Baer's extension problem for multidimensional linear systems

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Abstract. Within an algebraic analysis approach, the purpose of this paper is to constructively solve the following problem: given two fixed multidimensional linear systems $\mathcal{B}_1$ and $\mathcal{B}_2$, parametrize the multidimensional linear systems $\mathcal{B}$ which contain $\mathcal{B}_1$ as a subsystem and satisfy that $\mathcal{B}/\mathcal{B}_1$ is isomorphic to $\mathcal{B}_2$. In particular, we parametrize the equivalence classes of multidimensional linear systems $\mathcal{B}$ which admit a fixed parametrizable subsystem $\mathcal{B}_p$ and satisfy that $\mathcal{B}/\mathcal{B}_p$ is isomorphic to a fixed autonomous system $\mathcal{B}_a$.

Keywords. Multidimensional linear systems, behavioural approach, Baer extensions, differential time-delay systems, constructive algebra, module theory.

1 Introduction

A well-known result due to R. E. Kalman states that any time-invariant 1-D linear system defined by a state-space representation can be decomposed into the direct sum of its controllable (i.e., parametrizable) and autonomous subsystems ([11]). Within the behavioural approach, this result was extended by J. C. Willems to time-invariant polynomial linear systems ([16]). Using an algebraic analysis approach, M. Fliess generalized this result in [10] to time-varying linear systems of ordinary differential equations whose coefficients belong to a differential field. However, it is well-known that this result does not admit a generalization for multidimensional linear systems.

In the recent works [20, 21], we constructively characterized when a mul-

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tidimensional linear system decomposed into a direct sum of its parametrizable subsystem and the system formed by its autonomous elements. The corresponding algorithm was implemented in the library OreModules ([6, 7]) and illustrated by different explicit examples. Moreover, we applied these results to the *Monge problem* which questions the existence of parametrizations of the solutions of multidimensional linear systems and to optimal control and variational problems ([20, 21]).

Within an algebraic analysis approach, we constructively solve here the more general problem consisting in parametrizing all the multidimensional linear systems $\mathcal{C}$ whose parametrizable subsystems are isomorphic to a given parametrizable system $\mathcal{B}_p$ and such that $\mathcal{C}/\mathcal{B}_p$ are isomorphic to a given autonomous system $\mathcal{B}_a$, i.e., $\mathcal{C}/\mathcal{B}_p \cong \mathcal{B}_a$. In particular, $\mathcal{B}_p$ (resp., $\mathcal{B}_a$) can be chosen as the parametrizable subsystem (resp., the system formed by the autonomous elements) of a multidimensional linear system $\mathcal{B}$. Solving this last problem allows us to parametrize all the multidimensional linear systems which have the same parametrizable subsystem and autonomous system as $\mathcal{B}$. We then show how that result allows us to find again those obtained in [20, 21]. Our results are based on the important concept of *Baer extensions* developed in homological algebra and its connections with the extension abelian group $\mathrm{ext}^1_D(M, N)$ ([5, 12, 23]). This problem was pointed out to us by S. Shankar (Chennai Mathematical Institute) ([24]). We would like to thank him.

The plan of the paper is the following one: In Section 2, we recall Baer's interpretation of the elements of the abelian group $\mathrm{ext}^1_D(M, N)$ in terms of equivalence classes of extensions of N by M. In Section 3, we explicitly characterize $\mathrm{ext}^1_D(M, N)$ as an abelian group, which allows us in Section 4 to parametrize the equivalence classes of multidimensional linear systems $\mathcal{B}$ which admit as a subsystem the system $\mathcal{B}_1$ defined by M and satisfy that $\mathcal{B}/\mathcal{B}_1$ are isomorphic to the system $\mathcal{B}_2$ defined by N. In Section 5, the previous results are applied to the particular situation where $N = t(P)$ is the torsion left D-submodule of a given finitely presented left D-module P and $M = P/t(P)$. We finally explain how to find again the results of [20, 21].

In what follows, we refer to [6, 13, 18, 25] and the references therein for the concepts relevant to the module-theoretic approach to systems theory.

2 Baer extensions and Baer sums

We refer to [5, 12, 23] for the classical definitions of an exact sequence and a complex.

Let us first introduce the concept of *Baer extensions* which will play an important role in what follows.

Definition 1 ([5, 12, 23]). We have the following definitions:

1. Let M and N be two left D-modules. An *extension of N by M* is an exact sequence e of left D-modules of the form:

$$e: \; 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0. \tag{1}$$

2. Two extensions of N by M, $e_i : 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$, $i = 1, 2$, are said to be *equivalent*, denoted by $e_1 \sim e_2$, if there exists a D-isomorphism $\phi : E_1 \longrightarrow E_2$ such that we have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & N & \xrightarrow{f_1} & E_1 & \xrightarrow{g_1} & M & \longrightarrow 0 \\ & \| & & \downarrow \phi & & \| & \\ 0 \longrightarrow & N & \xrightarrow{f_2} & E_2 & \xrightarrow{g_2} & M & \longrightarrow 0, \end{array}$$

or, equivalently, such that $f_2 = \phi \circ f_1$ and $g_1 = g_2 \circ \phi$ hold.

