We study a time-varying linear OD system which corresponds to a linear system of differential algebraic equations (DAE). See G. Le Vey, *Some remarks on solvability and various indices for implicit differential equations*, Numerical Algorithms, vol. 19 (1998), pp. 127-145.

```
> with(Ore_algebra):
> with(OreModules):
```

Define the Weyl algebra $Alg = A_1$, where Dt acts as differential operator w.r.t. the variable t:

```
> Alg := DefineOreAlgebra(diff=[Dt,t], polynom=[t]):
```

The time-varying linear system is defined by means of the following matrix (cf. (Le Vey, 1998, p. 135)):

```
> R := evalm([[-t*Dt+1,t^2*Dt,-1,0], [-Dt,t*Dt+1,0,-1]]);
```

$$R := \begin{bmatrix} -t\,Dt+1 & t^2\,Dt & -1 & 0 \\ -Dt & t\,Dt+1 & 0 & -1 \end{bmatrix}$$

Compute the formal adjoint of R:

```
> R_adj := Involution(R, Alg);
```

$$R\_adj := \begin{bmatrix} t\,Dt+2 & Dt \\ -t^2\,Dt-2\,t & -t\,Dt \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

We compute the first extension module ext^1 with values in *Alg* of the left *Alg*-module N which is associated with *R_adj*:

```
> Ext1 := Exti(R_adj, Alg, 1);
```

$$Ext1 := \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & t & 1 & -t \\ 0 & 0 & -Dt & t\,Dt \end{bmatrix}, \begin{bmatrix} t & 1 \\ 1 & 0 \\ 0 & -t\,Dt+1 \\ 0 & -Dt \end{bmatrix} \right]$$

Since *Ext1*[1] is the identity matrix, we conclude that ext^1 of the module N defined by *R_adj* is the zero module. Hence, the torsion submodule t(M) of the module M which is associated with R is zero. Equivalently, R is parametrizable and *Ext1*[3] gives a parametrization of the system. Of course, a necessary condition for *Ext1*[3] to be a parametrization of the system is (R o *Ext1*[3]) = 0:

```
> Mult(R, Ext1[3], Alg);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The command *Parametrization* provides a more direct way to obtain a parametrization of the system. The result is given in terms of free functions ξ_1, ξ_2:

```
> Parametrization(R, Alg);
```

$$\begin{bmatrix} t\,\xi_1(t)+\xi_2(t) \\ \xi_1(t) \\ \xi_2(t)-t\,(\frac{d}{dt}\,\xi_2(t)) \\ -(\frac{d}{dt}\,\xi_2(t)) \end{bmatrix}$$

We try to compute a left-inverse of the parametrization:

```
>  S := LeftInverse(Ext1[3], Alg);
```

$$S := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -t \end{bmatrix}$$

We found a left-inverse of *Ext1*[3], which means that this parametrization is injective. In the context of linear control systems, every left-inverse of a parametrization defines a *flat output* of the system. More precisely, $(\xi 1 : \xi 2)^T = S\ (x1 : x2 : u1 : u2)^T$ is a flat output of the system which satisfies $(x1 : x2 : u1 : u2)^T = Ext1[3]\,(\xi 1 : \xi 2)^T$.

Now, we consider a linear system of ordinary differential equations with polynomial coefficients that arises in the study of a linear differential algebraic system of index 2. See G. Le Vey, *Some remarks on solvability and various indices for implicit differential equations*, Numerical Algorithms, vol. 19 (1998), pp. 127-145.

We enter the matrix of the linear system of ordinary differential equations with polynomial coefficients (cf. (Le Vey, 1998, p. 136)):

```
>  R2 := evalm([[Dt+1,-1,0,-1,-1,0], [1,Dt-1,t,0,-1,0], [-t,0,Dt-1,-t,0,0],
>  [0,t-1,1,Dt,-t,0], [t^2,(1-t)^2,t-2,0,0,-1]]);
```

$$R2 := \begin{bmatrix} Dt+1 & -1 & 0 & -1 & -1 & 0 \\ 1 & Dt-1 & t & 0 & -1 & 0 \\ -t & 0 & Dt-1 & -t & 0 & 0 \\ 0 & t-1 & 1 & Dt & -t & 0 \\ t^2 & (1-t)^2 & t-2 & 0 & 0 & -1 \end{bmatrix}$$

The formal adjoint *R2_ adj* of the system is defined by:

```
>  R2_adj := Involution(R2, Alg);
```

$$R2\_adj := \begin{bmatrix} -Dt+1 & 1 & -t & 0 & t^2 \\ -1 & -Dt-1 & 0 & t-1 & 1-2\,t+t^2 \\ 0 & t & -Dt-1 & 1 & t-2 \\ -1 & 0 & -t & -Dt & 0 \\ -1 & -1 & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Let us first compute the rank of the left *Alg*-module *M2* associated with *R2*.

```
>  OreRankRat(R2, Alg);
```

$$1$$

Now, let us check whether the system defined by the matrix *R2* is parametrizable over the ring ordinary differential operators in *Dt* with rational coefficients in *t*.

