^4\,m2^4\,g^3\\
&- L1^2\,L2^4\,m2^3\,k2^2\,k1\,m1 + 2\,L2^4\,m2^3\,d1^2\,g\,k2\,m1\,L1
\end{aligned}
$$

$$+ 2\, L2^{4}\, m2^{3}\, k2\, d1\, d2\, g\, m1\, L1 - 4\, m1^{2}\, L1^{3}\, L2^{4}\, m2^{3}\, k1\, k2\, g$$
$$- L2^{4}\, m2^{3}\, d1\, g\, k1\, d2\, m1\, L1 + m1^{2}\, L1^{2}\, L2^{4}\, m2^{3}\, d1\, g^{2}\, d2 + m1^{2}\, L1^{3}\, L2^{4}\, m2^{3}\, k2^{2}\, g$$
$$+ L2^{3}\, m2^{2}\, k2\, d1\, k1\, d2\, m1\, L1 - m1^{2}\, L1^{2}\, L2^{3}\, m2^{2}\, k2\, d1\, g\, d2$$
$$- L2^{3}\, m2^{2}\, k2^{2}\, d1^{2}\, m1\, L1 - L1^{2}\, L2^{6}\, m2^{4}\, g^{2}\, k1\, M\, m1$$
$$- 3\, L1^{2}\, L2^{3}\, m2^{2}\, d2^{2}\, g\, k1\, M\, m1 - 3\, m1^{2}\, L1^{2}\, L2^{6}\, m2^{4}\, k1\, g^{2}$$
$$+ m1^{2}\, L1^{2}\, L2^{6}\, m2^{4}\, g^{2}\, k2 + L2^{6}\, m2^{4}\, k1^{2}\, g\, m1\, L1 + 2\, L2^{7}\, m2^{5}\, k1\, g^{2}\, m1\, L1$$
$$- m1^{2}\, L1^{2}\, L2^{7}\, m2^{5}\, g^{3} - m1^{2}\, L1^{2}\, L2^{7}\, m2^{4}\, g^{3}\, M + 2\, m1^{2}\, L2^{7}\, m2^{4}\, k1\, g^{2}\, L1$$
$$- m1^{3}\, L1^{2}\, L2^{7}\, m2^{4}\, g^{3} + 2\, L2^{7}\, m2^{4}\, k1\, g^{2}\, M\, m1\, L1 - L2^{7}\, m2^{4}\, k1^{2}\, g\, m1$$
$$- L1^{2}\, L2^{6}\, m2^{5}\, g^{2}\, k1\, m1 + m1^{2}\, L1^{3}\, L2^{6}\, m2^{5}\, g^{3} + 2\, L2^{5}\, m2^{3}\, d1\, g\, d2\, k1\, m1$$
$$- 3\, m1^{3}\, L1^{3}\, L2^{5}\, m2^{3}\, k2\, g^{2} - 2\, m1^{2}\, L1^{3}\, L2^{5}\, m2^{3}\, k2\, g^{2}\, M$$
$$+ 4\, m1^{2}\, L1^{2}\, L2^{5}\, m2^{3}\, k2\, g\, k1) Dt^{2} / (m1\, L1\, \%1)]$$
$$\%1 := L2^{6}\, m2^{4}\, k1^{2}\, g^{2} - L2^{5}\, m2^{4}\, d1^{2}\, g^{3} + L2^{4}\, m2^{2}\, k2^{2}\, k1^{2} + L2^{2}\, m2\, d1^{2}\, k2^{3}$$
$$- 2\, L2^{6}\, m2^{4}\, k1\, g^{3}\, m1\, L1 + L2^{6}\, m2^{4}\, m1^{2}\, L1^{2}\, g^{4} - 2\, L2^{5}\, m2^{3}\, m1^{2}\, L1^{2}\, g^{3}\, k2$$
$$- 2\, L2^{5}\, m2^{4}\, g^{4}\, m1^{2}\, L1^{3} - 2\, L2^{5}\, m2^{3}\, k1^{2}\, k2\, g + 2\, L2^{5}\, m2^{4}\, g^{3}\, L1^{2}\, k1\, m1$$
$$+ 4\, L2^{5}\, m2^{3}\, k1\, k2\, m1\, L1\, g^{2} - 2\, L2^{4}\, m2^{2}\, k1\, k2^{2}\, m1\, L1\, g$$
$$- 6\, L2^{4}\, m2^{3}\, g^{2}\, L1^{2}\, k2\, k1\, m1 + 3\, L2^{4}\, m2^{3}\, d1^{2}\, g^{2}\, k2 + L2^{4}\, m2^{4}\, g^{4}\, L1^{4}\, m1^{2}$$
$$+ L2^{4}\, m2^{3}\, d1\, g^{3}\, d2\, m1\, L1 - L2^{4}\, m2^{3}\, d1\, d2\, k1\, g^{2} + 6\, L2^{4}\, m2^{3}\, g^{3}\, L1^{3}\, k2\, m1^{2}$$
$$+ L2^{4}\, m2^{2}\, m1^{2}\, L1^{2}\, g^{2}\, k2^{2} + L2^{3}\, m2^{3}\, d1\, g^{3}\, L1^{2}\, m1\, d2 + 2\, L2^{3}\, m2^{2}\, d1\, d2\, k2\, k1\, g$$
$$- 2\, L2^{3}\, m2^{2}\, d1\, g^{2}\, L1\, m1\, k2\, d2 - 6\, L2^{3}\, m2^{2}\, g^{2}\, L1^{3}\, m1^{2}\, k2^{2}$$
$$- 4\, L2^{3}\, m2^{3}\, g^{3}\, L1^{4}\, k2\, m1^{2} - 3\, L2^{3}\, m2^{2}\, d1^{2}\, g\, k2^{2} + 6\, L2^{3}\, m2^{2}\, g\, L1^{2}\, m1\, k2^{2}\, k1$$
$$- L2^{2}\, m2^{2}\, g^{3}\, L1^{3}\, d2^{2}\, m1^{2} - 2\, L2^{2}\, m2\, L1^{2}\, m1\, k2^{3}\, k1 - L2^{2}\, m2\, d1\, d2\, k2^{2}\, k1$$
$$+ L2^{2}\, m2^{2}\, g^{2}\, L1^{2}\, m1\, d2^{2}\, k1 - 3\, L2^{2}\, m2^{2}\, d1\, g^{2}\, L1^{2}\, m1\, k2\, d2$$
$$+ 6\, L2^{2}\, m2^{2}\, g^{2}\, L1^{4}\, m1^{2}\, k2^{2} + 2\, L2^{2}\, m2\, g\, L1^{3}\, m1^{2}\, k2^{3} + L2^{2}\, m2\, d1\, g\, L1\, m1\, k2^{2}\, d2$$
$$- 4\, L2\, m2\, g\, L1^{4}\, m1^{2}\, k2^{3} - 2\, L2\, m2\, g\, L1^{2}\, m1\, d2^{2}\, k2\, k1$$
$$+ 2\, L2\, m2\, g^{2}\, L1^{3}\, m1^{2}\, d2^{2}\, k2 + 3\, L2\, m2\, d1\, g\, L1^{2}\, m1\, k2^{2}\, d2 + L1^{4}\, m1^{2}\, k2^{4}$$
$$- g\, L1^{3}\, m1^{2}\, d2^{2}\, k2^{2} - d1\, k2^{3}\, m1\, L1^{2}\, d2 + L1^{2}\, m1\, d2^{2}\, k2^{2}\, k1$$
$$\%2 := L1^{4}\, k2^{2}\, k1^{2}\, m1^{2} + L1^{2}\, m1\, d2^{2}\, k1^{3} - g^{3}\, m1^{4}\, L1^{5}\, d2^{2} + g^{2}\, L1^{6}\, k2^{2}\, m1^{4} + k1^{4}\, m2^{2}\, L2^{4}$$
$$- d1\, L1^{2}\, m1\, k2\, d2\, k1^{2} - L2^{2}\, m2\, d1\, d2\, k1^{3} - 2\, L2^{2}\, m2\, k1^{3}\, m1\, L1^{2}\, k2$$
$$+ L2^{2}\, m2\, k1^{2}\, d1^{2}\, k2 + g^{4}\, m1^{4}\, L1^{4}\, m2^{2}\, L2^{4} - 2\, g^{4}\, m1^{4}\, L1^{5}\, m2^{2}\, L2^{3}$$
$$+ g^{4}\, m1^{4}\, L1^{6}\, m2^{2}\, L2^{2} - 4\, g^{3}\, m1^{3}\, L1^{3}\, k1\, m2^{2}\, L2^{4} + 6\, g^{3}\, L2^{3}\, m2^{2}\, m1^{3}\, L1^{4}\, k1$$
$$- g^{3}\, L2^{3}\, m2^{2}\, m1^{2}\, L1^{2}\, d1^{2} - 2\, g^{3}\, L2^{2}\, m2^{2}\, m1^{3}\, L1^{5}\, k1 + g^{3}\, L2^{2}\, m2\, m1^{3}\, L1^{3}\, d1\, d2$$
$$+ 2\, g^{3}\, L2^{2}\, m2\, m1^{4}\, L1^{5}\, k2 - 2\, g^{3}\, L2\, m2\, m1^{4}\, L1^{6}\, k2 + g^{3}\, L2\, m2\, d1\, d2\, m1^{3}\, L1^{4}$$
$$+ 6\, g^{2}\, L1^{2}\, k1^{2}\, m1^{2}\, m2^{2}\, L2^{4} - 6\, g^{2}\, L2^{3}\, m2^{2}\, L1^{3}\, k1^{2}\, m1^{2}$$
$$+ 2\, g^{2}\, L2^{3}\, m2^{2}\, k1\, d1^{2}\, m1\, L1 + g^{2}\, L2^{2}\, L1^{4}\, k1^{2}\, m2^{2}\, m1^{2}$$
$$- 6\, g^{2}\, L2^{2}\, m2\, k1\, m1^{3}\, L1^{4}\, k2 + g^{2}\, L2^{2}\, m2\, d1^{2}\, k2\, L1^{2}\, m1^{2}$$
$$- 3\, g^{2}\, L2^{2}\, m2\, d1\, L1^{2}\, m1^{2}\, d2\, k1 + 4\, g^{2}\, L2\, m2\, m1^{3}\, L1^{5}\, k1\, k2$$
$$- 2\, g^{2}\, L2\, m2\, d1\, d2\, k1\, m1^{2}\, L1^{3} - g^{2}\, d1\, d2\, m1^{3}\, L1^{4}\, k2 + 3\, g^{2}\, k1\, m1^{3}\, L1^{4}\, d2^{2}$$
$$- 4\, g\, k1^{3}\, L1\, m1\, m2^{2}\, L2^{4} + 2\, g\, L2^{3}\, m2^{2}\, k1^{3}\, m1\, L1^{2} - g\, L2^{3}\, m2^{2}\, k1^{2}\, d1^{2}$$
$$+ 3\, g\, L2^{2}\, m2\, k1^{2}\, L1\, m1\, d1\, d2 + 6\, g\, L2^{2}\, m2\, k1^{2}\, L1^{3}\, m1^{2}\, k2$$
$$- 2\, g\, L2^{2}\, m2\, d1^{2}\, k2\, L1\, m1\, k1 - 2\, g\, L2\, m2\, L1^{4}\, k2\, k1^{2}\, m1^{2}$$
$$+ g\, L2\, m2\, d1\, L1^{2}\, m1\, d2\, k1^{2} - 2\, g\, L1^{5}\, k2^{2}\, k1\, m1^{3} + 2\, g\, d1\, d2\, k1\, L1^{3}\, m1^{2}\, k2$$
$$- 3\, g\, k1^{2}\, L1^{3}\, m1^{2}\, d2^{2}$$