3. We denote by $[e]$ the equivalence class of the extension e for the equivalence relation $\sim$. The set of all equivalence classes of extensions of N by M is denoted by $\mathrm{e}_D(M, N)$.

4. A short exact sequence of the form (1) is said to *split* if $E \cong M \oplus N$, where $\oplus$ (resp., $\cong$) denotes the direct sum (resp., that two modules are isomorphic).

Let us introduce the concept of *Baer sum* of two extensions ([5, 12, 23]).

Definition 2 ([5]). Let $e_i : 0 \longrightarrow N \xrightarrow{f_i} E_i \xrightarrow{g_i} M \longrightarrow 0$, $i = 1, 2$, be two extensions of N by M and let us define the following two D-morphisms:

$$\begin{array}{rcl} -f_1 \oplus f_2 : N & \longrightarrow & E_1 \oplus E_2 \\ n & \longmapsto & (-f_1(n), f_2(n)) \end{array} \qquad \begin{array}{rcl} (g_1, -g_2) : E_1 \oplus E_2 & \longrightarrow & M \\ (a_1, a_2) & \longmapsto & g_1(a_1) - g_2(a_2). \end{array}$$

Then, the *Baer sum* of the extensions e_1 and e_2, denoted by $e_1 + e_2$, is defined by the left D-module $E_3 = \ker(g_1, -g_2)/\mathrm{im}\,(-f_1 \oplus f_2)$, i.e., by the short exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & N & \xrightarrow{f_3} & E_3 & \xrightarrow{g_3} & M & \longrightarrow 0, \\ & n & \longmapsto & \varpi(f_1(n), 0) = \varpi(0, f_2(n)) & & & \\ & & & \varpi(a_1, a_2) & \longmapsto & g_1(a_1) = g_2(a_2) & \end{array}$$

where $\varpi : \ker(g_1, -g_2) \longrightarrow E_3$ denotes the canonical projection onto E_3.

We have the following classical but important result on extensions.

Theorem 3 ([5, 12, 23]). The set $\mathrm{e}_D(M, N)$ equipped with the *Baer sum* forms an abelian group: the equivalence class of the split short exact sequence

$$0 \longrightarrow N \xrightarrow{i_2} M \oplus N \xrightarrow{p_1} M \longrightarrow 0$$

defines the zero element of $\mathrm{e}_D(M, N)$ and the inverse of the equivalence class $[e]$ of (1) is defined by the equivalence class of the following two equivalent extensions:

$$0 \longrightarrow N \xrightarrow{-f} E \xrightarrow{g} M \longrightarrow 0, \qquad 0 \longrightarrow N \xrightarrow{f} E \xrightarrow{-g} M \longrightarrow 0.$$

3 Computing extensions of finitely presented modules

In this section, we show how to compute the abelian group $\mathrm{ext}^1_D(M,N)$, when M and N are two finitely generated left D-modules over a *noetherian domain* D ([23]).

By assumption, the left D-module M admits the *finite free resolution*

$$\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0, \tag{2}$$

namely, (2) is an exact sequence of left D-modules where $R_i \in D^{p_i\times p_{i-1}}$ and $(.R_i)(\lambda) = \lambda\, R_i$, for all $\lambda \in D^{1\times p_i}$. Applying the *contravariant left exact functor* $\hom_D(\,\cdot\,,N)$ to the complex $\ldots \xrightarrow{.R_3} D^{1\times p_2} \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \longrightarrow 0$, we obtain the following complex of abelian groups

$$\ldots \xleftarrow{R_3.} N^{p_2} \xleftarrow{R_2.} N^{p_1} \xleftarrow{R_1.} N^{p_0} \longleftarrow 0, \tag{3}$$

where $(R_i.)(\eta) = R_i\, \eta$, for all $\eta \in N^{p_{i-1}}$. For more details, see, e.g., [5, 12, 19, 23].

Applying the *covariant right exact functor* $D^m \otimes_D \cdot$ to the *finite presentation* (i.e., to the exact sequence) $D^{1\times t} \xrightarrow{.S} D^{1\times s} \xrightarrow{\delta} N \longrightarrow 0$ of the left D-module N, and using the fact that D^m is a *free* right D-module, and thus, a *flat* right D-module, we obtain the following exact sequence:

$$D^{m\times t} \xrightarrow{.S} D^{m\times s} \xrightarrow{\mathrm{id}_m\otimes\delta} N^m \longrightarrow 0. \tag{4}$$

For more details, see, e.g., [5, 12, 19, 23].