```
>  st := time():  Ext2 := ExtiRat(R2_adj, Alg, 1):  time()-st; Ext2[1];
```

$$14.651$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, we obtain that the left *Alg*-module *M2* associated with the matrix *R2* is torsion-free, and thus, free because the ring of ordinary differential operators with rational coefficients is a left principal ideal domain. In particular, the system is parametrizable and a parametrization is defined by:

```
> map(collect, Ext2[3], Dt);
```

$$[\%1\, Dt^3 + (56\,t^7 - 113\,t^5 + 143\,t^6 + 2\,t^9 + 3\,t^8 + 39\,t^2 + 82\,t^3 - 149\,t^4)\, Dt^2 + (38\,t^8 + 353\,t^2 + 71\,t^7 - 725\,t^4 + 126\,t - 4\,t^9 + 1175\,t^5 - 870\,t^3 + 479\,t^6)\, Dt + 90 - 4\,t^{11} - 1340\,t^2 + 375\,t + 134\,t^7 - 39\,t^8 + 19\,t^9 + 363\,t^6 - 8\,t^{10} - 1285\,t^3 + 2722\,t^4 + 1248\,t^5]$$

$$[\%1\, Dt^3 + (-118\,t^5 + 150\,t^6 + 61\,t^7 - 2\,t^8 + 39\,t^2 + 82\,t^3 - 152\,t^4)\, Dt^2 + (12\,t^8 + 353\,t^2 - 42\,t^7 - 2\,t^{11} + 126\,t + 1287\,t^5 - 779\,t^4 - 7\,t^{10} + 547\,t^6 - 897\,t^3)\, Dt + 90 - 4\,t^{11} - 1385\,t^2 + 375\,t + 80\,t^7 - 3\,t^8 - 59\,t^9 - 176\,t^6 - 32\,t^{10} - 1395\,t^3 + 3092\,t^4 + 1430\,t^5]$$

$$[(3\,t^4 + 7\,t^9 + 2\,t^{10} + 8\,t^5 - 2\,t^6 - 12\,t^7)\, Dt^2 + (27\,t^3 - 50\,t^5 + 20\,t^7 + 84\,t^4 + 148\,t^8 - 188\,t^6 + 46\,t^9)\, Dt - 604\,t^5 + 170\,t^3 + 45\,t^2 + 695\,t^7 - 4\,t^{10} - 8\,t^9 + 173\,t^6 - 210\,t^4 + 255\,t^8]$$

$$[(5\,t^8 - 7\,t^6 + 3\,t^4 - 5\,t^7 + 2\,t^9 + 5\,t^5)\, Dt^3 + (-157\,t^5 + 155\,t^7 - 4\,t^9 + 56\,t^8 - 111\,t^6 + 39\,t^3 + 79\,t^4)\, Dt^2 + (450\,t^7 + 126\,t^2 + 1380\,t^6 - 674\,t^5 + 326\,t^3 - 92\,t^8 - 957\,t^4)\, Dt - 1525\,t^3 - 1080\,t^4 + 3444\,t^5 - 461\,t^7 + 7\,t^8 - 15\,t^9 + 982\,t^6 + 8\,t^{10} + 90\,t + 4\,t^{11} + 330\,t^2]$$

$$[\%1\, Dt^4 + (102\,t^3 + 77\,t^7 - 2\,t^8 - 187\,t^4 - 148\,t^5 + 48\,t^2 + 185\,t^6)\, Dt^3 + (-65\,t^7 + 2\,t^9 - 1505\,t^3 + 5\,t^8 + 204\,t + 2195\,t^5 - 1366\,t^4 + 975\,t^6 + 599\,t^2)\, Dt^2 + (4684\,t^5 + 101\,t^7 + 1081\,t + 43\,t^8 - 2\,t^{11} - t^{10} - 588\,t^6 + 15\,t^9 - 4076\,t^2 + 9608\,t^4 + 216 - 4484\,t^3) Dt + 375 - 4\,t^{11} - 4140\,t^2 - 2770\,t + 495\,t^6 - 28\,t^{10} - 1448\,t^5 + 203\,t^7 + 12523\,t^3 + 6950\,t^4 + 128\,t^8 + 13\,t^9]$$

$$[(6\,t^9 + t^7 - 18\,t^8 + 3\,t^3 - t^4 - 11\,t^5 + 19\,t^6 + 4\,t^{10})\, Dt^3 + (107\,t^9 - 244\,t^4 + 39\,t^2 + 337\,t^5 + 4\,t^{10} + 157\,t^8 + 97\,t^6 - 448\,t^7 + 4\,t^{11} + 4\,t^3)\, Dt^2 + (126\,t + 101\,t^2 - 3205\,t^6 + 1262\,t^5 + 1098\,t^7 + 61\,t^9 + 846\,t^8 - 1405\,t^3 + 8\,t^{11} + 89\,t^{10} - 2\,t^{13} - 3\,t^{12} + 1580\,t^4) Dt + 90 - 2045\,t^2 + 195\,t + 1830\,t^3 - 8\,t^{13} - 32\,t^{12} + 1893\,t^7 + 209\,t^8 + 432\,t^9 + 1828\,t^6 + 44\,t^{10} + 16\,t^{11} - 6436\,t^5 + 3747\,t^4]$$

$$\%1 := 5\,t^7 + 3\,t^3 - 5\,t^6 + 2\,t^8 + 5\,t^4 - 7\,t^5$$

The fact that the left *Alg*-module *M2* associated with *R2* is free implies that it is in particular projective. However, *R2* has full row rank as we can easily check it by computing a free resolution of *M2*.

```
> FreeResolutionRat(R2, Alg);
```

$$\text{table}([1 = \begin{bmatrix} Dt+1 & -1 & 0 & -1 & -1 & 0 \\ 1 & Dt-1 & t & 0 & -1 & 0 \\ -t & 0 & Dt-1 & -t & 0 & 0 \\ 0 & t-1 & 1 & Dt & -t & 0 \\ t^2 & (1-t)^2 & t-2 & 0 & 0 & -1 \end{bmatrix}, 2 = \text{INJ}(5)])$$

Therefore, *M2* is a projective left *Alg*-module iff *R2* admits a right-inverse. Let us check that such a right-inverse exists.