The fact that *M2* is generically a projective *Alg2*-module implies that *M2* is also a free *Alg2*-module as the system is time-invariant. This result can directly be verified by checking whether or not the parametrization *Ext2*[3] of the system admits a left-inverse:

```
> st := time(): S2 := LeftInverse(Ext2[3], Alg2): time()-st;
```

$$193.500$$

```
> Mult(S2, Ext2[3], Alg2);
```

$$[\ 1\ ]$$

Then, a left-inverse *S2* of *Ext2*[3] is defined by:

```
> S2fact := map(collect, S2, Dt);
```

$$
\begin{aligned}
&\mathit{S2fact} := \\
&[-(-L1^6\, m1^3\, k2^2\, k1\, d2 - L1^6\, m1^3\, k2^3\, d1 + 2\, L1^4\, L2^2\, m1^2\, m2\, k2^2\, k1\, d1 \\
&+ L1^4\, m1^2\, d2\, k2^2\, d1^2 - L1^2\, L2^2\, m1\, m2\, d1^3\, k2^2 + g\, L1^7\, m1^4\, k2^2\, d2 \\
&+ 2\, g\, L1^6\, L2\, m1^3\, m2\, k2\, k1\, d2 + 3\, g\, L1^6\, L2\, m1^3\, m2\, k2^2\, d1 \\
&+ 2\, g\, L1^2\, L2^3\, m1\, m2^2\, d1^3\, k2 - 2\, g\, L1^5\, L2^2\, m1^3\, m2\, k2^2\, d1 \\
&- 4\, g\, L1^4\, L2^3\, m1^2\, m2^2\, k2\, k1\, d1 - 2\, g\, L1^4\, L2\, m1^2\, m2\, d2\, k2\, d1^2 \\
&+ 2\, g^2\, L1^4\, L2^4\, m1^2\, m2^3\, k1\, d1 + 4\, g^2\, L1^5\, L2^3\, m1^3\, m2^2\, d1\, k2 \\
&+ g^2\, L1^4\, L2^2\, m1^2\, m2^2\, d2\, d1^2 - 2\, g^2\, L1^7\, L2\, m1^4\, m2\, k2\, d2 \\
&- g^2\, L1^6\, L2^2\, m1^3\, m2^2\, k1\, d2 - 3\, g^2\, L1^6\, L2^2\, m1^3\, m2^2\, d1\, k2 \\
&- g^2\, L1^2\, L2^4\, m1\, m2^3\, d1^3 + g^3\, L1^7\, L2^2\, m1^4\, m2^2\, d2 + g^3\, L1^6\, L2^3\, m1^3\, m2^3\, d1 \\
&- 2\, g^3\, L1^5\, L2^4\, m1^3\, m2^3\, d1)\mathit{Dt}/(\%1\, m1\, L1) - (-3\, g^3\, k2\, m2^2\, m1^4\, L2^2\, L1^7 \\
&+ 2\, g^3\, L1^6\, L2^3\, k2\, m2^2\, m1^4 - g^3\, L1^6\, L2^3\, k1\, m2^3\, m1^3 + 2\, g^3\, L1^5\, k1\, m2^3\, m1^3\, L2^4 \\
&+ 2\, g^3\, L1^5\, d1\, d2\, m2^2\, m1^3\, L2^2 + g^3\, d1^2\, m2^3\, m1^2\, L2^3\, L1^4 \\
&- 3\, g^3\, d1^2\, m2^3\, m1^2\, L2^4\, L1^3 + 3\, g^2\, k2^2\, m2\, m1^4\, L2\, L1^7 - g^2\, L1^6\, L2^2\, k2^2\, m2\, m1^4 \\
&+ 3\, g^2\, m1^3\, L1^6\, L2^2\, m2^2\, k1\, k2 - 4\, g^2\, L1^5\, k2\, k1\, m2^2\, m1^3\, L2^3 \\
&- 4\, g^2\, L1^5\, d1\, k2\, d2\, m2\, m1^3\, L2 - g^2\, m1^2\, L1^4\, L2^4\, m2^3\, k1^2 \\
&- 2\, g^2\, L1^4\, L2^2\, m1^2\, m2^2\, k1\, d1\, d2 - 3\, g^2\, m1^2\, L1^4\, L2^2\, m2^2\, d1^2\, k2 \\
&+ 6\, g^2\, m1^2\, m2^2\, k2\, d1^2\, L2^3\, L1^3 + 3\, g^2\, L1^2\, L2^4\, m2^3\, k1\, d1^2\, m1 \\
&+ g^2\, L1^2\, d1^3\, d2\, m2^2\, m1\, L2^2 - 3\, g\, k2^2\, k1\, m2\, m1^3\, L2\, L1^6 \\
&+ 2\, g\, L1^5\, k2^2\, k1\, m2\, m1^3\, L2^2 + 2\, g\, L1^5\, k2^2\, d1\, d2\, m1^3 \\
&+ 2\, g\, L1^4\, k2\, k1^2\, m2^2\, m1^2\, L2^3 + 4\, g\, L1^4\, L2\, m1^2\, m2\, k2\, k1\, d1\, d2 \\
&+ 3\, g\, L1^4\, L2\, m1^2\, m2\, k2^2\, d1^2 - 3\, g\, m1^2\, m2\, k2^2\, d1^2\, L2^2\, L1^3 \\
&- 6\, g\, L1^2\, k2\, k1\, d1^2\, m2^2\, m1\, L2^3 - 2\, g\, L1^2\, k2\, d1^3\, d2\, m2\, m1\, L2 \\
&+ 2\, g\, d1^4\, k2\, m2^2\, L2^3 - L1^4\, k1^2\, k2^2\, m2\, m1^2\, L2^2 - 2\, L1^4\, m1^2\, k2^2\, k1\, d1\, d2 \\
&+ 3\, L1^2\, k2^2\, k1\, d1^2\, m2\, m1\, L2^2 + L1^2\, k2^2\, d1^3\, d2\, m1 + g^4\, m2^3\, m1^4\, L2^3\, L1^7 \\
&- g^4\, m2^3\, m1^4\, L2^4\, L1^6 - g^2\, L2^4\, m2^3\, d1^4 - d1^4\, k2^2\, m2\, L2^2 + k2^3\, k1\, m1^3\, L1^6 \\
&- L1^4\, m1^2\, k2^3\, d1^2 - g\, k2^3\, m1^4\, L1^7)/(\%1\, m1\, L1), -(-g^2\, L1^4\, L2^2\, m1^3\, m2\, d2^3
\end{aligned}
$$

