Flat multidimensional linear systems with constant coefficients are equivalent to controllable 1-D linear systems

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***Abstract*— Based on constructive proofs of the Quillen-Suslin theorem, the purpose of this paper is to show that every flat multidimensional linear system with constant coefficients is equivalent to a controllable 1-D linear system. This result looks like the classical result in non-linear control theory stating that every flat ordinary differential non-linear system is equivalent to a controllable ordinary differential linear system. In particular, we prove that every flat differential time-delay linear system is equivalent to the ordinary differential controllable linear system obtained by setting all the delay amplitudes to 0. This result allows to transfer synthesis problems onto the equivalent ordinary differential linear system without delays, which sometimes simplifies the construction of stabilizing controllers. Finally, using algorithmic versions of the Quillen-Suslin theorem, we give a constructive proof of Pommaret's proof of the Lin-Bose conjecture and we show how to compute (weakly) doubly coprime factorizations of rational transfer matrices. All the results are illustrated on explicit examples and the different algorithms have been implemented in OREMODULES.**

***Keywords*— Multidimensional linear systems, flatness, differential time-delay systems, Lin-Bose conjecture, (weakly) doubly coprime factorization, Quillen-Suslin theorem, stabilizing controllers, symbolic computation.**

I. INTRODUCTION

A non-linear ordinary differential control system defined by $\dot{x} = f(x, u)$ is said to be *flat* if there exist some outputs y of the form $y = h(x, u, \dot{u}, \ldots, u^{(r)})$ such that we have:

$$\begin{cases} x = \phi(y, \dot{y}, \ldots, y^{(s)}), \\ u = \varphi(y, \dot{y}, \ldots, y^{(s)}). \end{cases}$$

The outputs y is then called *flat outputs* of the control system $\dot{x} = f(x, u)$. See [5], [6] and the references therein for more details and references. We can prove that the trajectories of a flat system are in a one-to-one correspondence with those of a controllable linear ordinary differential system having an arbitrary state dimension but the same number of inputs [6]. In particular, we say that a flat non-linear system is *Lie-Bäcklund equivalent* to a controllable linear ordinary differential system. This last result, as well as the fact that many classes of non-linear control systems commonly used in the literature were proved to be flat, has popularized this class of systems in the control theory community. The motion planning problem was shown to be easily tractable for flat systems and it was illustrated on several examples in the literature ([5], [6]). Finally, the fact that the trajectories of a flat non-linear systems are in a one-to-one correspondence with the ones of a linear controllable system can be used in order to construct controllers which stabilize a flat non-linear system around a given trajectory (tracking problem) ([5], [6]). See also [24] for applications to optimal control. Unfortunately, no general algorithm is known for checking whether or not a non-linear control system is flat and for the computation of flat outputs despite many effort devoted to the mathematical and control theory literatures. We refer the reader to [39] for a historical account of the main developments of the underlying mathematical problem called the *Monge problem*.

We illustrate these definitions on the model of a vertical take-off and landing aircraft used in [6]

$$\begin{cases} \ddot{x}(t) = u_1(t)\, \sin\theta(t) - \varepsilon\, u_2(t)\, \cos\theta(t), \\ \ddot{z}(t) = u_1(t)\, \cos\theta(t) + \varepsilon\, u_2(t)\, \sin\theta(t) - 1, \\ \ddot{\theta}(t) = u_2(t), \end{cases} \tag{1}$$

where ε is a small parameter. It is proved in [6] that the smooth solutions of (1) can be parametrized by means of the following non-linear differential operator

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{cases} x = y_1 - \varepsilon \dfrac{\ddot{y}_1}{\sqrt{(\ddot{y}_1)^2 + (\ddot{y}_2 + 1)^2}}, \\ z = y_2 - \varepsilon \dfrac{\ddot{y}_2 + 1}{\sqrt{(\ddot{y}_1)^2 + (\ddot{y}_2 + 1)^2}}, \\ \theta = \arctan\left(\dfrac{\ddot{y}_1}{\ddot{y}_2 + 1}\right), \end{cases} \tag{2}$$

where y_1 and y_2 are two arbitrary smooth functions satisfying the condition $(\ddot{y}_1(t))^2 + (\ddot{y}_2(t) + 1)^2 \neq 0$. Moreover, the arbitrary functions y_1 and y_2 can be expressed in terms of the system variables as:

$$\begin{cases} y_1 = x + \varepsilon\, \sin\theta, \\ y_2 = z + \varepsilon\, \cos\theta. \end{cases} \tag{3}$$