Using the notations $p = p_0$, $q = p_1$, $r = p_2$, $R = R_1$ and combining (3) and (4), we obtain the following commutative diagram of abelian groups with exact columns:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\uparrow & & \uparrow & & \uparrow \\
N^r & \xleftarrow{R_2.} & N^q & \xleftarrow{R.} & N^p \\
\uparrow {\scriptstyle \mathrm{id}_r\otimes\delta} & & \uparrow {\scriptstyle \mathrm{id}_q\otimes\delta} & & \uparrow {\scriptstyle \mathrm{id}_p\otimes\delta} \\
D^{r\times s} & \xleftarrow{R_2.} & D^{q\times s} & \xleftarrow{R.} & D^{p\times s} \\
\uparrow {\scriptstyle .S} & & \uparrow {\scriptstyle .S} & & \uparrow {\scriptstyle .S} \\
D^{r\times t} & \xleftarrow{R_2.} & D^{q\times t} & \xleftarrow{R.} & D^{p\times t}.
\end{array} \tag{5}$$

Let us now introduce the abelian group $\mathrm{ext}^1_D(M,N) = \ker_N(R_2.)/\mathrm{im}_N(R.)$, where:

$$\ker_N(R_2.) = \{\eta \in N^q \mid R_2\, \eta = 0\} = \{\eta = (\mathrm{id}_q \otimes \delta)(A) \mid A \in D^{q\times s} :\ R_2\, \eta = 0\},$$
$$\mathrm{im}_N(R.) = R\, N^p = \{\eta = (\mathrm{id}_q \otimes \delta)(A) \mid \exists\, B \in D^{p\times s} :\ \eta = R\,((\mathrm{id}_p \otimes \delta)(B))\}.$$

From (5), we get $(R_2.)\circ(\mathrm{id}_q\otimes\delta) = (\mathrm{id}_r\otimes\delta)\circ(R_2.)$ and $(R.)\circ(\mathrm{id}_p\otimes\delta) = (\mathrm{id}_q\otimes\delta)\circ(R.)$. Hence, using the exactness of the columns of (5), we obtain:

$$R_2((\mathrm{id}_q \otimes \delta)(A)) = (\mathrm{id}_r \otimes \delta)(R_2\, A) = 0 \;\Leftrightarrow\; \exists\, B \in D^{r\times t} :\ R_2\, A = B\, S.$$

$$(\mathrm{id}_q \otimes \delta)(A) = R\,((\mathrm{id}_p \otimes \delta)(X)) = (\mathrm{id}_q \otimes \delta)(R\,X)$$
$$\Leftrightarrow\ (\mathrm{id}_q \otimes \delta)(A - R\,X) = 0\ \Leftrightarrow\ \exists\, Y \in D^{q\times t}:\ A = R\,X + Y\,S.$$

Hence, we obtain the following results.

Lemma 4. *With the previous notations, we have:*

$$\begin{aligned} \ker_N(R_2.) &= \{(\mathrm{id}_q \otimes \delta)(A) \mid A \in D^{q\times s},\ \exists\, B \in D^{r\times t}:\ R_2\,A = B\,S\}, && (6)\\ \mathrm{im}_N(R.) &= \{(\mathrm{id}_q \otimes \delta)(A) \mid \exists\, X \in D^{p\times s},\ \exists\, Y \in D^{q\times t}:\ A = R\,X + Y\,S\} \\ &= (R\,D^{p\times s} + D^{q\times t}\,S)/(D^{q\times t}\,S). && (7)\end{aligned}$$

Moreover, if we define the abelian group

$$\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t}:\ R_2\,A = B\,S\}, \tag{8}$$

then we have the following isomorphism of abelian groups

$$\begin{array}{rcl} \mathrm{ext}^1_D(M,N) = \ker_N(R_2.)/\mathrm{im}_N(R.) & \xrightarrow{\ \iota\ } & \Omega/(R\,D^{p\times s} + D^{q\times t}\,S), \\ \rho((\mathrm{id}_q \otimes \delta)(A)) & \longmapsto & \varepsilon(A), \end{array} \tag{9}$$

where $\rho : \ker_N(R_2.) \longrightarrow \mathrm{ext}^1_D(M,N)$ *(resp.,* $\varepsilon : \Omega \longrightarrow \Omega/(R\,D^{p\times s} + D^{q\times t}\,S)$*) denotes the canonical projection onto* $\mathrm{ext}^1_D(M,N)$ *(resp.,* $\Omega/(R\,D^{p\times s} + D^{q\times t}\,S)$*).*

We let the reader check that ι is well-defined and bijective ([22]).

We recall that the abelian group $\mathrm{ext}^1_D(M,N)$ characterizes the obstructions for the existence of $\xi \in N^q$ satisfying the inhomogeneous linear system $R\,\xi = \zeta$, where $\zeta \in N^p$ satisfies the compatibility condition $R_2\,\zeta = 0$. In particular, the vanishing of $\mathrm{ext}^1_D(M,N)$ implies that $R_2\,\zeta = 0$ is a necessary and sufficient condition for the existence of $\xi \in N^q$ satisfying $R\,\xi = \zeta$. For more details, see [6, 7, 18, 19].