```
> T2 := map(collect, RightInverseRat(R2, Alg), Dt);
```

$T2 :=$

$\Big[(26\,t^4 + 263\,t^{10} - 30\,t^3 - 16\,t^{11} - 9\,t^2 + 49\,t^{14} + 148\,t^5 + 4\,t^{16} + 28\,t^{15} - 163\,t^8$

$+ 23\,t^6 - 236\,t^7 + 168\,t^9 - 180\,t^{12} - 48\,t^{13})Dt^2/(t^4\,\%1^3) + (58\,t^{11} + 18\,t - 35\,t^{15}$

$- 191\,t^{14} + 381\,t^5 - 782\,t^7 - 250\,t^4 + 99\,t^2 + 149\,t^{10} + 242\,t^{12} - 603\,t^8 + 16\,t^3$

$+ 611\,t^6 + 16\,t^{16} + 480\,t^9 + 4\,t^{17} - 78\,t^{13})Dt/(t^4\,\%1^3) + (-18 - 126\,t^{11} - 69\,t^2$

$- 99\,t - 461\,t^5 - 20\,t^7 - 2114\,t^8 - 498\,t^9 + 309\,t^{13} - 726\,t^{12} + 437\,t^{14} - 88\,t^{16}$

$+ 654\,t^3 - 30\,t^{15} - 16\,t^{17} + 1206\,t^{10} + 456\,t^4 + 528\,t^6)/(t^4\,\%1^3),\ \dfrac{\%2\,Dt^2}{t^4\,\%1^3}$

$+ (-404\,t^5 + 1100\,t^7 + 306\,t^4 - 18\,t - 99\,t^2 + 113\,t^{12} + 752\,t^8 + 630\,t^{11} - 466\,t^{10}$

$+ 18\,t^{14} - 1210\,t^9 + 5\,t^3 - 638\,t^6 - 159\,t^{13} + 8\,t^{16} + 44\,t^{15})Dt/(t^4\,\%1^3) + (18 - 54\,t^{11}$

$+ 69\,t^2 + 99\,t + 362\,t^9 - 68\,t^6 - 1035\,t^{10} + 8\,t^{17} - 149\,t^{13} + 475\,t^{12} - 243\,t^{14}$

$+ 884\,t^5 + 44\,t^{16} - 677\,t^3 - 352\,t^4 + 10\,t^{15} - 239\,t^7 + 1301\,t^8)/(t^4\,\%1^3), -(82\,t^4$

$- 56\,t^{12} - 51\,t^3 + 252\,t^5 + 49\,t^{13} - 18\,t^2 - 6\,t^{10} + 4\,t^{15} - 216\,t^{11} + 28\,t^{14} + 140\,t^8$

$- 162\,t^6 - 452\,t^7 + 424\,t^9)Dt^2/(t^4\,\%1^3) - (491\,t^5 - 949\,t^7 - 775\,t^4 + 229\,t^{11} + 36\,t$

$- 1456\,t^8 - 35\,t^{14} - 34\,t^3 + 800\,t^{10} - 194\,t^{12} + 198\,t^2 + 1518\,t^6 + 16\,t^{15} + 461\,t^9$

$+ 4\,t^{16} - 211\,t^{13})Dt/(t^4\,\%1^3) - (-36 + 162\,t^{11} - 138\,t^2 - 198\,t - 120\,t^8 - 911\,t^9$

$- 208\,t^6 - 1364\,t^{10} + 150\,t^{13} + 1169\,t^{12} - 307\,t^{14} - 46\,t^{15} + 8\,t^{17} + 36\,t^{16} + 1352\,t^3$

$+ 598\,t^4 - 2128\,t^5 + 1447\,t^7)/(t^4\,\%1^3), -(-39\,t^3 + 155\,t^{10} + 41\,t^{13} + 5\,t^4 + 396\,t^9$

$- 92\,t^{12} + 186\,t^5 - 206\,t^{11} - 378\,t^7 - 149\,t^8 + 4\,t^{15} + 28\,t^{14} - 9\,t^2 + 67\,t^6)Dt^2/(t^4$

$\%1^3) - (447\,t^7 - 161\,t^5 - 1522\,t^8 - 395\,t^4 + 539\,t^{11} + 18\,t - 319\,t^{12} + t^{14} + 46\,t^3$

$+ 1100\,t^{10} + 24\,t^{15} + 99\,t^2 + 1074\,t^6 - 229\,t^{13} - 627\,t^9 + 4\,t^{16})Dt/(t^4\,\%1^3)$

$- (-18 + 68\,t^{11} - 69\,t^2 - 99\,t - 929\,t^9 - 706\,t^6 - 719\,t^{10} - 48\,t^{15} + 193\,t^{13} + 389\,t^{12}$

$- 44\,t^{14} - 614\,t^5 - 8\,t^{16} + 499\,t^3 + 584\,t^4 + 709\,t^7 + 491\,t^8)/(t^4\,\%1^3),\ 0\Big]$

$\Big[- (-26\,t^4 + 180\,t^{12} + 30\,t^3 + 16\,t^{11} + 163\,t^8 - 263\,t^{10} - 28\,t^{15} + 9\,t^2 - 49\,t^{14} - 4\,t^{16}$

$+ 236\,t^7 - 168\,t^9 + 48\,t^{13} - 23\,t^6 - 148\,t^5)Dt^2/(t^4\,\%1^3) - (143\,t^{14} + 205\,t^{11} - 355\,t^5$

$+ 805\,t^7 - 99\,t^2 - 25\,t^3 + 19\,t^{10} - 258\,t^{12} + 220\,t^4 - 18\,t - 463\,t^6 - 643\,t^9 + 84\,t^{15}$

$- 102\,t^{13} + 367\,t^8 + 12\,t^{16})Dt/(t^4\,\%1^3) - (18 - 572\,t^{11} + 78\,t^2 + 99\,t - 385\,t^4$

$+ 375\,t^5 + 81\,t^7 + 1979\,t^8 + 427\,t^{13} + 725\,t^{12} - 362\,t^{14} - 1102\,t^{10} - 336\,t^{15} + 77\,t^{17}$

$- 15\,t^{16} + 32\,t^{18} + 4\,t^{19} - 591\,t^3 + 780\,t^9 - 529\,t^6)/(t^4\,\%1^3),\ \dfrac{\%2\,Dt^2}{t^4\,\%1^3} + (-339\,t^5$

$+ 937\,t^7 + 285\,t^4 - 18\,t - 99\,t^2 - 58\,t^{12} + 449\,t^8 + 497\,t^{11} - 149\,t^{10} + 62\,t^{14}$