Hence, (y_1, y_2) is a flat output of the non-linear system (1) and its knowledge gives a way to generate the trajectories of (1). Finally, the flat ordinary differential system (1) is Lie-Bäcklund equivalent to the Brunovsky linear system defined by

$$\begin{cases} y_1^{(4)} = v_1, \\ y_2^{(4)} = v_2, \end{cases} \tag{4}$$

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under the invertible transformation ($\eta_1 = u_1 - \varepsilon\,\dot{\theta}^2$ and $\eta_2 = \dot{\eta}_1$):

$$\begin{cases} y_1 = x + \varepsilon\,\sin\theta, \\ y_2 = z + \varepsilon\,\cos\theta, \\ v_1 = \dot{\eta}_2\,\sin\theta + 2\,\eta_2\,\dot{\theta}\,\cos\theta + \eta_1\,u_2\,\cos\theta - \eta_1\,\dot{\theta}^2\,\sin\theta, \\ v_2 = \dot{\eta}_2\,\cos\theta - 2\,\eta_2\,\dot{\theta}\,\sin\theta - \eta_1\,u_2\,\sin\theta - \eta_1\,\dot{\theta}^2\,\cos\theta. \end{cases}$$

The study of flat linear ordinary differential time-delay systems has recently been initiated in [7], [15]. As for non-linear ordinary differential systems, this class of systems shares some interesting mathematical properties which can be used to do some motion planning and tracking as shown in [15] and the references therein on explicit examples. However, the theory of flat linear ordinary differential time-delay systems is still in its infancy and some concepts developed for non-linear ordinary differential systems seem to have no counterparts for this second class of systems. In particular, for flat linear differential time-delay systems, we can wonder which kind of linear systems could play a similar role as the one played by the linear controllable systems for flat non-linear systems. To answer this question, we first need to find the concept of equivalence corresponding to the Lie-Bäcklund equivalence for differential time-delay linear systems. To our knowledge, these important questions have not be tackled in the literature till now. This paper aims at constructively answer these questions.

As the differential time-delay systems is a particular (but important) class of multidimensional systems, we can define the concept of a flat multidimensional linear system in terms of the existence of an injective parametrization of the trajectories of the system ([3], [21], [37]). In terms of the module-theoretic/behaviour approach recently developed for multidimensional linear systems ([3], [18], [16], [37], [38]), it means that the module M intrinsically associated with the multidimensional linear system is free over the commutative polynomial ring D of functional operators, i.e., M admits a basis over D. The bases of the D-module M are then in a one-to-one correspondence with flat outputs of the multidimensional linear system. A well-known constructive test for flatness of multidimensional linear systems with constant coefficients consists in checking if some minors of the system matrix (*Fitting ideals*) do not simultaneously vanish on complex common zeros ([9], [34]). This result is based on the famous proofs by Quillen and Suslin of Serre's conjecture asserting that a projective module over a commutative polynomial ring with coefficients in a field is free ([31], [33], [34]). However, there is no easy way for obtaining the flat outputs of the system, and thus, we are led to use constructive versions of the Quillen-Suslin theorem developed in the symbolic algebra community ([8], [11], [13], [17]) for computing a basis of the free module intrinsically associated with the flat multidimensional linear system. We finally refer the reader to [28], [30] for a constructive algorithm which computes bases/flat outputs of multidimensional linear systems with varying coefficients. See also [2] for an implementation.