$$
\begin{aligned}
&+ g^3\,L1^3\,L2^6\,m1^3\,m2^3\,d2 + 2\,g\,L1^3\,L2^2\,m1^2\,m2\,k1\,d2^3 \\
&- 2\,g\,L1^2\,L2^5\,m1\,m2^3\,k1^2\,d2 - 2\,g^3\,L1^4\,L2^5\,m1^3\,m2^3\,d2 - L2^6\,m2^3\,k1^3\,d2 \\
&- g^2\,L1^2\,L2^6\,m1^2\,m2^3\,d1\,k2 + 2\,g\,L1\,L2^6\,m1\,m2^3\,k1\,d1\,k2 + g\,L2^7\,m2^4\,d1\,k1^2 \\
&- L1^2\,L2^2\,m1\,m2\,k1^2\,d2^3 - 2\,g^2\,L1\,L2^7\,m1\,m2^4\,d1\,k1 \\
&+ 2\,g^2\,L1^4\,L2^4\,m1^3\,m2^2\,k2\,d2 + 4\,g^2\,L1^3\,L2^5\,m1^2\,m2^3\,k1\,d2 \\
&+ g^2\,L1^2\,L2^4\,m1^2\,m2^2\,d2^2\,d1 + 3\,g\,L1\,L2^6\,m1\,m2^3\,k1^2\,d2 \\
&- 4\,g\,L1^3\,L2^4\,m1^2\,m2^2\,k2\,k1\,d2 - 2\,g\,L1\,L2^4\,m1\,m2^2\,d2^2\,d1\,k1 \\
&+ g^3\,L1^2\,L2^7\,m1^2\,m2^4\,d1 - L2^6\,m2^3\,k1^2\,d1\,k2 + L2^4\,m2^2\,d2^2\,d1\,k1^2 \\
&+ 2\,L1^2\,L2^4\,m1\,m2^2\,k2\,k1^2\,d2 - 3\,g^2\,L1^2\,L2^6\,m1^2\,m2^3\,k1\,d2)Dt/(\%1\,m1\,L1) \\
&-(-L1^2\,k1^2\,k2^2\,m2^2\,m1\,L2^4 + 3\,L1^2\,k2\,k1^2\,d2^2\,m2\,m1\,L2^2 - g^4\,m2^4\,m1^3\,L2^6\,L1^4 \\
&+ g^4\,m2^4\,m1^3\,L2^7\,L1^3 + 2\,g^3\,L1^4\,k2\,m2^3\,m1^3\,L2^5 - 3\,g^3\,L1^4\,d2^2\,m2^2\,m1^3\,L2^3 \\
&- g^3\,L1^3\,L2^6\,k2\,m2^3\,m1^3 + 2\,g^3\,L1^3\,L2^6\,k1\,m2^4\,m1^2 + g^3\,L1^3\,d2^2\,m2^2\,m1^3\,L2^4 \\
&- 3\,g^3\,L1^2\,k1\,m2^4\,m1^2\,L2^7 + 2\,g^3\,L1^2\,d1\,d2\,m2^3\,m1^2\,L2^5 \\
&- g^2\,L1^4\,k2^2\,m2^2\,m1^3\,L2^4 + 3\,g^2\,L1^4\,k2\,d2^2\,m2\,m1^3\,L2^2 \\
&- 4\,g^2\,m1^2\,L1^3\,L2^5\,m2^3\,k1\,k2 + 6\,g^2\,L1^3\,k1\,d2^2\,m2^2\,m1^2\,L2^3 \\
&+ 3\,g^2\,L1^2\,L2^6\,k2\,k1\,m2^3\,m1^2 - g^2\,L1^2\,L2^6\,k1^2\,m2^4\,m1 \\
&- 3\,g^2\,L1^2\,L2^4\,k1\,d2^2\,m2^2\,m1^2 - 2\,g^2\,m1^2\,L1^2\,L2^4\,m2^2\,d2\,k2\,d1 \\
&+ g^2\,m1^2\,L1^2\,L2^2\,m2\,d2^3\,d1 + 3\,g^2\,L1\,k1^2\,m2^4\,m1\,L2^7 \\
&- 4\,g^2\,L2^5\,m2^3\,k1\,d2\,d1\,m1\,L1 + 2\,g\,L1^3\,k2^2\,k1\,m2^2\,m1^2\,L2^4 \\
&- 6\,g\,L1^3\,k2\,k1\,d2^2\,m2\,m1^2\,L2^2 + 2\,g\,L1^3\,d2^4\,k1\,m1^2 + 2\,g\,L1^2\,k2\,k1^2\,m2^3\,m1\,L2^5 \\
&- 3\,g\,L1^2\,k1^2\,d2^2\,m2^2\,m1\,L2^3 - 3\,g\,L1\,L2^6\,m2^3\,k1^2\,k2\,m1 \\
&+ 3\,g\,L1\,L2^4\,m2^2\,k1^2\,d2^2\,m1 + 4\,g\,L1\,L2^4\,m2^2\,k2\,d1\,d2\,k1\,m1 \\
&- 2\,g\,L1\,L2^2\,m2\,d1\,d2^3\,k1\,m1 + 2\,g\,L2^5\,k1^2\,m2^3\,d2\,d1 - g^2\,L1^4\,d2^4\,m1^3 \\
&+ k1^3\,m2^3\,k2\,L2^6 - L1^2\,d2^4\,k1^2\,m1 - g\,k1^3\,m2^4\,L2^7 - L2^4\,m2^2\,k1^3\,d2^2 \\
&+ d2^3\,m2\,L2^2\,k1^2\,d1 - 2\,L2^4\,k1^2\,m2^2\,d2\,d1\,k2)/(\%1\,m1\,L1), \\
&-(-L1^5\,m1^3\,k2^2\,k1\,d2 + 2\,L1^3\,L2^2\,m1^2\,m2\,k2\,k1^2\,d2 + 2\,L1^3\,L2^2\,m1^2\,m2\,k2^2\,k1\,d1 \\
&+ L1^3\,m1^2\,k2^2\,d1^2\,d2 - L1^5\,m1^3\,k2^3\,d1 - L1^3\,m1^2\,k1^2\,d2^3 - g^2\,L1^5\,d2^3\,m1^4 \\
&- L1\,L2^4\,m1\,m2^2\,k1^3\,d2 - L1\,L2^4\,m1\,m2^2\,k2\,k1^2\,d1 + L1\,L2^2\,m1\,m2\,k1^2\,d1\,d2^2 \\
&- L1\,L2^2\,m1\,m2\,k2^2\,d1^3 + g\,k2^2\,d2\,m1^4\,L1^6 + g^3\,m2^2\,d2\,m1^4\,L2^2\,L1^6 \\
&- 2\,g^3\,L1^5\,L2^3\,m2^2\,d2\,m1^4 + g^3\,L1^5\,L2^3\,d1\,m2^3\,m1^3 + g^3\,L1^4\,L2^4\,m2^2\,d2\,m1^4 \\
&- 2\,g^3\,L1^4\,L2^4\,d1\,m2^3\,m1^3 + g^3\,d1\,m2^3\,m1^3\,L2^5\,L1^3 - 2\,g^2\,k2\,d2\,m2\,m1^4\,L2\,L1^6 \\
&+ 2\,g^2\,L1^5\,L2^2\,k2\,d2\,m2\,m1^4 - g^2\,L1^5\,L2^2\,m1^3\,m2^2\,k1\,d2 \\
&- 3\,g^2\,L1^5\,L2^2\,m1^3\,m2^2\,d1\,k2 + 4\,g^2\,L1^4\,L2^3\,m1^3\,m2^2\,k1\,d2 \\
&+ 4\,g^2\,L1^4\,L2^3\,m1^3\,m2^2\,d1\,k2 - 3\,g^2\,L1^3\,L2^4\,m1^3\,m2^2\,k1\,d2 \\
&- g^2\,L1^3\,L2^4\,m1^3\,m2^2\,d1\,k2 + 2\,g^2\,L1^3\,L2^4\,m1^2\,m2^3\,k1\,d1 \\
&+ g^2\,L1^3\,L2^2\,d1\,d2^2\,m2\,m1^3 + g^2\,L1^3\,L2^2\,d1^2\,d2\,m2^2\,m1^2 \\
&- 2\,g^2\,m1^2\,L1^2\,k1\,d1\,m2^3\,L2^5 - g^2\,d1^3\,m2^3\,m1\,L2^4\,L1 \\
&+ 2\,g\,L1^5\,L2\,m1^3\,m2\,k2\,k1\,d2 + 3\,g\,L1^5\,L2\,m1^3\,m2\,k2^2\,d1
\end{aligned}
$$

$$- 4 g L1^{4} L2^{2} m1^{3} m2 k2 k1 d2 - 2 g L1^{4} L2^{2} m1^{3} m2 k2^{2} d1 + 2 g L1^{4} k1 d2^{3} m1^{3}$$
$$- 2 g L1^{3} L2^{3} m1^{2} m2^{2} k1^{2} d2 - 4 g L1^{3} L2^{3} m1^{2} m2^{2} k2 k1 d1$$
$$- 2 g L1^{3} k2 d1^{2} d2 m2 m1^{2} L2 + 3 g L1^{2} L2^{4} m1^{2} m2^{2} k1^{2} d2$$
$$+ 2 g L1^{2} L2^{4} m1^{2} m2^{2} k2 k1 d1 - 2 g L1^{2} k1 d1 d2^{2} m2 m1^{2} L2^{2}$$
$$+ g L1 k1^{2} d1 m2^{3} m1 L2^{5} + 2 g L1 k2 d1^{3} m2^{2} m1 L2^{3}) Dt/(\%1 m1 L1) - ($$
$$-g k2^{3} m1^{4} L1^{6} + k2^{3} k1 m1^{3} L1^{5} - g^{3} L1^{5} m2 d2^{2} m1^{4} L2 - g^{3} L1^{4} L2^{4} k2 m2^{2} m1^{4}$$
$$+ 4 g^{3} L1^{4} k1 m2^{3} m1^{3} L2^{4} + g^{3} L1^{4} d1 d2 m2^{2} m1^{3} L2^{2} - 3 g^{3} L1^{3} k1 m2^{3} m1^{3} L2^{5}$$
$$+ g^{3} L1^{3} d1 d2 m2^{2} m1^{3} L2^{3} - g^{3} d1^{2} m2^{3} m1^{2} L2^{4} L1^{2} + 3 g^{2} k2^{2} m2 m1^{4} L2 L1^{6}$$
$$- 2 g^{2} L1^{5} L2^{2} k2^{2} m2 m1^{4} + 3 g^{2} L1^{5} L2^{2} k2 k1 m2^{2} m1^{3} + g^{2} L1^{5} k2 d2^{2} m1^{4}$$
$$- 8 g^{2} L1^{4} k2 k1 m2^{2} m1^{3} L2^{3} + 2 g^{2} L1^{4} L2 m1^{3} m2 k1 d2^{2}$$
$$- 2 g^{2} L1^{4} d1 k2 d2 m2 m1^{3} L2 + 3 g^{2} L1^{3} L2^{4} k2 k1 m2^{2} m1^{3}$$
$$- 2 g^{2} L1^{3} k1^{2} m2^{3} m1^{2} L2^{4} - g^{2} L1^{3} L2^{2} d1 k2 d2 m2 m1^{3}$$
$$- g^{2} L1^{3} L2^{2} m1^{2} m2^{2} k1 d1 d2 + 3 g^{2} L1^{2} k1^{2} m2^{3} m1^{2} L2^{5}$$
$$- 2 g^{2} L1^{2} L2^{3} m1^{2} m2^{2} k1 d1 d2 + 2 g^{2} L1^{2} m1^{2} m2^{2} k2 d1^{2} L2^{3}$$
$$+ g^{2} k1 d1^{2} m2^{3} m1 L2^{4} L1 - 3 g k2^{2} k1 m2 m1^{3} L2 L1^{5}$$
$$+ 4 g L1^{4} k2^{2} k1 m2 m1^{3} L2^{2} - 2 g L1^{4} m1^{3} k2 k1 d2^{2} + g L1^{4} k2^{2} d1 d2 m1^{3}$$
$$+ 4 g L1^{3} k2 k1^{2} m2^{2} m1^{2} L2^{3} - g L1^{3} L2 m1^{2} m2 k1^{2} d2^{2}$$
$$+ 2 g L1^{3} L2 m1^{2} m2 k2 k1 d1 d2 - 3 g L1^{2} k2 k1^{2} m2^{2} m1^{2} L2^{4}$$
$$+ 2 g L1^{2} L2^{2} m1^{2} m2 k2 k1 d1 d2 - g L1^{2} m1^{2} m2 k2^{2} d1^{2} L2^{2}$$
$$- g L1 k1^{3} m2^{3} m1 L2^{5} + g L1 L2^{3} m1 m2^{2} k1^{2} d1 d2$$
$$- 2 g L1 k2 k1 d1^{2} m2^{2} m1 L2^{3} + g^{4} m2^{3} m1^{4} L2^{3} L1^{6} - 2 g^{4} m2^{3} m1^{4} L2^{4} L1^{5}$$
$$+ g^{4} m2^{3} m1^{4} L2^{5} L1^{4} - 3 g^{3} k2 m2^{2} m1^{4} L2^{2} L1^{6} + 4 g^{3} L1^{5} L2^{3} k2 m2^{2} m1^{4}$$
$$+ L1^{3} m1^{2} k2 k1^{2} d2^{2} - 2 L1^{3} k1^{2} k2^{2} m2 m1^{2} L2^{2} - L1^{3} m1^{2} k2^{2} k1 d1 d2$$
$$+ L1 k1^{3} k2 m2^{2} m1 L2^{4} - L1 L2^{2} m1 m2 k2 k1^{2} d1 d2 + L1 k2^{2} k1 d1^{2} m2 m1 L2^{2}$$
$$- g^{3} L1^{5} L2^{3} k1 m2^{3} m1^{3})/(\%1 m1 L1), 0]$$
$$\%1 := -L2^{2} m2 k1^{3} d1 k2^{2} d2 - 12 g^{3} L1^{2} m1^{2} L2^{5} m2^{3} k2 k1^{2} - 2 g k2^{4} k1 m1^{3} L1^{5}$$
$$+ L2^{2} m2 k1^{2} k2^{3} d1^{2} + L2^{4} m2^{2} k1^{4} k2^{2} + k1^{2} k2^{4} m1^{2} L1^{4} + g^{2} k2^{4} m1^{4} L1^{6}$$
$$+ g^{2} L2^{6} m2^{4} k1^{4} - 4 g L2 m2 k1^{2} k2^{3} m1^{2} L1^{4} - 3 g L1^{3} m1^{2} k1^{2} k2^{2} d2^{2}$$
$$+ 2 g L1^{3} m1^{2} k1 d1 k2^{3} d2 + 6 g L1^{3} m1^{2} L2^{2} m2 k1^{2} k2^{3}$$
$$- 2 g L1^{2} m1 L2 m2 k1^{3} k2 d2^{2} + 3 g L1^{2} m1 L2 m2 k1^{2} d1 k2^{2} d2$$
$$+ 6 g L1^{2} m1 L2^{3} m2^{2} k1^{3} k2^{2} + 3 g L1 L2^{2} m1 m2 k1^{2} d1 k2^{2} d2$$
$$- 2 g L1 m1 L2^{2} m2 k1 d1^{2} k2^{3} - 4 g L1 m1 L2^{4} m2^{2} k1^{3} k2^{2}$$
$$+ 2 g L2^{3} m2^{2} k1^{3} d1 k2 d2 - 2 g L2^{5} m2^{3} k2 k1^{4} - 3 g L2^{3} m2^{2} k2^{2} d1^{2} k1^{2}$$
$$- 3 g^{3} L1^{2} m1^{2} L2^{3} m2^{2} k2^{2} d1^{2} + g^{3} L1^{2} m1 L2^{3} m2^{3} k1^{2} d1 d2$$
$$+ 2 g^{3} L1^{2} m1 L2^{5} m2^{4} k1^{3} + 3 g^{3} L1 L2^{4} m1 m2^{3} k1^{2} d1 d2$$
$$- 4 g^{3} L1 m1 L2^{6} m2^{4} k1^{3} - 6 g^{3} L1 m1 L2^{4} m2^{3} k2 k1 d1^{2} - g^{3} L2^{5} m2^{4} d1^{2} k1^{2}$$
$$+ 8 g^{2} L2 m2 k2^{3} k1 m1^{3} L1^{5} + 3 g^{2} L1^{4} m1^{3} k2^{2} k1 d2^{2} - g^{2} L1^{4} m1^{3} k2^{3} d2 d1$$
$$- 6 g^{2} L1^{4} m1^{3} L2^{2} m2 k2^{3} k1 + 6 g^{2} L1^{4} L2^{2} m2^{2} k1^{2} k2^{2} m1^{2}$$
$$+ 6 g^{2} L1^{3} m1^{2} L2 m2 k1^{2} k2 d2^{2} - 6 g^{2} L1^{3} m1^{2} L2 m2 k1 d1 k2^{2} d2$$