If $\ker_D(.R) = 0$, i.e., $R_2 = 0$, we then get $\Omega = D^{q\times s}$. Another simple case is $N = D^{1\times s}$, i.e., $S = 0$, for which we have $\Omega = \{A \in D^{q\times s} \mid R_2\,A = 0\}$ (see [4]).

If D is a commutative ring and $\otimes$ denotes the *Kronecker product*, then using the identity $U\,V\,W = \mathrm{row}(V)\,(U^T \otimes W)$, where $\mathrm{row}(V)$ is obtained by concatenating the rows of V, we have $\Omega/(R\,D^{p\times s} + D^{q\times t}\,S) \cong D^{1\times u}\,Z/(D^{1\times(p\,s+q\,t)}\,X)$, where

$$X = \begin{pmatrix} R^T \otimes I_s \\ I_q \otimes S \end{pmatrix} \in D^{(p\,s+q\,t)\times q\,s}, \quad Y = \begin{pmatrix} R_2^T \otimes I_s \\ I_r \otimes S \end{pmatrix} \in D^{(q\,s+r\,t)\times r\,s},$$

and $Z \in D^{u\times q\,s}$ is defined by $\ker_D(.Y) = D^{1\times u}\,(Z \quad -T)$ and $T \in D^{u\times r\,t}$. Moreover, if D is a polynomial ring over a computable field k (e.g., $k = \mathbb{Q}$, $\mathbb{F}_p$), then, using Gröbner or Janet bases, we can explicitly describe the D-module $\mathrm{ext}^1_D(M,N)$ by means of generators and relations ([3, 8]). For the implementations of the corresponding algorithms, see the packages `homalg` ([2, 3]) and OreMorphisms ([9]).

Example 5. Let us consider the commutative polynomial ring $D = \mathbb{Q}(\alpha)\,[\partial, \delta]$ of differential time-delay operators, where $\alpha \in \mathbb{R}$, and the following two matrices:

$$R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1+\delta^2 & -\alpha\,\partial\,\delta \end{pmatrix} \in D^{2\times 3}, \quad S = \begin{pmatrix} \partial & -\partial \\ \partial\,\delta^2 & -\partial \end{pmatrix} \in D^{2\times 2}. \tag{10}$$

Let us define the D-modules $M = D^{1\times 3}/(D^{1\times 2}\, R)$ and $N = D^{1\times 2}/(D^{1\times 2}\, S)$. We have $R_2 = 0$, and thus, $\Omega = D^{2\times 2}$, $\mathrm{ext}^1_D(M, N) \cong D^{2\times 2}/(R\, D^{3\times 2} + D^{2\times 2}\, S)$ and:

$$\mathrm{ext}^1_D(M, N) \cong D^{1\times 4}/\left(D^{1\times 10}\left(\begin{array}{c} R^T \otimes I_2 \\ I_2 \otimes S \end{array}\right)\right). \tag{11}$$

We denote by L the matrix appearing in (11) and $\epsilon : D^{1\times 4} \longrightarrow P = D^{1\times 4}/(D^{1\times 10}\, L)$ the canonical projection onto P. Denoting by $v_i = \epsilon(g_i)$ the residue class in P of the i^{th} vector of the standard basis $\{g_i\}_{1\le i\le 4}$ of $D^{1\times 4}$, we obtain:

$$v_i = 0, \quad i = 1,\, 2, \quad (1+\delta^2)\, v_i = 0, \quad i = 3,\, 4, \quad \partial\, v_i = 0, \quad i = 3,\, 4.$$

Hence, the D-module P is generated by $v_3 = \epsilon((0, 0, 1, 0))$ and $v_4 = \epsilon((0, 0, 0, 1))$. Transforming back the row vectors g_3 and g_4 into 2×2 matrices, we obtain that the D-module $D^{2\times 2}/(R\, D^{3\times 2} + D^{2\times 2}\, S)$ is generated by $\varepsilon(A_1)$ and $\varepsilon(A_2)$, where:

$$A_1 = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right), \quad A_2 = \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array}\right). \tag{12}$$

It is a torsion D-module as we have $(1+\delta^2)\, \varepsilon(A_i) = 0$ and $\partial\, \varepsilon(A_i) = 0$, $i = 1, 2$. Using (9), we obtain that the $\rho((\mathrm{id}_2 \otimes \delta)(A_i))$'s generate the D-module $\mathrm{ext}^1_D(M, N) = N^2/(R\, N^3)$ and satisfy $(1+\delta^2)\, \rho((\mathrm{id}_2 \otimes \delta)(A_i)) = 0$, $\partial\, \rho((\mathrm{id}_2 \otimes \delta)(A_i)) = 0$, $i = 1, 2$.