$- 995\,t^9 - 4\,t^3 - 495\,t^6 - 133\,t^{13} + 8\,t^{16} + 52\,t^{15})Dt/(t^4\,\%1^3) + (18 + 280\,t^{11}$

$+ 78\,t^2 + 99\,t - 96\,t^9 - 65\,t^6 - 843\,t^{10} + 8\,t^{17} - 261\,t^{13} + 429\,t^{12} - 267\,t^{14} + 707\,t^5$

$+ 44\,t^{16} - 614\,t^3 - 305\,t^4 + 10\,t^{15} + 123\,t^7 + 1135\,t^8)/(t^4\,\%1^3), (51\,t^3 - 49\,t^{13}$

$- 82\,t^4 + 6\,t^{10} - 252\,t^5 + 216\,t^{11} + 56\,t^{12} - 4\,t^{15} - 28\,t^{14} + 18\,t^2 - 140\,t^8 + 162\,t^6$

$+ 452\,t^7 - 424\,t^9)Dt^2/(t^4\,\%1^3) + (-409\,t^5 + 724\,t^4 - 36\,t - 198\,t^2 + 787\,t^7$

$+ 1004\,t^8 - 235\,t^{11} - 376\,t^{10} - 22\,t^{12} - 321\,t^9 + 16\,t^3 - 1266\,t^6 + 84\,t^{14} + 155\,t^{13}$

$+ 12\,t^{15})Dt/(t^4\,\%1^3)$

$-(-36 + 421\,t^{11} - 246\,t^2 - 216\,t - 1178\,t^9 - 151\,t^6 - 659\,t^{10} + 8\,t^{17} - 74\,t^{13}$
$+ (36 + 169\,t^{11} + 156\,t^2 + 198\,t + 513\,t^8 + 176\,t^9 + 27\,t^6$
$+ 592\,t^{10} - 124\,t^{13} - 345\,t^{12} - 87\,t^{14} + 24\,t^{17} + 25\,t^{16} + 4\,t^{18} - 1226\,t^3 - 516\,t^4$
$+ 1688\,t^5 - 634\,t^7 - 61\,t^{15})/(t^4\,\%1^3), (39\,t^3 - 41\,t^{13} - 5\,t^4 - 155\,t^{10} - 186\,t^5$
$+ 206\,t^{11} + 92\,t^{12} - 4\,t^{15} - 28\,t^{14} + 9\,t^2 + 149\,t^8 - 67\,t^6 + 378\,t^7 - 396\,t^9)Dt^2/(t^4$
$\%1^3) + (166\,t^5 + 356\,t^4 - 18\,t - 99\,t^2 - 380\,t^7 + 1144\,t^8 - 384\,t^{11} - 704\,t^{10}$
$+ 113\,t^{12} + 478\,t^9 - 55\,t^3 - 888\,t^6 + 40\,t^{14} + 137\,t^{13} + 4\,t^{15})Dt/(t^4\,\%1^3) + (18$
$+ 183\,t^{11} + 78\,t^2 + 99\,t + 398\,t^8 + 445\,t^9 + 338\,t^6 - 468\,t^{10} - 135\,t^{13} + 551\,t^{12}$
$- 350\,t^{14} + 32\,t^{17} + 69\,t^{16} + 4\,t^{18} - 436\,t^3 - 498\,t^4 + 424\,t^5 - 300\,t^7 - 59\,t^{15})/(t^4$
$\%1^3), 0\Big]$

$$\Bigg[\frac{(17\,t^5 - 3\,t - 8\,t^2 - 28\,t^7 - 5\,t^8 + 16\,t^9 + 2\,t^{11} + 11\,t^{10} + 26\,t^4 + 5\,t^3 - 15\,t^6)\,Dt}{t^2\,\%1^2} +$$
$$\frac{3 - 4\,t^{11} + 12\,t^2 + 16\,t + 4\,t^5 + 11\,t^7 + 24\,t^8 - 18\,t^9 - 52\,t^6 - 20\,t^{10} + 22\,t^3 + 65\,t^4}{t^2\,\%1^2},$$
$$\frac{(16\,t^5 - 5\,t^7 - 16\,t^8 + 3\,t + 5\,t^2 - 4\,t^9 - 26\,t^4 - 16\,t^3 + 31\,t^6)\,Dt}{t^2\,\%1^2}$$
$$+ \frac{-3 + 4\,t^{11} - 7\,t^2 - 16\,t - 42\,t^8 + 13\,t^9 + 79\,t^6 + 20\,t^{10} - 7\,t^7 - 8\,t^3 - 53\,t^4 + 8\,t^5}{t^2\,\%1^2},$$
$$-\frac{(-8\,t^5 - 13\,t^2 - 9\,t^7 + 16\,t^8 - 6\,t + 11\,t^9 + 2\,t^{10} + 41\,t^4 + 18\,t^3 - 40\,t^6)\,Dt}{t^2\,\%1^2}$$
$$- \frac{6 + 7\,t^2 + 32\,t + 5\,t^7 - 13\,t^8 - 18\,t^9 - 15\,t^6 - 4\,t^{10} - 24\,t^3 + 44\,t^4 + 28\,t^5}{t^2\,\%1^2},$$
$$-\frac{(-33\,t^6 + 16\,t^5 - 21\,t^7 + 12\,t^8 - 3\,t - 11\,t^2 + 11\,t^9 + 2\,t^{10} + 33\,t^4)\,Dt}{t^2\,\%1^2}$$
$$- \frac{3 + 14\,t^2 + 16\,t - 16\,t^8 - 2\,t^9 + 25\,t^6 + 60\,t^5 - 30\,t^7 - 21\,t^3 - 4\,t^4}{t^2\,\%1^2}, 0\Bigg]$$

$\Big[(26\,t^4 + 263\,t^{10} - 30\,t^3 - 16\,t^{11} - 9\,t^2 + 49\,t^{14} + 148\,t^5 + 4\,t^{16} + 28\,t^{15} - 163\,t^8$
$+ 23\,t^6 - 236\,t^7 + 168\,t^9 - 180\,t^{12} - 48\,t^{13})Dt^2/(t^3\,\%1^3) + (108\,t^2 + 207\,t^5 - 569\,t^7$
$- 192\,t^{14} - 189\,t^{11} - 282\,t^{10} - 204\,t^8 - 246\,t^4 + 55\,t^3 + 440\,t^6 + 438\,t^{12} - 112\,t^{15}$
$+ 475\,t^9 + 18\,t + 150\,t^{13} - 16\,t^{16})Dt/(t^3\,\%1^3) + (-18 + 1041\,t^{11} - 123\,t^2 - 108\,t$
$- 1977\,t^8 - 1832\,t^9 + 592\,t^6 + 880\,t^{10} - 355\,t^{13} - 423\,t^{12} + 206\,t^{14} + 371\,t^4$
$+ 156\,t^{15} + 24\,t^{16} + 571\,t^3 - 713\,t^5 + 871\,t^7)/(t^3\,\%1^3), \dfrac{\%2\,Dt^2}{t^3\,\%1^3} + (634\,t^7 - 196\,t^5$