$$
\begin{aligned}
&-18\,g^2\,L1^3\,m1^2\,L2^3\,m2^2\,k1^2\,k2^2 - 3\,g^2\,L1^2\,L2^2\,m1^2\,m2\,k2^2\,k1\,d1\,d2\\
&+ g^2\,L1^2\,m1^2\,L2^2\,m2\,d1^2\,k2^3 + 6\,g^2\,L1^2\,m1^2\,L2^4\,m2^2\,k2^2\,k1^2\\
&+ g^2\,L1^2\,m1\,L2^2\,m2^2\,k1^3\,d2^2 - 3\,g^2\,L1^2\,m1\,L2^2\,m2^2\,k1^2\,d1\,k2\,d2\\
&- 6\,g^2\,L1^2\,m1\,L2^4\,m2^3\,k1^3\,k2 - 6\,g^2\,L1\,L2^3\,m1\,m2^2\,k2\,k1^2\,d1\,d2\\
&+ 6\,g^2\,L1\,m1\,L2^3\,m2^2\,k2^2\,k1\,d1^2 + 8\,g^2\,L1\,m1\,L2^5\,m2^3\,k1^3\,k2\\
&- g^2\,L2^4\,m2^3\,k1^3\,d1\,d2 + 3\,g^2\,L2^4\,m2^3\,k2\,d1^2\,k1^2 + L1^2\,m1\,k1^3\,k2^2\,d2^2\\
&- L1^2\,m1\,k1^2\,d1\,k2^3\,d2 - 2\,L1^2\,m1\,L2^2\,m2\,k1^3\,k2^3 + g^6\,m2^4\,L2^4\,m1^4\,L1^6\\
&- 2\,g^6\,L2^5\,m2^4\,m1^4\,L1^5 + g^6\,L2^6\,m2^4\,m1^4\,L1^4 - 4\,g^5\,k2\,m2^3\,L2^3\,m1^4\,L1^6\\
&- g^5\,L1^5\,d2^2\,m2^2\,m1^4\,L2^2 + 6\,g^5\,L1^5\,m1^4\,L2^4\,k2\,m2^3 - 2\,g^5\,L1^5\,L2^4\,m2^4\,k1\,m1^3\\
&- 2\,g^5\,L1^4\,L2^5\,k2\,m2^3\,m1^4 + g^5\,L1^4\,m1^3\,d1\,m2^3\,L2^3\,d2 + 6\,g^5\,L1^4\,m1^3\,L2^5\,m2^4\,k1\\
&+ g^5\,L1^3\,L2^4\,d1\,d2\,m2^3\,m1^3 - 4\,g^5\,L1^3\,m1^3\,L2^6\,m2^4\,k1 - g^5\,d1^2\,m2^4\,m1^2\,L2^5\,L1^2\\
&+ 6\,g^4\,k2^2\,m2^2\,L2^2\,m1^4\,L1^6 + 2\,g^4\,L1^5\,m1^4\,k2\,m2\,L2\,d2^2\\
&- 6\,g^4\,L1^5\,m1^4\,L2^3\,k2^2\,m2^2 + 8\,g^4\,L1^5\,L2^3\,m2^3\,k2\,k1\,m1^3\\
&+ g^4\,L1^4\,L2^4\,k2^2\,m2^2\,m1^4 + 3\,g^4\,L1^4\,m1^3\,L2^2\,k1\,m2^2\,d2^2\\
&- 3\,g^4\,L1^4\,m1^3\,L2^2\,m2^2\,d1\,k2\,d2 - 18\,g^4\,L1^4\,m1^3\,L2^4\,m2^3\,k2\,k1\\
&+ g^4\,L1^4\,m2^4\,L2^4\,k1^2\,m1^2 - 2\,g^4\,L1^3\,L2^3\,d1\,k2\,d2\,m2^2\,m1^3\\
&+ 8\,g^4\,L1^3\,m1^3\,L2^5\,m2^3\,k2\,k1 - 2\,g^4\,L1^3\,m1^2\,L2^3\,m2^3\,k1\,d1\,d2\\
&- 6\,g^4\,L1^3\,m1^2\,L2^5\,m2^4\,k1^2 - 3\,g^4\,L1^2\,L2^4\,m1^2\,m2^3\,k1\,d1\,d2\\
&+ 6\,g^4\,L1^2\,m1^2\,k1^2\,m2^4\,L2^6 + 3\,g^4\,L1^2\,m1^2\,L2^4\,m2^3\,k2\,d1^2\\
&+ 2\,g^4\,L2^5\,k1\,d1^2\,m2^4\,m1\,L1 - 4\,g^3\,k2^3\,m2\,L2\,m1^4\,L1^6 - g^3\,L1^5\,m1^4\,k2^2\,d2^2\\
&+ 2\,g^3\,L1^5\,m1^4\,L2^2\,m2\,k2^3 - 12\,g^3\,L1^5\,L2^2\,m2^2\,k2^2\,k1\,m1^3\\
&- 6\,g^3\,L1^4\,m1^3\,L2\,m2\,k1\,k2\,d2^2 + 3\,g^3\,L1^4\,m1^3\,L2\,m2\,k2^2\,d1\,d2\\
&+ 18\,g^3\,L1^4\,m1^3\,L2^3\,m2^2\,k2^2\,k1 - 4\,g^3\,L1^4\,L2^3\,m2^3\,k2\,k1^2\,m1^2\\
&+ g^3\,L1^3\,L2^2\,k2^2\,d1\,d2\,m2\,m1^3 - 4\,g^3\,L1^3\,m1^3\,L2^4\,m2^2\,k2^2\,k1\\
&- 3\,g^3\,L1^3\,m1^2\,L2^2\,m2^2\,k1^2\,d2^2 + 6\,g^3\,L1^3\,m1^2\,L2^2\,m2^2\,k1\,d1\,k2\,d2\\
&+ 18\,g^3\,L1^3\,m1^2\,L2^4\,m2^3\,k2\,k1^2 + 6\,g^3\,L1^2\,L2^3\,m1^2\,m2^2\,k2\,k1\,d1\,d2
\end{aligned}
$$

Therefore, a generic flat output of the system is defined by $\xi = S2\ (x1 : x2 : x3 : u)^T$, where $x1$, $x2$, $x3$ and u satisfy $(x1 : x2 : x3 : u)^T = Ext2[3]\ \xi$.