Closely following some proofs of the Quillen-Suslin theorem ([31], [33], [34]), the purpose of this paper is to prove that flat shift-invariant multidimensional linear system with constant coefficients is equivalent to a linear controllable 1-D system obtained by setting all but one functional operator to 0 in the system matrix. Despite the fact that this result is nothing else than a straightforward consequence of the Quillen-Suslin theorem, its consequence in terms of flat multidimensional systems seems to be ignored and it answers the previous open questions in the theory of flat systems. In particular, the algebraic equivalence we use is the natural equivalence developed in module theory, namely, two multidimensional linear systems are said to be equivalent if their canonical associated modules are isomorphic over the underlying commutative ring of functional operators D. This equivalence is nothing else than the natural substitute to the Lie-Bäcklund equivalence for multidimensional linear systems. In the case of ordinary differential linear systems, we already know that Lie-Bäcklund transformations correspond to morphisms of the underlying modules [6].

Finally, using the Quillen-Suslin theorem, we give constructive proofs of Pommaret's proof of the Lin-Bose conjecture ([10], [20], [35], [36]) and we show how to compute (weakly) doubly coprime factorizations of rational transfer matrices. These two last results solve open questions in the literature of multidimensional systems theory. All the results are illustrated on explicit examples and the different algorithms have been implemented in the library OREMODULES ([2]).

II. A MODULE-THEORETIC APPROACH TO SYSTEMS THEORY

Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k and $R \in D^{q \times p}$. We recall that a matrix R is said to have *full row rank* if the *first syzygy module* of the D-module $(D^{1 \times q}\,R)$, namely,

$$\ker_D(.R) \triangleq \{\lambda \in D^{1 \times p} \mid \lambda\,R = 0\},$$

is reduced to 0. In other words, $\lambda\,R = 0$ implies $\lambda = 0$ or the rows of R are D-linearly independent.

The following definitions of primeness are very classical in the multidimensional systems theory.

Definition 1 ([16], [34], [38]): Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k, $R \in D^{q \times p}$ a full row rank matrix, J the ideal generated by the $q \times q$ minors of R and the algebraic variety defined by:

$$V(J) = \{\xi \in \mathbb{C}^n \mid P(\xi) = 0, \quad \forall\, P \in J\}.$$

1) R is called *minor left-prime* if

$$\dim_{\mathbb{C}} V(J) \leq n - 2,$$

i.e., the greatest common divisor of the $q \times q$ minors of R is 1.

2) R is called *weakly zero left-prime* if

$$\dim_{\mathbb{C}} V(J) \leq 0,$$

i.e., the $q \times q$ minors of R may only vanish simultaneously in a finite number of points of $\mathbb{C}^n$.

3) R is called *zero left-prime* if

$$\dim_{\mathbb{C}} V(J) = -1,$$

i.e., the $q \times q$ minors of R do not vanish simultaneously in $\mathbb{C}^n$.

The previous classification plays an important role in multidimensional systems theory. See [16], [34], [38] and the references therein for more details. The purpose of this section is two fold. We first recall how we can generalize the previous classification for general multidimensional linear systems (i.e., not only defined by means of full row rank matrices). We also explain the duality existing between the behavioural approach to multidimensional systems ([18], [16], [37], [38]) and the module-theoretic one ([21], [22], [23]).

In what follows, D will denote a commutative polynomial ring with coefficients in a field k. In particular, we shall be interested in commutative polynomial rings of functional operators such as partial differential operators, differential time-delay operators or shift operators. Let us consider a matrix $R \in D^{q \times p}$ and a D-module $\mathcal{F}$, namely:

$$\forall\, f_1,\, f_2 \in \mathcal{F}, \quad \forall\, a_1,\, a_2 \in D: \quad a_1\, f_1 + a_2\, f_2 \in \mathcal{F}.$$

If we define the D-morphism (namely, D-linear map)

$$\begin{array}{rcl} .R : D^{1\times q} & \xrightarrow{.R} & D^{1 \times p}, \\ \lambda = (\lambda_1, \ldots, \lambda_q) & \longmapsto & \lambda\, R, \end{array}$$

where $D^{1\times p}$ denotes the D-module of row vectors of length p, then the cokernel of $.R$ is defined by:

$$M = D^{1\times p}/(D^{1\times q}\, R).$$

The D-module M is said to be presented by R or simply *finitely presented* ([3], [31]). Moreover, we can also define the *system* or *behaviour*:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}.$$

As it was noticed by Malgrange in [14], the D-module M and the system $\ker_{\mathcal{F}}(R.)$ are closely related. As this relation will play an important role in what follows, we shall explain it in details. In order to do that, let us first introduce a few classical definitions of homological algebra. We refer the reader to [31] for more details.