If D is a non-commutative ring, then $\mathrm{ext}^1_D(M, N)$ is an abelian group, but not a left D-module. If D is a k-algebra, where k is a field contained in the center of D, then $\mathrm{ext}^1_D(M, N)$ is a k-vector space. If M and N are two finite-dimensional k-vector spaces or two *holonomic* left modules over the k-algebra of differential operators with k-polynomial (resp., k-rational) coefficients (the so-called *Weyl algebras* $A_n(k)$ and $B_n(k)$), then we can compute a k-basis of $\mathrm{ext}^1_D(M, N)$ (see [8] and the references therein). However, $\mathrm{ext}^1_D(M, N)$ is generally an infinite-dimensional k-vector space. If D is a non-commutative polynomial ring over which Gröbner or Janet bases exist (e.g., the Weyl algebras, certain classes of *Ore algebras* [6]), then we can compute the k-vector space formed by the matrices $A \in D^{q\times s}$ with a fixed order in the functional operators and a fixed degree (resp., fixed degrees) in the polynomial (resp., rational) coefficients which satisfy $R_2\, A \in D^{r\times t}\, S$. See [8] for more details and the package OreMorphisms ([9]) for an implementation.

4 An explicit description of $\mathrm{ext}^1_D(M, N)$

The following theorem is an important result in homological algebra which can be traced back to the pioneering work of R. Baer ([1]).

Theorem 6 ([5, 12, 23]). *Let M and N be two left D-modules. Then, the abelian groups $\mathrm{ext}^1_D(M, N)$ and $\mathrm{e}_D(M, N)$ are isomorphic.*

The explicit description of $\mathrm{ext}^1_D(M, N)$ − being proved by making Theorem 6 constructive for the interesting class of modules in systems theory − can be given now. For the sake of brevity, we refer to [22, Theorem 3] for the proof.

Theorem 7. *Let $R \in D^{q\times p}$ and $S \in D^{t\times s}$ be two matrices with entries in D and $M = D^{1\times p}/(D^{1\times q}\, R)$ and $N = D^{1\times s}/(D^{1\times t}\, S)$ the left D-modules finitely presented by R resp. S. Let us denote by $R_2 \in D^{r\times q}$ a matrix satisfying $\ker_D(.R) = D^{1\times r}\, R_2$. Then, every equivalence class of extensions of N by M is represented by*

$$e:\, 0 \longrightarrow N \stackrel{\alpha}{\longrightarrow} E \stackrel{\beta}{\longrightarrow} M \longrightarrow 0, \tag{13}$$

where the left D-module E is defined by

$$D^{1\times(q+t)} \stackrel{.Q}{\longrightarrow} D^{1\times(p+s)} \stackrel{\varrho}{\longrightarrow} E \longrightarrow 0, \quad Q = \left(\begin{array}{cc} R & -T \\ 0 & S \end{array}\right) \in D^{(q+t)\times(p+s)}, \tag{14}$$

and T is a certain element of $\Omega = \{A \in D^{q\times s} \mid \exists\, B \in D^{r\times t} : R_2\, A = B\, S\}$.

Finally, the equivalence class $[e]$ only depends on the residue class $\varepsilon(T)$ of $T \in \Omega$ in $\Omega/(R\, D^{p\times s} + D^{q\times t}\, S) = \iota^{-1}(\mathrm{ext}^1_D(M,N))$, where ι is defined in (9).

Example 8. Let us consider again Example 5. Theorem 7 says there exist two non-trivial equivalence classes of extensions of N by M respectively defined by $E_i = D^{1\times 5}/\left(D^{1\times 4}\left(\begin{array}{cc} R & -T_i \\ 0 & S \end{array}\right)\right)$, where the matrices R and S are given by (10) and the matrices $T_1 = A_1$ and $T_2 = A_2$ by (12). Finally, the trivial extension of N by M (i.e., the split extension) is defined by the D-module E_0 where $T_0 = 0$.

Let $\mathcal{F}$ be a left D-module. Applying the contravariant left exact functor $\hom_D(\cdot\,,\mathcal{F})$ to (13), we obtain the following results [22, Corollary 1].

Corollary 9. *With the previous notations, we have the following results:*

1. $\ker_{\mathcal{F}}(S.) \stackrel{\alpha^\star}{\longleftarrow} \ker_{\mathcal{F}}(Q.) \stackrel{\beta^\star}{\longleftarrow} \ker_{\mathcal{F}}(R.) \longleftarrow 0$ *is an exact sequence, where the D-morphism $\beta^\star$ (resp., $\alpha^\star$) is defined by $\beta^\star(\xi) = (\xi^T \;\; 0^T)^T$, for all $\xi \in \ker_{\mathcal{F}}(R.)$ (resp., $\alpha^\star(\eta) = \eta_2$, for all $\eta = (\eta_1^T \;\; \eta_2^T)^T$, $\eta_1 \in \mathcal{F}^p$ and $\eta_2 \in \mathcal{F}^s$).*

2. *If $\mathcal{F}$ is an injective left D-module ([23]), then we have the exact sequence:*

$$0 \longleftarrow \ker_{\mathcal{F}}(S.) \stackrel{\alpha^\star}{\longleftarrow} \ker_{\mathcal{F}}(Q.) \stackrel{\beta^\star}{\longleftarrow} \ker_{\mathcal{F}}(R.) \longleftarrow 0. \tag{15}$$

Moreover, if $\mathcal{F}$ is cogenerator ([23]), then (15) is exact if and only if (13) is.