Now, let us compute the obstructions of flatness. In order to do that, let us compute where the denominators of $S2$ vanish.

```
> denoms := map(denom, S2fact):
> denoms[1,1];
```

$$
\begin{aligned}
&(-L2^{2}\, m2\, k1^{3}\, d1\, k2^{2}\, d2 - 12\, g^{3}\, L1^{2}\, m1^{2}\, L2^{5}\, m2^{3}\, k2\, k1^{2} - 2\, g\, k2^{4}\, k1\, m1^{3}\, L1^{5} \\
&+ L2^{2}\, m2\, k1^{2}\, k2^{3}\, d1^{2} + L2^{4}\, m2^{2}\, k1^{4}\, k2^{2} + k1^{2}\, k2^{4}\, m1^{2}\, L1^{4} + g^{2}\, k2^{4}\, m1^{4}\, L1^{6} \\
&+ g^{2}\, L2^{6}\, m2^{4}\, k1^{4} - 4\, g\, L2\, m2\, k1^{2}\, k2^{3}\, m1^{2}\, L1^{4} - 3\, g\, L1^{3}\, m1^{2}\, k1^{2}\, k2^{2}\, d2^{2} \\
&+ 2\, g\, L1^{3}\, m1^{2}\, k1\, d1\, k2^{3}\, d2 + 6\, g\, L1^{3}\, m1^{2}\, L2^{2}\, m2\, k1^{2}\, k2^{3} \\
&- 2\, g\, L1^{2}\, m1\, L2\, m2\, k1^{3}\, k2\, d2^{2} + 3\, g\, L1^{2}\, m1\, L2\, m2\, k1^{2}\, d1\, k2^{2}\, d2 \\
&+ 6\, g\, L1^{2}\, m1\, L2^{3}\, m2^{2}\, k1^{3}\, k2^{2} + 3\, g\, L1\, L2^{2}\, m1\, m2\, k1^{2}\, d1\, k2^{2}\, d2 \\
&- 2\, g\, L1\, m1\, L2^{2}\, m2\, k1\, d1^{2}\, k2^{3} - 4\, g\, L1\, m1\, L2^{4}\, m2^{2}\, k1^{3}\, k2^{2} \\
&+ 2\, g\, L2^{3}\, m2^{2}\, k1^{3}\, d1\, k2\, d2 - 2\, g\, L2^{5}\, m2^{3}\, k2\, k1^{4} - 3\, g\, L2^{3}\, m2^{2}\, k2^{2}\, d1^{2}\, k1^{2} \\
&- 3\, g^{3}\, L1^{2}\, m1^{2}\, L2^{3}\, m2^{2}\, k2^{2}\, d1^{2} + g^{3}\, L1^{2}\, m1\, L2^{3}\, m2^{3}\, k1^{2}\, d1\, d2 \\
&+ 2\, g^{3}\, L1^{2}\, m1\, L2^{5}\, m2^{4}\, k1^{3} + 3\, g^{3}\, L1\, L2^{4}\, m1\, m2^{3}\, k1^{2}\, d1\, d2 \\
&- 4\, g^{3}\, L1\, m1\, L2^{6}\, m2^{4}\, k1^{3} - 6\, g^{3}\, L1\, m1\, L2^{4}\, m2^{3}\, k2\, k1\, d1^{2} - g^{3}\, L2^{5}\, m2^{4}\, d1^{2}\, k1^{2} \\
&+ 8\, g^{2}\, L2\, m2\, k2^{3}\, k1\, m1^{3}\, L1^{5} + 3\, g^{2}\, L1^{4}\, m1^{3}\, k2^{2}\, k1\, d2^{2} - g^{2}\, L1^{4}\, m1^{3}\, k2^{3}\, d2\, d1 \\
&- 6\, g^{2}\, L1^{4}\, m1^{3}\, L2^{2}\, m2\, k2^{3}\, k1 + 6\, g^{2}\, L1^{4}\, L2^{2}\, m2^{2}\, k1^{2}\, k2^{2}\, m1^{2} \\
&+ 6\, g^{2}\, L1^{3}\, m1^{2}\, L2\, m2\, k1^{2}\, k2\, d2^{2} - 6\, g^{2}\, L1^{3}\, m1^{2}\, L2\, m2\, k1\, d1\, k2^{2}\, d2 \\
&- 18\, g^{2}\, L1^{3}\, m1^{2}\, L2^{3}\, m2^{2}\, k1^{2}\, k2^{2} - 3\, g^{2}\, L1^{2}\, L2^{2}\, m1^{2}\, m2\, k2^{2}\, k1\, d1\, d2 \\
&+ g^{2}\, L1^{2}\, m1^{2}\, L2^{2}\, m2\, d1^{2}\, k2^{3} + 6\, g^{2}\, L1^{2}\, m1^{2}\, L2^{4}\, m2^{2}\, k2^{2}\, k1^{2} \\
&+ g^{2}\, L1^{2}\, m1\, L2^{2}\, m2^{2}\, k1^{3}\, d2^{2} - 3\, g^{2}\, L1^{2}\, m1\, L2^{2}\, m2^{2}\, k1^{2}\, d1\, k2\, d2 \\
&- 6\, g^{2}\, L1^{2}\, m1\, L2^{4}\, m2^{3}\, k1^{3}\, k2 - 6\, g^{2}\, L1\, L2^{3}\, m1\, m2^{2}\, k2\, k1^{2}\, d1\, d2 \\
&+ 6\, g^{2}\, L1\, m1\, L2^{3}\, m2^{2}\, k2^{2}\, k1\, d1^{2} + 8\, g^{2}\, L1\, m1\, L2^{5}\, m2^{3}\, k1^{3}\, k2 \\
&- g^{2}\, L2^{4}\, m2^{3}\, k1^{3}\, d1\, d2 + 3\, g^{2}\, L2^{4}\, m2^{3}\, k2\, d1^{2}\, k1^{2} + L1^{2}\, m1\, k1^{3}\, k2^{2}\, d2^{2} \\
&- L1^{2}\, m1\, k1^{2}\, d1\, k2^{3}\, d2 - 2\, L1^{2}\, m1\, L2^{2}\, m2\, k1^{3}\, k2^{3} + g^{6}\, m2^{4}\, L2^{4}\, m1^{4}\, L1^{6} \\
&- 2\, g^{6}\, L2^{5}\, m2^{4}\, m1^{4}\, L1^{5} + g^{6}\, L2^{6}\, m2^{4}\, m1^{4}\, L1^{4} - 4\, g^{5}\, k2\, m2^{3}\, L2^{3}\, m1^{4}\, L1^{6} \\
&- g^{5}\, L1^{5}\, d2^{2}\, m2^{2}\, m1^{4}\, L2^{2} + 6\, g^{5}\, L1^{5}\, m1^{4}\, L2^{4}\, k2\, m2^{3} - 2\, g^{5}\, L1^{5}\, L2^{4}\, m2^{4}\, k1\, m1^{3} \\
&- 2\, g^{5}\, L1^{4}\, L2^{5}\, k2\, m2^{3}\, m1^{4} + g^{5}\, L1^{4}\, m1^{3}\, d1\, m2^{3}\, L2^{3}\, d2 + 6\, g^{5}\, L1^{4}\, m1^{3}\, L2^{5}\, m2^{4}\, k1 \\
&+ g^{5}\, L1^{3}\, L2^{4}\, d1\, d2\, m2^{3}\, m1^{3} - 4\, g^{5}\, L1^{3}\, m1^{3}\, L2^{6}\, m2^{4}\, k1 - g^{5}\, d1^{2}\, m2^{4}\, m1^{2}\, L2^{5}\, L1^{2} \\
&+ 6\, g^{4}\, k2^{2}\, m2^{2}\, L2^{2}\, m1^{4}\, L1^{6} + 2\, g^{4}\, L1^{5}\, m1^{4}\, k2\, m2\, L2\, d2^{2} \\
&- 6\, g^{4}\, L1^{5}\, m1^{4}\, L2^{3}\, k2^{2}\, m2^{2} + 8\, g^{4}\, L1^{5}\, L2^{3}\, m2^{3}\, k2\, k1\, m1^{3} \\
&+ g^{4}\, L1^{4}\, L2^{4}\, k2^{2}\, m2^{2}\, m1^{4} + 3\, g^{4}\, L1^{4}\, m1^{3}\, L2^{2}\, k1\, m2^{2}\, d2^{2} \\
&- 3\, g^{4}\, L1^{4}\, m1^{3}\, L2^{2}\, m2^{2}\, d1\, k2\, d2 - 18\, g^{4}\, L1^{4}\, m1^{3}\, L2^{4}\, m2^{3}\, k2\, k1 \\
&+ g^{4}\, L1^{4}\, m2^{4}\, L2^{4}\, k1^{2}\, m1^{2} - 2\, g^{4}\, L1^{3}\, L2^{3}\, d1\, k2\, d2\, m2^{2}\, m1^{3} \\
&+ 8\, g^{4}\, L1^{3}\, m1^{3}\, L2^{5}\, m2^{3}\, k2\, k1 - 2\, g^{4}\, L1^{3}\, m1^{2}\, L2^{3}\, m2^{3}\, k1\, d1\, d2 \\
&- 6\, g^{4}\, L1^{3}\, m1^{2}\, L2^{5}\, m2^{4}\, k1^{2} - 3\, g^{4}\, L1^{2}\, L2^{4}\, m1^{2}\, m2^{3}\, k1\, d1\, d2 \\
&+ 6\, g^{4}\, L1^{2}\, m1^{2}\, k1^{2}\, m2^{4}\, L2^{6} + 3\, g^{4}\, L1^{2}\, m1^{2}\, L2^{4}\, m2^{3}\, k2\, d1^{2} \\
&+ 2\, g^{4}\, L2^{5}\, k1\, d1^{2}\, m2^{4}\, m1\, L1 - 4\, g^{3}\, k2^{3}\, m2\, L2\, m1^{4}\, L1^{6} - g^{3}\, L1^{5}\, m1^{4}\, k2^{2}\, d2^{2} \\
&+ 2\, g^{3}\, L1^{5}\, m1^{4}\, L2^{2}\, m2\, k2^{3} - 12\, g^{3}\, L1^{5}\, L2^{2}\, m2^{2}\, k2^{2}\, k1\, m1^{3} \\
&- 6\, g^{3}\, L1^{4}\, m1^{3}\, L2\, m2\, k1\, k2\, d2^{2} + 3\, g^{3}\, L1^{4}\, m1^{3}\, L2\, m2\, k2^{2}\, d1\, d2 \\
&+ 18\, g^{3}\, L1^{4}\, m1^{3}\, L2^{3}\, m2^{2}\, k2^{2}\, k1 - 4\, g^{3}\, L1^{4}\, L2^{3}\, m2^{3}\, k2\, k1^{2}\, m1^{2} \\
&+ g^{3}\, L1^{3}\, L2^{2}\, k2^{2}\, d1\, d2\, m2\, m1^{3} - 4\, g^{3}\, L1^{3}\, m1^{3}\, L2^{4}\, m2^{2}\, k2^{2}\, k1 \\
&- 3\, g^{3}\, L1^{3}\, m1^{2}\, L2^{2}\, m2^{2}\, k1^{2}\, d2^{2} + 6\, g^{3}\, L1^{3}\, m1^{2}\, L2^{2}\, m2^{2}\, k1\, d1\, k2\, d2 \\
&+ 18\, g^{3}\, L1^{3}\, m1^{2}\, L2^{4}\, m2^{3}\, k2\, k1^{2} + 6\, g^{3}\, L1^{2}\, L2^{3}\, m1^{2}\, m2^{2}\, k2\, k1\, d1\, d2)\, m1\, L1
\end{aligned}
$$

```
> simplify(denoms[1,2]-denoms[1,1]);
```