$- 32\,t^{12} + 350\,t^4 - 18\,t - 108\,t^2 + 70\,t^{14} + 664\,t^8 + 326\,t^{11} - 282\,t^{10} - 89\,t^{13}$
$- 678\,t^9 - 25\,t^3 - 658\,t^6 + 52\,t^{15} + 8\,t^{16})Dt/(t^3\,\%1^3) + (18 - 824\,t^{11} + 123\,t^2$
$+ 108\,t - 281\,t^6 - 1219\,t^{10} - 176\,t^{14} - 16\,t^{16} + 252\,t^{13} + 489\,t^{12} + 982\,t^5 - 837\,t^7$
$- 600\,t^3 - 313\,t^4 - 112\,t^{15} + 1593\,t^8 + 1284\,t^9)/(t^3\,\%1^3), -(82\,t^4 - 56\,t^{12} - 51\,t^3$
$+ 252\,t^5 + 49\,t^{13} - 18\,t^2 - 6\,t^{10} + 4\,t^{15} - 216\,t^{11} + 28\,t^{14} + 140\,t^8 - 162\,t^6 - 452\,t^7$
$+ 424\,t^9)Dt^2/(t^3\,\%1^3) - (157\,t^5 - 335\,t^7 - 806\,t^4 + 36\,t + 216\,t^2 + 78\,t^{12} - 1144\,t^8$
$+ 451\,t^{11} + 382\,t^{10} - 112\,t^{14} - 103\,t^9 + 35\,t^3 + 1428\,t^6 - 204\,t^{13} - 16\,t^{15})Dt/(t^3\,\%1^3)$

$$+627\,t^{12}-141\,t^{14}-2049\,t^{5}+52\,t^{16}+1228\,t^{3}+705\,t^{4}+62\,t^{15}+1569\,t^{7}-582\,t^{8})$$
$$/(t^{3}\,\%1^{3}),-(-39\,t^{3}+155\,t^{10}+41\,t^{13}+5\,t^{4}+396\,t^{9}-92\,t^{12}+186\,t^{5}-206\,t^{11}$$
$$-378\,t^{7}-149\,t^{8}+4\,t^{15}+28\,t^{14}-9\,t^{2}+67\,t^{6})Dt^{2}/(t^{3}\,\%1^{3})-(758\,t^{7}-352\,t^{5}$$
$$-68\,t^{14}-21\,t^{12}-361\,t^{4}+18\,t+108\,t^{2}-874\,t^{9}-995\,t^{8}+590\,t^{11}+549\,t^{10}$$
$$-178\,t^{13}+94\,t^{3}+821\,t^{6}-8\,t^{15})Dt/(t^{3}\,\%1^{3})-(-18+87\,t^{11}-123\,t^{2}-108\,t$$
$$-373\,t^{10}+75\,t^{12}+52\,t^{14}+419\,t^{3}+98\,t^{13}-510\,t^{5}+611\,t^{7}+326\,t^{8}+621\,t^{4}$$
$$-724\,t^{6}+8\,t^{15}-888\,t^{9})/(t^{3}\,\%1^{3}),\;0\Big]$$

$$[-(-732\,t^{7}+1927\,t^{8}+135\,t^{4}+9\,t^{5}+228\,t^{12}+27\,t^{3}+883\,t^{17}-2011\,t^{16}-2898\,t^{15}$$
$$+2500\,t^{9}+1733\,t^{14}-8\,t^{22}-218\,t^{20}+19\,t^{19}-76\,t^{21}-829\,t^{6}-4361\,t^{11}$$
$$-2018\,t^{10}+4588\,t^{13}+1021\,t^{18})Dt^{3}/(t^{5}\,\%1^{4})-(-11477\,t^{7}-108\,t^{2}-2846\,t^{8}$$
$$-23509\,t^{11}+870\,t^{20}+1096\,t^{15}+3493\,t^{10}-1907\,t^{18}+3229\,t^{5}+252\,t^{21}-540\,t^{4}$$
$$+24\,t^{22}+3253\,t^{14}+286\,t^{16}-702\,t^{3}+1735\,t^{6}+10811\,t^{13}-3110\,t^{17}+22558\,t^{9}$$
$$-4429\,t^{12}+697\,t^{19})Dt^{2}/(t^{5}\,\%1^{4})-(-28073\,t^{11}+75403\,t^{10}+1521\,t^{2}+6\,t^{17}$$
$$+17867\,t^{7}-586\,t^{21}+216\,t-9405\,t^{16}-14584\,t^{5}-8263\,t^{4}+11244\,t^{9}+19989\,t^{6}$$
$$+41915\,t^{14}-1445\,t^{20}-8\,t^{23}+2019\,t^{3}+11112\,t^{13}-76065\,t^{12}-42613\,t^{8}$$
$$-108\,t^{22}+2327\,t^{18}-827\,t^{19}+4595\,t^{15})Dt/(t^{5}\,\%1^{4})-(-216+40758\,t^{11}-492\,t^{20}$$
$$-2178\,t^{2}-1521\,t-60447\,t^{13}-61260\,t^{12}+1988\,t^{14}+8\,t^{25}+226\,t^{23}+76\,t^{24}$$
$$+33\,t^{22}-915\,t^{21}+10581\,t^{15}-2815\,t^{17}-6009\,t^{16}+6081\,t^{18}+4040\,t^{19}+7203\,t^{3}$$
$$+20427\,t^{4}-26577\,t^{5}+36151\,t^{7}+65662\,t^{8}-6384\,t^{9}-61816\,t^{6}+35857\,t^{10})/(t^{5}$$
$$\%1^{4}),(6313\,t^{12}-153\,t^{5}-232\,t^{18}+108\,t^{4}+27\,t^{3}-946\,t^{6}+169\,t^{7}+3173\,t^{8}$$
$$+1519\,t^{16}-886\,t^{15}+962\,t^{13}-5807\,t^{10}+161\,t^{9}+488\,t^{17}-16\,t^{20}-128\,t^{19}$$
$$-4139\,t^{14}-667\,t^{11})Dt^{3}/(t^{5}\,\%1^{4})+(939\,t^{6}-36284\,t^{11}+28662\,t^{9}-4636\,t^{14}$$
$$-13375\,t^{7}-4791\,t^{10}+144\,t^{21}+3688\,t^{5}-675\,t^{3}+558\,t^{8}-108\,t^{2}+1882\,t^{16}$$
$$-11448\,t^{15}+6945\,t^{12}+27233\,t^{13}-807\,t^{18}-192\,t^{19}+344\,t^{20}-297\,t^{4}+16\,t^{22}$$
$$+2283\,t^{17})Dt^{2}/(t^{5}\,\%1^{4})+(-12132\,t^{16}+2303\,t^{18}-11134\,t^{15}-29786\,t^{11}$$