5 Applications to multidimensional systems theory

The purpose of this section is to parametrize all equivalence classes of multidimensional linear systems which have fixed parametrizable subsystem and autonomous system. Let $R \in D^{q\times p}$ be a matrix with entries in a noetherian domain D. If $M = D^{1\times p}/(D^{1\times q}\, R)$ denotes the left D-module finitely presented by R, then $t(M) = \{m \in M \mid \exists\; 0 \neq a \in D :\;\; a\, m = 0\}$ is a left D-submodule of M and we have the following canonical short exact sequence (see, e.g., [5, 12, 23]):

$$0 \longrightarrow t(M) \stackrel{\iota}{\longrightarrow} M \stackrel{\tau}{\longrightarrow} M/t(M) \longrightarrow 0. \tag{16}$$

An element of $t(M)$ is called a *torsion element* of M and M is said to be *torsion-free* if $t(M) = 0$ and *torsion* if $t(M) = M$ (see, e.g., [23]). Constructive results developed in [6, 7, 18] show that there exists a matrix $R' \in D^{q' \times p}$ satisfying:

$$t(M) = (D^{1\times q'} R')/(D^{1\times q} R), \quad M/t(M) = D^{1\times p}/(D^{1\times q'} R').$$

If $\mathcal{F}$ is an injective left D-module, applying the exact functor $\hom_D(\,\cdot\,, \mathcal{F})$ to the exact sequence (16), we then get the exact sequence of abelian groups:

$$0 \longleftarrow \hom_D(t(M), \mathcal{F}) \xleftarrow{\iota^\star} \hom_D(M, \mathcal{F}) \xleftarrow{\tau^\star} \hom_D(M/t(M), \mathcal{F}) \longleftarrow 0.$$

The linear system $\ker_{\mathcal{F}}(R'.) = \{\zeta \in \mathcal{F}^p \mid R'\,\zeta = 0\} \cong \hom_D(M/t(M), \mathcal{F})$ is the *parametrizable subsystem* of $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F})$ as there always exists a matrix $Q' \in D^{p\times m}$ such that $\ker_{\mathcal{F}}(R'.) = Q'\,\mathcal{F}^m$, i.e., any solution $\eta \in \mathcal{F}^p$ of the system $R'\,\eta = 0$ has the form $\eta = Q'\,\xi$ for a certain $\xi \in \mathcal{F}^m$. For more details, see [6, 15, 18, 25]. For certain classes of multidimensional systems, $\ker_{\mathcal{F}}(R'.)$ is also called the *controllable subsystem* of $\ker_{\mathcal{F}}(R.)$ (see, e.g., [6, 15, 17, 25]).

If we denote by $R'' \in D^{q\times q'}$ (resp., $R_2' \in D^{r'\times q'}$) a matrix satisfying $R = R''\,R'$ (resp., $\ker_D(.R') = D^{1\times r'}\,R_2'$), then we have the following D-isomorphism ([8, 21]):

$$t(M) \cong D^{1\times q'} / \left(D^{1\times (q+r')} \begin{pmatrix} R'' \\ R_2' \end{pmatrix} \right). \tag{17}$$

The *autonomous system* defined by $\ker_{\mathcal{F}}((R''^T\ R_2'^T)^T.) \cong \hom_D(t(M), \mathcal{F})$ satisfies:

$$\ker_{\mathcal{F}}((R''^T\ R_2'^T)^T.) \cong \ker_{\mathcal{F}}(R.)/\tau^\star(\ker_{\mathcal{F}}(R'.)).$$

This last system will be called the *autonomous quotient* of the system $\ker_{\mathcal{F}}(R.)$.

If M and N are respectively a torsion-free and a torsion left D-module defined by two finite presentations, Theorem 7 parametrizes the equivalence classes of extensions of N by M. Moreover, if $\mathcal{F}$ is an injective left D-module, by Corollary 9, we then obtain the equivalence classes of systems admitting $\hom_D(M, \mathcal{F})$ as a parametrizable subsystem and $\hom_D(N, \mathcal{F})$ as autonomous quotient. If we consider the left D-module $P = M \oplus N$, we then have $t(P) \cong N$ and $P/t(P) \cong M$ and the previous problem can be reduced to the case where we only consider the extensions of $t(P)$ by $P/t(P)$ for a finitely presented left D-module P.