$$0$$

```
> simplify(denoms[1,3]-denoms[1,1]);
```

$$0$$

```
> denoms[1,4];
```

$$1$$

Therefore, the entries of $S2$ has either $denoms$[1,1] or 1 as denominators. Let us find the algebraic variety corresponding to $denoms[1,1] = 0$:

```
> Sol := solve(denoms[1,1]):  nops([Sol]);
```

$$6$$

Hence, the solutions of $denoms[1,1] = 0$ split into 6 different groups:

```
> Sol[1];
```

$$\{g = g,\ L1 = L1,\ L2 = L2,\ m2 = m2,\ d1 = d1,\ d2 = d2,\ k1 = k1,\ k2 = k2,\ m1 = 0\}$$

This solution corresponds to $m1 = 0$. This solution is not physically admissible if we consider two pendula with non-zero masses.

```
> Sol[2];
```

$$\{g = g,\ L2 = L2,\ m2 = m2,\ d1 = d1,\ d2 = d2,\ k1 = k1,\ k2 = k2,\ m1 = m1,\ L1 = 0\}$$

This solution corresponds to $L1 = 0$. This solution is not physically admissible if we consider two pendula with non-zero lengths.

```
> Sol[3];
```

$$\{k2 = m2\, L2\, g,\ g = g,\ L1 = L1,\ L2 = L2,\ m2 = m2,\ d1 = d1,\ d2 = d2,\ k1 = k1,\ m1 = m1\}$$

This solution corresponds to $k2 = m2\, L2\, g$. This is a physically admissible solution.

```
> Sol[4];
```

$$\{k1 = m1\, L1\, g,\ g = g,\ L1 = L1,\ L2 = L2,\ m2 = m2,\ d1 = d1,\ d2 = d2,\ k2 = k2,\ m1 = m1\}$$

This solution corresponds to $k1 = m1\, L1\, g$. This is a physically admissible solution. Moreover, we obtain the following two solutions:

```
> Sol[5];
```

$$\begin{aligned}
&\{g = g,\ L1 = L1,\ L2 = L2,\ m2 = m2,\ d2 = d2,\ k1 = k1,\ k2 = k2,\ m1 = m1,\\
&d1 = (-L1^2\, m1\, k2\, d2 + g\, L1\, L2^2\, d2\, m2\, m1 + g\, m2\, L2\, d2\, m1\, L1^2 - L2^2\, m2\, k1\, d2\\
&+(8\, L1^2\, L2^6\, m2^4\, g^2\, k1\, m1 + 4\, L2^7\, m2^4\, k1^2\, g + L2^4\, m2^2\, k1^2\, d2^2 - 4\, L2^6\, m2^3\, k1^2\, k2\\
&- 8\, m1^2\, L1^3\, L2^6\, m2^4\, g^3 - 4\, m1^2\, L1^2\, L2^6\, m2^3\, g^2\, k2 - 16\, L1^2\, L2^5\, m2^3\, k2\, g\, k1\, m1\\
&+ 16\, m1^2\, L1^3\, L2^5\, m2^3\, k2\, g^2 + 2\, m1^2\, L1^3\, L2^2\, m2\, d2^2\, k2\, g\\
&- 2\, L1^2\, L2^2\, m2\, d2^2\, k2\, k1\, m1 + 4\, m1^2\, L1^2\, L2^7\, m2^4\, g^3 + 2\, L1^2\, L2^3\, m2^2\, d2^2\, g\, k1\, m1\\
&- 2\, m1^2\, L1^3\, L2^3\, m2^2\, d2^2\, g^2 + 8\, L1^2\, L2^4\, m2^2\, k2^2\, k1\, m1\\
&- 2\, L2^4\, m2^2\, k1\, d2^2\, g\, m1\, L1 + m1^2\, L1^2\, L2^4\, m2^2\, g^2\, d2^2 - 8\, m1^2\, L1^3\, L2^4\, m2^2\, k2^2\, g\\
&+ 8\, L2^6\, m2^3\, g\, k2\, k1\, m1\, L1 - 8\, L2^7\, m2^4\, k1\, g^2\, m1\, L1 - 4\, k2^3\, m2\, L2^2\, m1^2\, L1^4\\
&- 12\, L2^4\, m2^3\, g^2\, k2\, m1^2\, L1^4 + 12\, L2^3\, m2^2\, g\, m1^2\, k2^2\, L1^4 + 4\, L2^5\, m2^4\, g^3\, L1^4\, m1^2\\
&+ L1^4\, m1^2\, k2^2\, d2^2 + g^2\, L1^4\, L2^2\, m1^2\, m2^2\, d2^2 - 2\, g\, L1^4\, L2\, m1^2\, m2\, k2\, d2^2)^{(1/2)})/(2\\
&(L2^3\, m2^2\, g - k2\, m2\, L2^2))\}
\end{aligned}$$

```
> Sol[6];
```

$$\{g = g,\ L1 = L1,\ L2 = L2,\ m2 = m2,\ d2 = d2,\ k1 = k1,\ k2 = k2,\ m1 = m1,$$
$$d1 = (-L1^2\, m1\, k2\, d2 + g\, L1\, L2^2\, d2\, m2\, m1 + g\, m2\, L2\, d2\, m1\, L1^2 - L2^2\, m2\, k1\, d2$$
$$-(8\, L1^2\, L2^6\, m2^4\, g^2\, k1\, m1 + 4\, L2^7\, m2^4\, k1^2\, g + L2^4\, m2^2\, k1^2\, d2^2 - 4\, L2^6\, m2^3\, k1^2\, k2$$
$$-\, 8\, m1^2\, L1^3\, L2^6\, m2^4\, g^3 - 4\, m1^2\, L1^2\, L2^6\, m2^3\, g^2\, k2 - 16\, L1^2\, L2^5\, m2^3\, k2\, g\, k1\, m1$$
$$+\, 16\, m1^2\, L1^3\, L2^5\, m2^3\, k2\, g^2 + 2\, m1^2\, L1^3\, L2^2\, m2\, d2^2\, k2\, g$$
$$-\, 2\, L1^2\, L2^2\, m2\, d2^2\, k2\, k1\, m1 + 4\, m1^2\, L1^2\, L2^7\, m2^4\, g^3 + 2\, L1^2\, L2^3\, m2^2\, d2^2\, g\, k1\, m1$$
$$-\, 2\, m1^2\, L1^3\, L2^3\, m2^2\, d2^2\, g^2 + 8\, L1^2\, L2^4\, m2^2\, k2^2\, k1\, m1$$
$$-\, 2\, L2^4\, m2^2\, k1\, d2^2\, g\, m1\, L1 + m1^2\, L1^2\, L2^4\, m2^2\, g^2\, d2^2 - 8\, m1^2\, L1^3\, L2^4\, m2^2\, k2^2\, g$$
$$+\, 8\, L2^6\, m2^3\, g\, k2\, k1\, m1\, L1 - 8\, L2^7\, m2^4\, k1\, g^2\, m1\, L1 - 4\, k2^3\, m2\, L2^2\, m1^2\, L1^4$$
$$-\, 12\, L2^4\, m2^3\, g^2\, k2\, m1^2\, L1^4 + 12\, L2^3\, m2^2\, g\, m1^2\, k2^2\, L1^4 + 4\, L2^5\, m2^4\, g^3\, L1^4\, m1^2$$
$$+\, L1^4\, m1^2\, k2^2\, d2^2 + g^2\, L1^4\, L2^2\, m1^2\, m2^2\, d2^2 - 2\, g\, L1^4\, L2\, m1^2\, m2\, k2\, d2^2)^{(1/2)})/(2$$
$$(L2^3\, m2^2\, g - k2\, m2\, L2^2))\}$$

It seems to be difficult to know when these two solutions are physically admissible.

The last two solutions are the roots of the following second order equation in *d1*:

```
> d15 := rhs(select(has, Sol[5], d1)[1]):
> d16 := rhs(select(has, Sol[6], d1)[1]):
> sumroots := simplify(d15+d16):
> prodroots := simplify(expand(d15*d16)):
> Relation := collect(simplify(d1^2-sumroots*d1+prodroots), d1);
```

$$Relation := -\frac{(-L2^3\, m2^2\, g + k2\, m2\, L2^2)\, d1^2}{m2\, L2^2\, (-k2 + m2\, L2\, g)}$$
$$-\,\frac{(-L1^2\, m1\, k2\, d2 + g\, L1\, L2^2\, d2\, m2\, m1 + g\, m2\, L2\, d2\, m1\, L1^2 - L2^2\, m2\, k1\, d2)\, d1}{m2\, L2^2\, (-k2 + m2\, L2\, g)}$$
$$-\,(g^2\, L1^4\, L2^2\, m2^2\, m1^2 - 2\, g^2\, L1^3\, m1^2\, L2^3\, m2^2 + g^2\, L1^2\, L2^4\, m1^2\, m2^2$$
$$-\, 2\, g\, k2\, m2\, L2\, m1^2\, L1^4 + 2\, g\, L1^3\, m1^2\, L2^2\, k2\, m2 - g\, m1^2\, L1^3\, d2^2$$
$$+\, 2\, g\, L1^2\, m1\, L2^3\, m2^2\, k1 + m1^2\, k2^2\, L1^4 - 2\, g\, L1\, L2^4\, m1\, m2^2\, k1 + L1^2\, m1\, k1\, d2^2$$
$$+\, L2^4\, m2^2\, k1^2 - 2\, L1^2\, m1\, L2^2\, m2\, k2\, k1)/(m2\, L2^2\, (-k2 + m2\, L2\, g))$$

```
> a := simplify(coeff(Relation, d1, 2)); b := simplify(coeff(Relation, d1, 1));
> c := simplify(coeff(Relation, d1, 0));
```