$$+24785\,t^{6}-14384\,t^{5}+23636\,t^{7}+350\,t^{9}+1956\,t^{3}-1048\,t^{20}-16\,t^{22}-8752\,t^{4}$$
$$+1521\,t^{2}+67285\,t^{10}+3592\,t^{17}-224\,t^{21}-46228\,t^{8}+40037\,t^{14}-66006\,t^{12}$$
$$-1400\,t^{19}+216\,t+29020\,t^{13})Dt/(t^{5}\,\%1^{4})+(-216+10259\,t^{11}-488\,t^{20}-2178\,t^{2}$$
$$-1521\,t-5425\,t^{15}+9559\,t^{17}-698\,t^{16}+3041\,t^{18}-20692\,t^{13}-32914\,t^{12}$$
$$+5595\,t^{14}+20887\,t^{4}-27917\,t^{5}+50881\,t^{7}+32\,t^{23}+272\,t^{22}+608\,t^{21}-14660\,t^{10}$$
$$+7272\,t^{3}+79325\,t^{8}-24082\,t^{9}-2150\,t^{19}-60244\,t^{6})/(t^{5}\,\%1^{4}),(3530\,t^{13}+243\,t^{4}$$
$$-365\,t^{7}+54\,t^{3}-3076\,t^{11}-117\,t^{5}+4631\,t^{8}+35\,t^{18}+1003\,t^{16}-2619\,t^{15}$$
$$-7370\,t^{10}+1658\,t^{9}+1133\,t^{17}-76\,t^{20}-218\,t^{19}-8\,t^{21}-3809\,t^{14}-1613\,t^{6}$$
$$+6930\,t^{12})Dt^{3}/(t^{5}\,\%1^{4})+(-47238\,t^{11}+45928\,t^{9}-25232\,t^{7}+10705\,t^{10}+24\,t^{21}$$
$$+7781\,t^{5}+4429\,t^{6}-8978\,t^{8}-216\,t^{2}-6398\,t^{16}-1377\,t^{3}-12019\,t^{12}+11649\,t^{14}$$
$$+1135\,t^{18}-5477\,t^{15}+260\,t^{20}+26165\,t^{13}+970\,t^{19}-702\,t^{4}-1652\,t^{17})Dt^{2}/(t^{5}\,\%1^{4})$$
$$+(55322\,t^{7}-3554\,t^{18}+23266\,t^{15}+123168\,t^{10}-28776\,t^{5}-16\,t^{23}$$
$$+14444\,t^{11}-39442\,t^{9}+3434\,t^{16}+102\,t^{20}-17978\,t^{4}+3840\,t^{3}+54386\,t^{6}$$
$$-8670\,t^{17}-152\,t^{22}+432\,t+3042\,t^{2}-82930\,t^{12}+850\,t^{19}-102628\,t^{8}$$
$$+25878\,t^{14}-18646\,t^{13}-404\,t^{21})Dt/(t^{5}\,\%1^{4})+(-432+75227\,t^{11}+137\,t^{20}$$
$$-4356\,t^{2}-3042\,t-46336\,t^{13}-3655\,t^{12}-27819\,t^{14}+60\,t^{23}+8\,t^{24}+114\,t^{22}$$

$-15\,t^{21} - 922\,t^{15} + 4132\,t^{17} + 11464\,t^{16} + 284\,t^{18} + 709\,t^{19} + 14538\,t^{3} + 41734\,t^{4}$
$- 56126\,t^{5} + 109412\,t^{7} + 143223\,t^{8} - 96897\,t^{9} - 116777\,t^{6} - 47608\,t^{10})/(t^{5}\,\%1^{4}),$
$-(-5872\,t^{13} - 162\,t^{4} - 117\,t^{5} + 1246\,t^{7} - 27\,t^{3} + 6085\,t^{11} - 3626\,t^{12} - 2339\,t^{8}$
$- 147\,t^{18} - 395\,t^{16} + 3530\,t^{15} + 3694\,t^{10} - 3789\,t^{9} - 1253\,t^{17} + 76\,t^{20} + 202\,t^{19}$
$+ 8\,t^{21} + 2012\,t^{14} + 901\,t^{6})Dt^{3}/(t^{5}\,\%1^{4}) - (-5636\,t^{6} + 22252\,t^{11} - 19366\,t^{9}$
$+ 10421\,t^{7} - 31030\,t^{10} - 8\,t^{21} - 3499\,t^{5} - 23134\,t^{14} + 17471\,t^{8} + 108\,t^{2} + 9152\,t^{16}$
$+ 729\,t^{3} + 34038\,t^{12} - 13925\,t^{13} - 1495\,t^{18} + 2781\,t^{15} - 132\,t^{20} + 810\,t^{4} - 754\,t^{19}$
$+ 1541\,t^{17})Dt^{2}/(t^{5}\,\%1^{4}) - (935\,t^{18} - 21031\,t^{15} + 53942\,t^{9} - 46217\,t^{10} + 16068\,t^{5}$
$- 38557\,t^{7} - 58017\,t^{11} - 2109\,t^{3} + 504\,t^{16} + 378\,t^{20} + 7198\,t^{4} - 216\,t - 15073\,t^{6}$
$+ 2966\,t^{17} + 8\,t^{22} + 27838\,t^{8} - 1521\,t^{2} + 45758\,t^{12} + 661\,t^{19} - 21291\,t^{14}$
$+ 44390\,t^{13} + 92\,t^{21})Dt/(t^{5}\,\%1^{4}) - (216 - 30604\,t^{11} - 1864\,t^{20} + 2178\,t^{2} + 1521\,t$
$+ 40866\,t^{13} - 33934\,t^{12} + 37701\,t^{14} + 84\,t^{23} + 8\,t^{24} + 246\,t^{22} - 193\,t^{21} - 7757\,t^{15}$
$+ 1605\,t^{17} - 14405\,t^{16} + 4495\,t^{18} - 1098\,t^{19} - 6738\,t^{3} - 17327\,t^{4} + 17185\,t^{5}$
$- 16995\,t^{7} - 76829\,t^{8} + 3216\,t^{9} + 53881\,t^{6} + 44002\,t^{10})/(t^{5}\,\%1^{4}),\ 0]$
$[-(-68\,t^{4} + 206\,t^{11} + 215\,t^{14} + 12\,t^{3} - 378\,t^{12} + 166\,t^{15} - 48\,t^{17} - 14\,t^{7} - 46\,t^{16}$