Let $L \in D^{m\times l}$ be a matrix with entries in a noetherian domain D and let us consider the finitely presented left D-module $P = D^{1\times l}/(D^{1\times m}\,L)$. As shown in [6, 18] and implemented in [7], computing the left D-module $\mathrm{ext}^1_D(N, D)$, where $N = D^m/(L\,D^l)$, gives us a matrix $L' \in D^{m'\times l}$ satisfying:

$$\begin{cases} t(P) = (D^{1\times m'}\,L')/(D^{1\times m}\,L), \\ P/t(P) = D^{1\times l}/(D^{1\times m'}\,L'). \end{cases} \tag{18}$$

We denote by $\epsilon : D^{1\times m} \longrightarrow P$ (resp., $\epsilon' : D^{1\times m} \longrightarrow P/t(P)$) the canonical projection onto P (resp., $P/t(P)$). In particular, we have the relation $\epsilon' = \tau \circ \epsilon$, where τ denotes the canonical projection $P \longrightarrow P/t(P)$ (see (16) with $M = P$).

Corollary 10. *Every class of extensions of $t(P)$ by $P/t(P)$ is defined by means of the left D-module $E = D^{1\times(l+m')}/(D^{1\times(m'+m+n')}\,Q)$, where the matrix Q has the form*

$$Q = \begin{pmatrix} L' & -T \\ 0 & L'' \\ 0 & L'_2 \end{pmatrix} \in D^{(m'+m+n')\times(l+m')} \tag{19}$$

(with L'' (resp., L'_2) playing the role of R'' (resp., R'_2) in (17)) and T is an element of the abelian group:

$$\Omega = \left\{ A \in D^{m'\times m'} \mid \exists\, B \in D^{n'\times(m+n')} :\; L'_2\, A = B \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right\}. \tag{20}$$

Finally, the equivalence classes of the extensions of $t(P)$ by $P/t(P)$ only depend on the residue classes $\varepsilon(T)$ in the abelian group (with ι as defined in (9)):

$$\Omega/\left(L'\, D^{l\times m'} + D^{m'\times(m+n')} \begin{pmatrix} L'' \\ L'_2 \end{pmatrix} \right) = \iota^{-1}(\mathrm{ext}^1_D(P/t(P), t(P))). \tag{21}$$

If $\mathcal{F}$ is an injective left D-module and $\ker_{\mathcal{F}}(L.) \cong \hom_D(P,\mathcal{F})$, then Corollaries 9 and 10 give a parametrization of the equivalence classes of linear systems $\ker_{\mathcal{F}}(Q.) \cong \hom_D(E,\mathcal{F})$ which admit $\ker_{\mathcal{F}}(L'.)$ as a parametrizable subsystem and $\ker_{\mathcal{F}}((L''^T\; L_2'^T)^T.)$ as an autonomous quotient.

Example 11. Let us consider the differential time-delay system ([14])

$$\begin{cases} \dot y_1(t) - \dot y_2(t-2\,h) + \alpha\, \ddot y_3(t-h) = 0, \\ \dot y_1(t-2\,h) - \dot y_2(t) + \alpha\, \ddot y_3(t-h) = 0, \end{cases} \tag{22}$$

where $\alpha \in \mathbb{R}$ and h is a strictly positive real number. We denote by $D = \mathbb{Q}(\alpha)\,[\partial, \delta]$ the commutative polynomial ring of differential time-delay operators, the matrix

$$L = \begin{pmatrix} \partial & -\partial\,\delta^2 & \alpha\,\partial^2\,\delta \\ \partial\,\delta^2 & -\partial & \alpha\,\partial^2\,\delta \end{pmatrix} \in D^{2\times 3},$$

and the D-module $P = D^{1\times 3}/(D^{1\times 2}\,L)$. Using a constructive algorithm developed in [6, 19] and implemented in [7], we get $L' = R \in D^{2\times 3}$ defined by (10). We can check that $\ker_D(.L') = 0$ and $L = L''\,L'$, where $L'' = S \in D^{2\times 2}$ is defined by (10). Hence, we obtain $t(P) \cong D^{1\times 2}/(D^{1\times 2}\,L'')$. Now, the equivalence classes of extensions of $t(P)$ by $P/t(P)$ are in 1-1 correspondence with the elements of the D-module $\mathrm{ext}^1_D(P/t(P), t(P))$. Using Examples 5 and 8, we obtain that the two non-trivial equivalence classes of extensions are defined by the D-modules E_1 and E_2 given in Example 8. They respectively correspond to the following systems:

$$\begin{cases} z_1(t) + z_2(t) = 0, \\ z_2(t) + z_2(t-2\,h) \\ \qquad -(\alpha\,\dot z_3(t-h) + z_4(t)) = 0, \\ \dot z_4(t) - \dot z_5(t) = 0, \\ \dot z_4(t-2\,h) - \dot z_5(t) = 0, \end{cases} \qquad \begin{cases} z_1(t) + z_2(t) = 0, \\ z_2(t) + z_2(t-2\,h) \\ \qquad -(\alpha\,\dot z_3(t-h) + z_5(t)) = 0, \\ \dot z_4(t) - \dot z_5(t) = 0, \\ \dot z_4(t-2\,h) - \dot z_5(t) = 0. \end{cases}$$