$$a := 1$$
$$b := -\frac{d2\, (-m1\, L1^2\, k2 + m2\, L2^2\, m1\, L1\, g + m1\, L1^2\, m2\, L2\, g - m2\, L2^2\, k1)}{m2\, L2^2\, (-k2 + m2\, L2\, g)}$$
$$c := -(g^2\, L1^4\, L2^2\, m2^2\, m1^2 - 2\, g^2\, L1^3\, m1^2\, L2^3\, m2^2 + g^2\, L1^2\, L2^4\, m1^2\, m2^2$$
$$-\, 2\, g\, k2\, m2\, L2\, m1^2\, L1^4 + 2\, g\, L1^3\, m1^2\, L2^2\, k2\, m2 - g\, m1^2\, L1^3\, d2^2$$
$$+\, 2\, g\, L1^2\, m1\, L2^3\, m2^2\, k1 + m1^2\, k2^2\, L1^4 - 2\, g\, L1\, L2^4\, m1\, m2^2\, k1 + L1^2\, m1\, k1\, d2^2$$
$$+\, L2^4\, m2^2\, k1^2 - 2\, L1^2\, m1\, L2^2\, m2\, k2\, k1)/(m2\, L2^2\, (-k2 + m2\, L2\, g))$$

Let us first consider the case where $k1 = m1\, L1\, g$ (the case $k2 = m2\, L2\, g$ can be treated similarly). The system matrix becomes:

```
> R3 := subs(k1=m1*L1*g, evalm(R2));
```

$$R3 := \begin{bmatrix} m1\,L1\,Dt^2 & m2\,L2\,Dt^2 & (M+m1+m2)\,Dt^2 & -1 \\ m1\,L1^2\,Dt^2 + d1\,Dt & 0 & m1\,L1\,Dt^2 & 0 \\ 0 & m2\,L2^2\,Dt^2 + d2\,Dt + k2 - m2\,L2\,g & m1\,L1\,Dt^2 & 0 \end{bmatrix}$$

```
> R3_adj := Involution(R3, Alg2):
> Ext3 := Exti(R3_adj, Alg2, 1):  Ext3[1];
```

$$\begin{bmatrix} Dt & 0 & 0 \\ 0 & Dt & 0 \\ 0 & 0 & Dt \end{bmatrix}$$

We find again that this system is not controllable. We parametrization of the controllable part is then given by:

```
> map(collect, Ext3[3], Dt);
```

$$[-m2\,L2^2\,m1\,L1\,Dt^3 - L1\,m1\,d2\,Dt^2 + (-m1\,k2\,L1 + m2\,L2\,m1\,L1\,g)\,Dt]$$
$$[-m1^2\,L1^3\,Dt^3 - L1\,m1\,d1\,Dt^2]$$
$$[m2\,L2^2\,L1^2\,m1\,Dt^3 + (L2^2\,d1\,m2 + m1\,L1^2\,d2)\,Dt^2$$
$$+ (m1\,L1^2\,k2 - m1\,L1^2\,m2\,L2\,g + d1\,d2)\,Dt + d1\,k2 - g\,d1\,m2\,L2]$$
$$[(L1^2\,L2^2\,m1\,m2^2 + M\,m1\,L1^2\,L2^2\,m2 - m2\,m1^2\,L1^3\,L2)\,Dt^5 + (M\,m1\,L1^2\,d2$$
$$+ L1^2\,d2\,m2\,m1 + L2^2\,d1\,m2\,m1 - L2\,d1\,m2\,m1\,L1 + M\,m2\,L2^2\,d1 + L2^2\,d1\,m2^2)$$
$$Dt^4 + (M\,d1\,d2 + d1\,d2\,m1 + d1\,d2\,m2 - g\,L1^2\,L2\,m1\,m2^2 + M\,m1\,L1^2\,k2$$
$$+ L1^2\,m1\,m2\,k2 - M\,m1\,L1^2\,m2\,L2\,g)Dt^3 +$$
$$(M\,d1\,k2 + k2\,d1\,m2 - g\,L2\,d1\,m2^2 + m1\,d1\,k2 - g\,L2\,d1\,m2\,m1 - M\,g\,d1\,m2\,L2)$$
$$Dt^2]$$

The torsion elements of the *Alg2*-module associated with *R4* are defined by:

```
> TorsionElements(R3, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$\left[\begin{bmatrix} \frac{d}{dt}\,\theta_1(t) = 0 \\ \frac{d}{dt}\,\theta_2(t) = 0 \\ \frac{d}{dt}\,\theta_3(t) = 0 \end{bmatrix},\right.$$
$$[\theta_1(t) = d1\,\mathrm{x1}(t) + m1\,L1^2\,(\tfrac{d}{dt}\,\mathrm{x1}(t)) + m1\,L1\,(\tfrac{d}{dt}\,\mathrm{x3}(t))]$$
$$\Big[\theta_2(t) = -d1^2\,L2\,\mathrm{x1}(t) + (-L2\,m2\,m1\,L1^3\,g + k2\,m1\,L1^3)\,\mathrm{x2}(t)$$
$$+ L1^3\,m1\,d2\,(\tfrac{d}{dt}\,\mathrm{x2}(t)) - d1\,m1\,L2\,L1\,(\tfrac{d}{dt}\,\mathrm{x3}(t))$$
$$+ (m1^2\,L1^4 - L2\,m2\,m1\,L1^3 - L1^3\,L2\,m1\,M)\,(\tfrac{d^2}{dt^2}\,\mathrm{x3}(t)) + m1\,L2\,L1^3\,\mathrm{u}(t)\Big]$$
$$\Big[\theta_3(t) = d1^2\,\mathrm{x1}(t) + (L2\,L1^2\,m2^2\,g - L1^2\,M\,k2 + L2\,L1^2\,M\,g\,m2 - L1^2\,m2\,k2)\,\mathrm{x2}(t)$$
$$+ (-L1^2\,M\,d2 - L1^2\,m2\,d2)\,(\tfrac{d}{dt}\,\mathrm{x2}(t))$$
$$+ (-L2^2\,L1^2\,m2\,M - L2^2\,L1^2\,m2^2 + L2\,m2\,m1\,L1^3)\,(\tfrac{d^2}{dt^2}\,\mathrm{x2}(t))$$
$$\left. + L1\,m1\,d1\,(\tfrac{d}{dt}\,\mathrm{x3}(t)) - m1\,L1^3\,\mathrm{u}(t)\Big]\right]$$

Therefore, the autonomous elements of the system are defined by:

```
> AutonomousElements(R3, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$\left[\left[\begin{array}{c} d1\,\theta_1(t) - \theta_3(t) = 0 \\ \theta_2(t) + L2\,\theta_3(t) = 0 \\ \frac{d}{dt}\,\theta_3(t) = 0 \end{array}\right], \left[\begin{array}{c} \theta_1 = \frac{\_C1}{d1} \\ \theta_2 = -L2\,\_C1 \\ \theta_3 = \_C1 \end{array}\right],\right.$$

$$[\theta_1 = d1\,\mathrm{x1}(t) + m1\,L1^2\,(\tfrac{d}{dt}\,\mathrm{x1}(t)) + m1\,L1\,(\tfrac{d}{dt}\,\mathrm{x3}(t))]$$

$$\Big[\theta_2 = -d1^2\,L2\,\mathrm{x1}(t) + (-L2\,m2\,m1\,L1^3\,g + k2\,m1\,L1^3)\,\mathrm{x2}(t) + L1^3\,m1\,d2\,(\tfrac{d}{dt}\,\mathrm{x2}(t))$$
$$-\,d1\,m1\,L2\,L1\,(\tfrac{d}{dt}\,\mathrm{x3}(t)) + (m1^2\,L1^4 - L2\,m2\,m1\,L1^3 - L1^3\,L2\,m1\,M)\,(\tfrac{d^2}{dt^2}\,\mathrm{x3}(t))$$
$$+\,m1\,L2\,L1^3\,\mathrm{u}(t)\Big]$$

$$\Big[\theta_3 = d1^2\,\mathrm{x1}(t) + (L2\,L1^2\,m2^2\,g - L1^2\,M\,k2 + L2\,L1^2\,M\,g\,m2 - L1^2\,m2\,k2)\,\mathrm{x2}(t)$$
$$+\,(-L1^2\,M\,d2 - L1^2\,m2\,d2)\,(\tfrac{d}{dt}\,\mathrm{x2}(t))$$
$$+\,(-L2^2\,L1^2\,m2\,M - L2^2\,L1^2\,m2^2 + L2\,m2\,m1\,L1^3)\,(\tfrac{d^2}{dt^2}\,\mathrm{x2}(t))$$
$$\left.+\,L1\,m1\,d1\,(\tfrac{d}{dt}\,\mathrm{x3}(t)) - m1\,L1^3\,\mathrm{u}(t)\Big]\right]$$

Finally, the first integral of motion is defined by:

```
> FirstIntegral(R3, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$\_C1\,(d1\,\mathrm{x1}(t) + m1\,L1^2\,(\tfrac{d}{dt}\,\mathrm{x1}(t)) + m1\,L1\,(\tfrac{d}{dt}\,\mathrm{x3}(t)))$$

We easily check that the first time-derivative of the previous function is 0.

As in J. W. Polderman, J. C. Willems, *Introduction to Mathematical Systems Theory. A Behavioral Approach*, Springer, 1998, p. 171, let us consider the case $M = m1 = m2 = 1$, $L1 = L2 = 1$, $d1 = d2$ and $k1 = k2$.