$+ 221\,t^{6} - 36\,t^{5} - 22\,t^{9} + 9\,t^{2} - 8\,t^{18} - 355\,t^{8} + 399\,t^{10} - 280\,t^{13})Dt^{2}/(t^{4}\,\%1^{3})$
$- (82\,t^{17} - 805\,t^{5} + 1163\,t^{7} + 331\,t^{16} - 413\,t^{14} - 78\,t^{13} - 735\,t^{12} - 18\,t + 4\,t^{15}$
$+ 1069\,t^{6} - 375\,t^{9} - 2914\,t^{8} - 63\,t^{2} - 28\,t^{18} + 2684\,t^{10} - 99\,t^{11} + 119\,t^{3} - 8\,t^{19}$
$+ 3\,t^{4})Dt/(t^{4}\,\%1^{3}) - (18 + 2868\,t^{11} + 24\,t^{20} - 66\,t^{2} + 63\,t + 326\,t^{15} - 203\,t^{17}$
$- 460\,t^{16} - 5\,t^{18} - 1154\,t^{13} - 2088\,t^{12} + 1094\,t^{14} - 197\,t^{5} + 2237\,t^{7} - 667\,t^{8}$
$- 3360\,t^{9} - 1807\,t^{6} + 3557\,t^{10} + 4\,t^{21} + 41\,t^{19} - 432\,t^{3} + 1071\,t^{4})/(t^{4}\,\%1^{3}), (-89\,t^{4}$
$+ 3\,t^{3} - 729\,t^{11} + 29\,t^{5} - 16\,t^{16} - 72\,t^{15} + 28\,t^{14} + 319\,t^{6} - 309\,t^{7} + 719\,t^{9} - 495\,t^{8}$
$- 102\,t^{12} + 350\,t^{13} + 337\,t^{10} + 9\,t^{2})Dt^{2}/(t^{4}\,\%1^{3}) + (4194\,t^{10} - 330\,t^{15} - 790\,t^{5}$
$+ 936\,t^{7} + 16\,t^{18} - 18\,t + 1124\,t^{13} - 2459\,t^{12} + 44\,t^{4} + 140\,t^{3} + 1146\,t^{6} - 52\,t^{16}$
$- 3470\,t^{8} + 72\,t^{17} + 668\,t^{14} + 434\,t^{9} - 63\,t^{2} - 1574\,t^{11})Dt/(t^{4}\,\%1^{3}) + (18$
$+ 2363\,t^{11} - 66\,t^{2} + 63\,t + 24\,t^{19} - 455\,t^{3} + 1167\,t^{4} - 190\,t^{5} - 736\,t^{13} - 4322\,t^{12}$
$+ 2331\,t^{14} - 1898\,t^{6} + 4384\,t^{10} + 116\,t^{18} + 19\,t^{15} - 74\,t^{17} - 845\,t^{16} + 2231\,t^{7}$
$- 722\,t^{8} - 2919\,t^{9})/(t^{4}\,\%1^{3}), (-644\,t^{11} + 182\,t^{14} - 148\,t^{4} + 14\,t^{5} + 271\,t^{13} - 48\,t^{16}$
$+ 15\,t^{3} - 8\,t^{17} - 376\,t^{7} + 760\,t^{9} - 46\,t^{15} + 502\,t^{6} - 364\,t^{12} + 574\,t^{10} + 18\,t^{2} - 720\,t^{8}$
$)Dt^{2}/(t^{4}\,\%1^{3}) + (-1590\,t^{12} - 8\,t^{18} - 1676\,t^{5} + 2265\,t^{7} + 176\,t^{4} + 304\,t^{3} - 587\,t^{13}$
$- 126\,t^{2} + 82\,t^{16} + 204\,t^{14} + 1688\,t^{6} - 603\,t^{9} - 4672\,t^{8} - 28\,t^{17} + 371\,t^{15} + 4166\,t^{10}$
$- 36\,t + 7\,t^{11})Dt/(t^{4}\,\%1^{3}) + (36 + 247\,t^{11} + 4\,t^{20} - 132\,t^{2} + 126\,t - 47\,t^{18} + 8\,t^{19}$
$- 908\,t^{3} + 2382\,t^{4} + 960\,t^{13} + 1014\,t^{12} - 1393\,t^{14} - 4101\,t^{9} - 4506\,t^{6} + 1701\,t^{10}$
$- 579\,t^{5} - 416\,t^{15} - t^{17} + 351\,t^{16} + 4513\,t^{7} + 1419\,t^{8})/(t^{4}\,\%1^{3}), (-276\,t^{11} + 238\,t^{14}$
$- 65\,t^{4} - 98\,t^{5} + 187\,t^{13} - 48\,t^{16} + 21\,t^{3} - 8\,t^{17} + 140\,t^{7} + 62\,t^{9} - 30\,t^{15} + 295\,t^{6}$
$- 630\,t^{12} + 935\,t^{10} + 9\,t^{2} - 741\,t^{8})Dt^{2}/(t^{4}\,\%1^{3}) + (-1576\,t^{12} - 8\,t^{18} - 667\,t^{5}$
$+ 2038\,t^{7} + 181\,t^{4} + 89\,t^{3} - 1557\,t^{13} - 63\,t^{2} + 26\,t^{16} + 394\,t^{14} - 53\,t^{6} - 3380\,t^{9}$
$- 933\,t^{8} - 44\,t^{17} + 479\,t^{15} + 2027\,t^{10} - 18\,t + 2984\,t^{11})Dt/(t^{4}\,\%1^{3}) + (18$
$+ 3810\,t^{11} + 4\,t^{20} - 66\,t^{2} + 63\,t + 17\,t^{18} + 24\,t^{19} - 277\,t^{3} + 660\,t^{4} - 1631\,t^{13}$
$+ 1139\,t^{12} - 172\,t^{14} - 2854\,t^{9} - 2032\,t^{6} - 2479\,t^{10} + 144\,t^{5} + 359\,t^{15} - 113\,t^{17}$
$- 85\,t^{16} + 988\,t^{7} + 2939\,t^{8})/(t^{4}\,\%1^{3}),\ -1]$
$\%1 := 5\,t^{4} - 7\,t^{2} + 3 - 5\,t^{3} + 2\,t^{5} + 5\,t$
$\%2 := 21\,t^{3} - 65\,t^{4} - 317\,t^{9} + 133\,t^{10} - 44\,t^{13} + 171\,t^{11} + 303\,t^{7} - 26\,t^{12} - 143\,t^{5} - 8\,t^{14}$
$+ 9\,t^{2} - 215\,t^{8} + 163\,t^{6}$