The trivial class of extensions of $t(P)$ by $P/t(P)$ can be defined by the system:

$$\begin{cases} z_1(t) + z_2(t) = 0, \\ z_2(t) + z_2(t - 2\,h) - \alpha\,\dot{z}_3(t - h) = 0, \\ \dot{z}_4(t) - \dot{z}_5(t) = 0, \\ \dot{z}_4(t - 2\,h) - \dot{z}_5(t) = 0. \end{cases}$$

Hence, the three systems admit the same parametrizable subsystem and the same autonomous quotient as (22).

Remark 12. The matrix Q defined by (19) with $T = I_{m'}$ was used in [20, 21] to parametrize the $\mathcal{F}$-solutions of the system $\ker_{\mathcal{F}}(L.)$ in terms of the $\mathcal{F}$-solutions of $\ker_{\mathcal{F}}(L'.)$ and $\ker_{\mathcal{F}}((L''^T \; L_2'^T)^T.)$. We first need to solve the following autonomous homogeneous linear system $\ker_{\mathcal{F}}((L''^T \; L_2'^T)^T.)$ corresponding to $\hom_D(t(P), \mathcal{F})$:

$$\begin{cases} L''\,\theta = 0, \\ L_2'\,\theta = 0. \end{cases} \tag{23}$$

Then, we need to solve the inhomogeneous system $L'\,\eta = \theta$, i.e., find a particular solution $\eta^\star \in \mathcal{F}^l$ of $L'\,\eta^\star = \theta$ and the general solution of the homogeneous system $L'\,\eta = 0$ associated with $\hom_D(P/t(P), \mathcal{F})$. As the subsystem $\hom_D(P/t(P), \mathcal{F})$ of $\hom_D(P, \mathcal{F})$ is parametrizable, we can compute a matrix $Q' \in D^{l \times k'}$ satisfying $\ker_{\mathcal{F}}(L'.) = Q'\,\mathcal{F}^{k'}$ whenever $\mathcal{F}$ is an injective left D-module ([6, 15, 19, 25]). Then, the solution of $L\,\eta = 0$ has the form $\eta = \eta^\star + Q'\,\xi$, for arbitrary $\xi \in \mathcal{F}^{k'}$. We refer to [21] for applications to variational and optimal control problems.

Next, we have a direct consequence of Remark 12. For more details, see [22].

Proposition 13. *The exact sequence* $0 \longrightarrow t(P) \xrightarrow{\iota} P \xrightarrow{\tau} P/t(P) \longrightarrow 0$ *splits iff* $\varepsilon(I_{m'}) = 0$, *i.e., iff there exist* $X \in D^{l \times m'}$, $Y \in D^{m' \times m}$ *and* $Z \in D^{m' \times n'}$ *satisfying:*

$$I_{m'} = L'\,X + Y\,L'' + Z\,L_2' \quad \Leftrightarrow \quad L' - L'\,X\,L' = Y\,L. \tag{24}$$

Remark 14. As shown in [20, 21], Proposition 13 gives a particular solution $\eta^\star \in \mathcal{F}^l$ of the inhomogeneous system $L'\,\eta = \theta$, where $\theta \in \mathcal{F}^{q'}$ is a general solution of the system (23): using (24), we get $\theta = L'\,X\,\theta + Y\,L''\,\theta + Z'\,L_2'\,\theta = L'\,(X\,\theta)$ as θ satisfies (23). If $\mathcal{F}$ is an injective left D-module, using Remark 12, we then obtain that the elements of $\ker_{\mathcal{F}}(L.)$ have the form $\eta = X\,\theta + Q'\,\xi$, for all $\xi \in \mathcal{F}^{k'}$.

The left D-module $P/t(P) = D^{1 \times l}/(D^{1 \times m'}\,L')$ is *stably free*, i.e., satisfies $P/t(P) \oplus D^{1 \times s} \cong D^{1 \times r}$ for non-negative integers r and s ([23]), iff there exists $X \in D^{l \times m'}$ such that $L'\,X\,L' = L'$ ([17]). Hence, if $P/t(P)$ is stably free, the equivalent statements of Proposition 13 hold. In particular, if $D = k[t]\,[\partial]$ is the *Weyl algebra* (k a field of characteristic 0) or a *left principal ideal domain* (e.g., $K\,[\partial]$, K a differential field), then every torsion-free left D-module is stably free and, in particular, $P/t(P)$ for any left D-module P. Hence, we find again Kalman's result ([11]) and its different generalizations ([10, 16]) described in the introduction.

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