Definition 2: 1) A sequence $(d_i : M_i \longrightarrow M_{i-1})_{i\in\mathbb{Z}}$ of morphisms $d_i : M_i \longrightarrow M_{i-1}$ between left D-modules is *a complex* if we have:

$$\forall\, i \in \mathbb{Z}, \quad \operatorname{im} d_i \subseteq \ker d_{i-1}.$$

We denote the previous complex by:

$$\ldots \xrightarrow{d_{i+2}} M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \ldots \quad (5)$$

2) The *defect of exactness of the complex (5) at M_i* is:

$$H(M_i) = \ker d_i / \operatorname{im} d_{i+1}.$$

3) The complex (5) is *exact at M_i* if we have:

$$H(M_i) = 0 \quad \Longleftrightarrow \quad \ker d_i = \operatorname{im} d_{i+1}.$$

4) The complex (5) is *exact* if:

$$\forall\, i \in \mathbb{Z}, \quad \ker d_i = \operatorname{im} d_{i+1}.$$

5) The complex (5) is a *split exact sequence* if it is exact and there exist morphisms $s_i : M_{i-1} \longrightarrow M_i$ satisfying the following conditions:

$$\forall\, i \geq 0, \quad \begin{cases} s_{i+1} \circ s_i = 0, \\ s_i \circ d_i + d_{i+1} \circ s_{i+1} = id_{M_i}. \end{cases}$$

6) A *finite free resolution* of a left D-module M is an exact sequence of the form

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} \ldots \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0. \quad (6)$$

where $p_i \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $R_i \in D^{p_i \times p_{i-1}}$,

$$\begin{array}{rcl} (.R_i) : D^{1\times p_i} & \longrightarrow & D^{1\times p_{i-1}} \\ \lambda & \longmapsto & (.R_i)(\lambda) = \lambda\, R_i. \end{array}$$

The next classical result of homological algebra will play a crucial role in the rest of the paper.

Theorem 1 ([31]): Let $\mathcal{F}$ be a D-module, M a D-module and (6) a finite free resolution of M. Then, the defects of exactness of the following complex

$$\ldots \xleftarrow{R_3.} \mathcal{F}^{p_2} \xleftarrow{R_2.} \mathcal{F}^{p_1} \xleftarrow{R_1.} \mathcal{F}^{p_0} \longleftarrow 0, \quad (7)$$

where $(R_i.) : \mathcal{F}^{p_{i-1}} \longrightarrow \mathcal{F}^{p_i}$ is defined by

$$\forall\, \eta \in \mathcal{F}^{p_{i-1}}, \quad (R_i.)\,\eta = R_i\,\eta,$$

only depend on the D-modules M and $\mathcal{F}$. Up to an isomorphism, we denote these defects of exactness by:

$$\begin{cases} \operatorname{ext}^0_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_1.), \\ \operatorname{ext}^i_D(M, \mathcal{F}) \cong \ker_{\mathcal{F}}(R_{i+1}.)/(R_i\,(\mathcal{F}^{p_i})), \quad i \geq 1. \end{cases}$$

Finally, we have $\operatorname{ext}^0_D(M, \mathcal{F}) = \hom_D(M, \mathcal{F})$, where $\hom_D(M, \mathcal{F})$ denotes the D-module of D-morphisms (namely, D-linear maps) from M to $\mathcal{F}$.

We refer the reader to Example 9 for explicit computations of $\operatorname{ext}^i_D(N, D)$.