```
> Mult(R2, T2, Alg);
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fact that the left *Alg*-module *M2* is free also implies that there exists an injective parametrization. Let us check whether or not the parametrization *Ext2*[3] that we have already computed is injective.

```
> S2 := map(factor, LeftInverseRat(Ext2[3], Alg));
```

$$S2 := \left[0\,,\ \frac{t^6+t^3-t^2+2\,t-1}{t^7\,\%1^2}\,,\ \frac{2\,t^5+2\,t^4-2\,t^3-4\,t^2-t+2}{t^7\,\%1^2}\,,\ \frac{t^3+2\,t^2-t-1}{t^5\,\%1^2}\,,\ 0\,,\ -\frac{(t^2+t-1)\,(t^2+t+1)}{t^7\,\%1^2}\right]$$

$$\%1 := 5\,t^4-7\,t^2+3-5\,t^3+2\,t^5+5\,t$$

```
> Mult(S2, Ext2[3], Alg);
```

$$[\ 1\ ]$$

Therefore, we know that $\xi = S2\ (x1 : x2 : x3 : x4 : x5 : u)^T$ is basis of the free left *Alg*-module *M2* associated with *R2*. More precisely, we have:

```
> xi(t)=ApplyMatrix(S2, [seq(x[i](t),i=1..5), u(t)],Alg)[1,1];
```

$$\xi(t) = \frac{(t^6+t^3-t^2+2\,t-1)\,x_2(t)}{t^7\,\%1^2} + \frac{(2\,t^5+2\,t^4-2\,t^3-4\,t^2-t+2)\,x_3(t)}{t^7\,\%1^2} + \frac{(t^3+2\,t^2-t-1)\,x_4(t)}{t^5\,\%1^2} - \frac{(t^2+t-1)\,(t^2+t+1)\,\mathrm{u}(t)}{t^7\,\%1^2}$$

$$\%1 := 5\,t^4-7\,t^2+3-5\,t^3+2\,t^5+5\,t$$

From the previous expression, we see that the singularities of the flat output are defined by:

```
> Sing := solve(denom(S2[1,2])):  evalf(Sing);
```

$0., 0., 0., 0., 0., 0., 0., 0.9634116577 + 0.3542530975\,I, -0.4436365743, -1.121258736, -2.861928005, 0.9634116577 - 0.3542530975\,I, 0.9634116577 + 0.3542530975\,I, -0.4436365743, -1.121258736, -2.861928005, 0.9634116577 - 0.3542530975\,I$

As the time t is real positive, the only singularity of the flat output is 0 with multiplicity 7.