```
> R4 := subs(M=1,m1=1,m2=1,L1=1,L2=1,d1=1,d2=1,k2=k1, evalm(R2));
```

$$R4 := \begin{bmatrix} Dt^2 & Dt^2 & 3\,Dt^2 & -1 \\ Dt^2 + Dt + k1 - g & 0 & Dt^2 & 0 \\ 0 & Dt^2 + Dt + k1 - g & Dt^2 & 0 \end{bmatrix}$$

```
> st := time(): Ext4 := Exti(Involution(R4, Alg2), Alg2, 1); time()-st;
```

$$Ext4 := \left[\begin{bmatrix} Dt^2 + Dt + k1 - g & 0 & 0 \\ 0 & Dt^2 + Dt + k1 - g & 0 \\ 0 & 0 & Dt^2 + Dt + k1 - g \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2\,g - 2\,Dt - 2\,k1 & Dt^2 & -1 \\ 0 & Dt^2 + 3\,Dt + 3\,k1 - 3\,g & 0 & 1 \end{bmatrix}, \begin{bmatrix} -Dt^2 \\ -Dt^2 \\ Dt^2 + Dt + k1 - g \\ Dt^4 + 3\,Dt^3 + 3\,k1\,Dt^2 - 3\,g\,Dt^2 \end{bmatrix}\right]$$

$$0.679$$

We find that the *Alg2*-module associated with *R4* is not torsion-free, and thus, the corresponding system is not controllable.

```
> TorsionElements(R4, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$\left[\left[\begin{array}{c} (k1-g)\,\theta_1(t)+(\frac{d}{dt}\,\theta_1(t))+(\frac{d^2}{dt^2}\,\theta_1(t))=0 \\ (k1-g)\,\theta_2(t)+(\frac{d}{dt}\,\theta_2(t))+(\frac{d^2}{dt^2}\,\theta_2(t))=0 \\ (k1-g)\,\theta_3(t)+(\frac{d}{dt}\,\theta_3(t))+(\frac{d^2}{dt^2}\,\theta_3(t))=0 \end{array}\right],\right.$$

$$\left.\left[\begin{array}{c} \theta_1(t)=\mathrm{x1}(t)-\mathrm{x2}(t) \\ \theta_2(t)=(-2\,k1+2\,g)\,\mathrm{x2}(t)-2\,(\frac{d}{dt}\,\mathrm{x2}(t))+(\frac{d^2}{dt^2}\,\mathrm{x3}(t))-\mathrm{u}(t) \\ \theta_3(t)=(3\,k1-3\,g)\,\mathrm{x2}(t)+3\,(\frac{d}{dt}\,\mathrm{x2}(t))+(\frac{d^2}{dt^2}\,\mathrm{x2}(t))+\mathrm{u}(t) \end{array}\right]\right]$$

Moreover, we obtain the following autonomous elements of the system:

```
> AutonomousElements(R4, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$\left[\left[\begin{array}{c} (-g^2-k1^2+2\,k1\,g)\,\theta_1(t)+(-k1+g+1)\,\theta_3(t)+(\frac{d}{dt}\,\theta_3(t))=0 \\ (k1-g)\,\theta_1(t)+(\frac{d}{dt}\,\theta_1(t))+\theta_3(t)=0 \\ \theta_2(t)+\theta_3(t)=0 \end{array}\right],\right.$$

$$\left[\theta_1=\frac{1}{2}(-\_C2\,(-2\,g-1+\sqrt{1-4\,k1+4\,g}+2\,k1)\,e^{((-1/2-\frac{\sqrt{1-4\,k1+4\,g}}{2})\,t)}\right.$$

$$\left.+\_C1\,e^{((-1/2+\frac{\sqrt{1-4\,k1+4\,g}}{2})\,t)}\,(2\,g+1+\sqrt{1-4\,k1+4\,g}-2\,k1))\Big/(k1-g)^2\right]$$

$$\left[\theta_2=-\_C1\,e^{((-1/2+\frac{\sqrt{1-4\,k1+4\,g}}{2})\,t)}-\_C2\,e^{((-1/2-\frac{\sqrt{1-4\,k1+4\,g}}{2})\,t)}\right]$$

$$\left[\theta_3=\_C1\,e^{((-1/2+\frac{\sqrt{1-4\,k1+4\,g}}{2})\,t)}+\_C2\,e^{((-1/2-\frac{\sqrt{1-4\,k1+4\,g}}{2})\,t)}\right],$$

$$\left.\left[\begin{array}{c} \theta_1=\mathrm{x1}(t)-\mathrm{x2}(t) \\ \theta_2=(-2\,k1+2\,g)\,\mathrm{x2}(t)-2\,(\frac{d}{dt}\,\mathrm{x2}(t))+(\frac{d^2}{dt^2}\,\mathrm{x3}(t))-\mathrm{u}(t) \\ \theta_3=(3\,k1-3\,g)\,\mathrm{x2}(t)+3\,(\frac{d}{dt}\,\mathrm{x2}(t))+(\frac{d^2}{dt^2}\,\mathrm{x2}(t))+\mathrm{u}(t) \end{array}\right]\right]$$

As in J. W. Polderman, J. C. Willems, *Introduction to Mathematical Systems Theory. A Behavioral Approach*, Springer, 1998, p. 171, we find that the autonomous element $\theta 1 = x1 - x2$ depends on $-\frac{1}{2}-\frac{1\,(1-4\,k1+4\,g)^{(\frac{1}{2})}}{2}$ and $-\frac{1}{2}+\frac{1\,(1-4\,k1+4\,g)^{(\frac{1}{2})}}{2}$. In particular, we easily check that if $g < k1$, then the system is stabilizable. Finally, let us compute the corresponding first integral of motion V:

```
> V := FirstIntegral(R4, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$
\begin{aligned}
V :=& -\frac{1}{2}\,\mathrm{x1}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)} - \frac{1}{2}\,\mathrm{x1}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)} + \left(\frac{d}{dt}\,\mathrm{x2}(t)\right)\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\\
&+ \left(\frac{d}{dt}\,\mathrm{x2}(t)\right)\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)} + \frac{1}{2}\,\mathrm{x2}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\\
&+ \frac{1}{2}\,\mathrm{x2}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)} - \left(\frac{d}{dt}\,\mathrm{x1}(t)\right)\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\\
&+ \frac{1}{2}\,\mathrm{x1}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\sqrt{1-4\,\mathit{k1}+4\,g}\\
&- \frac{1}{2}\,\mathrm{x1}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\sqrt{1-4\,\mathit{k1}+4\,g}\\
&- \frac{1}{2}\,\mathrm{x2}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\sqrt{1-4\,\mathit{k1}+4\,g}\\
&+ \frac{1}{2}\,\mathrm{x2}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\sqrt{1-4\,\mathit{k1}+4\,g} - \left(\frac{d}{dt}\,\mathrm{x1}(t)\right)\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\\
\%1 :=& \sqrt{1-4\,\mathit{k1}+4\,g}
\end{aligned}
$$

Let us check that V is a first integral of motion, i.e., its first time-derivative equals 0. First of all, let us differentiate V with respect to t.

```
> Vdot := simplify(diff(V,t));
```

$$
\begin{aligned}
\mathit{Vdot} :=& \left(\frac{d}{dt}\,\mathrm{x2}(t)\right)\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)} + \left(\frac{d}{dt}\,\mathrm{x2}(t)\right)\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\\
&- \left(\frac{d}{dt}\,\mathrm{x1}(t)\right)\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)} - \left(\frac{d}{dt}\,\mathrm{x1}(t)\right)\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\\
&+ \left(\frac{d^2}{dt^2}\,\mathrm{x2}(t)\right)\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)} + \left(\frac{d^2}{dt^2}\,\mathrm{x2}(t)\right)\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\\
&- \left(\frac{d^2}{dt^2}\,\mathrm{x1}(t)\right)\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)} - \left(\frac{d^2}{dt^2}\,\mathrm{x1}(t)\right)\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\\
&- \mathrm{x1}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\,\mathit{k1} + \mathrm{x1}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\,g\\
&- \mathrm{x1}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\,\mathit{k1} + \mathrm{x1}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\,g\\
&+ \mathrm{x2}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\,\mathit{k1} - \mathrm{x2}(t)\,\mathit{\_C1}\,e^{\left(\frac{(1+\%1)\,t}{2}\right)}\,g\\
&+ \mathrm{x2}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\,\mathit{k1} - \mathrm{x2}(t)\,\mathit{\_C2}\,e^{\left(-\frac{(-1+\%1)\,t}{2}\right)}\,g\\
\%1 :=& \sqrt{1-4\,\mathit{k1}+4\,g}
\end{aligned}
$$

Then, the left hand sides of the system equations are given by:

```
> Sys4 := ApplyMatrix(R4, [x1(t),x2(t),x3(t),u(t)], Alg2);
```

$$
\mathit{Sys4} := \begin{bmatrix} \left(\frac{d^2}{dt^2}\,\mathrm{x1}(t)\right) + \left(\frac{d^2}{dt^2}\,\mathrm{x2}(t)\right) + 3\left(\frac{d^2}{dt^2}\,\mathrm{x3}(t)\right) - \mathrm{u}(t)\\ (\mathit{k1}-g)\,\mathrm{x1}(t) + \left(\frac{d}{dt}\,\mathrm{x1}(t)\right) + \left(\frac{d^2}{dt^2}\,\mathrm{x1}(t)\right) + \left(\frac{d^2}{dt^2}\,\mathrm{x3}(t)\right)\\ (\mathit{k1}-g)\,\mathrm{x2}(t) + \left(\frac{d}{dt}\,\mathrm{x2}(t)\right) + \left(\frac{d^2}{dt^2}\,\mathrm{x2}(t)\right) + \left(\frac{d^2}{dt^2}\,\mathrm{x3}(t)\right) \end{bmatrix}
$$

We solve the second equation by respect of the second time-derivative of $\mathit{x1}$:

```
> Solvedform1 := solve(Sys4[2,1], diff(x1(t),t$2));
```

$$
\mathit{Solvedform1} := -\mathrm{x1}(t)\,\mathit{k1} + \mathrm{x1}(t)\,g - \left(\frac{d}{dt}\,\mathrm{x1}(t)\right) - \left(\frac{d^2}{dt^2}\,\mathrm{x3}(t)\right)
$$

We solve the third equation by respect of the second time-derivative of $\mathit{x2}$:

```
> Solvedform2 := solve(Sys4[3,1], diff(x2(t),t$2));
```

$$
\mathit{Solvedform2} := -\mathrm{x2}(t)\,\mathit{k1} + \mathrm{x2}(t)\,g - \left(\frac{d}{dt}\,\mathrm{x2}(t)\right) - \left(\frac{d^2}{dt^2}\,\mathrm{x3}(t)\right)
$$

We finally substitute the two previous results in *Vdot* and we obtain that, modulo the system equations, *Vdot* equals

```
> simplify(subs(diff(x1(t),t$2)=Solvedform1, diff(x2(t),t$2)=Solvedform2, Vdot));
```

$$0$$