Coming back to the D-module M, we have the following beginning of a finite free resolution of M:

$$\begin{array}{ccccc} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \\ \lambda & \longmapsto & \lambda R \end{array} \tag{8}$$

where π denotes the D-morphism which sends elements of $D^{1\times p}$ in their residue classes in M. If we "apply the left-exact functor" $\hom_D(\cdot, \mathcal{F})$ to (8), by Theorem 1, we obtain the following exact sequence:

$$\begin{array}{ccccc} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0. \\ R\,\eta & \longleftarrow\!\shortmid & \eta \end{array}$$

This implies the following important isomorphism [14]:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}). \tag{9}$$

For more details, see [3], [14], [16], [23], [37] and the references therein. In particular, (9) gives an intrinsic characterization of the $\mathcal{F}$-solutions of a linear system over D. It only depends on two objects:

1) The finitely presented D-module M which represents the equations of the linear system.

2) The D-module $\mathcal{F}$ which is the functional space where we seek the solutions.

If D is now a ring of functional operators (e.g., differential operators, time-delay operators, difference operators), then the issue of understanding which $\mathcal{F}$ is suitable for a particular linear system has been studied for a long time in functional analysis and is still a very active subject. It does not seem that constructive algebra and symbolic computation can propose new methods to handle this functional analysis problem. However, they are very useful for classifying $\hom_D(M, \mathcal{F})$ by means of the algebraic properties of the left D-module M. Indeed, a large classification of the properties of modules is developed in homological algebra. See [31] for more information. Let us recall a few of them.

Definition 3 ([31]): Let D be a commutative polynomial ring with coefficients in a field k and M a finitely generated left D-module. Then, we have:

1) M is called *free* if it is isomorphic to $D^{1\times r}$ for a certain non-negative integer r, i.e., if we have:
$$M \cong D^{1\times r} \quad r \in \mathbb{Z}_+ = \{0, 1, 2 \ldots\}.$$

2) M is called *stably free* if there exist two non-negative integers r and s such that:
$$M \oplus D^{1\times s} \cong D^{1\times r}.$$

3) M is called *projective* if there exist a D-module P and non-negative integer r such that:
$$M \oplus P \cong D^{1\times r}.$$

4) M is called *reflexive* if the canonical map
$$\varepsilon_M : M \longrightarrow \hom_D(\hom_D(M, D), D),$$
defined by
$$\forall\, m \in M,\ f \in \hom_D(M, D),\ \varepsilon_M(m)(f) = f(m),$$
is an isomorphism, where $\hom_D(M, D)$ denotes the D-module of all D-morphisms from M to D.

5) M is *torsion-free* if the submodule of M defined by
$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D :\ P\,m = 0\}$$
is the zero module. $t(M)$ is called the *torsion submodule* of M and the elements of $t(M)$ are the *torsion elements* of M.

6) M is *torsion* if $t(M) = M$, i.e., every element of M is a torsion element.

Let $K = Q(D) = k(x_1, \ldots, x_n)$ be the quotient field of D and M a finitely presented D-module. We call the *rank* of M, denoted by $\mathrm{rank}_D(M)$, the dimension of the K-vector space $K \otimes_D M$ obtained by extending the scalars of M from D to K, i.e., we have:

$$\mathrm{rank}_D(M) = \dim_K(K \otimes_D M).$$

We can check that if M is a torsion D-module, we then $K \otimes_D M = 0$, and thus, $\mathrm{rank}_D(M) = 0$ ([31]).

Let us recall some results about the notions introduced in Definition 3.

Theorem 2 ([31]): Let D be a commutative polynomial ring with coefficients in a field k.

1) We have the implications among the concepts:
$$\text{free} \Longrightarrow \text{stably free} \Longrightarrow \text{projective} \Longrightarrow$$
$$\text{reflexive} \Longrightarrow \text{torsion-free}.$$

2) If $D = k[x_1]$, then D is a *principal ideal domain* − namely, every ideal of D is *principal*, i.e., it can be generated by one element of D − then every finitely generated torsion-free D-module is free.

3) (Quillen-Suslin theorem) Every projective module over D is free.

The famous Quillen-Suslin theorem will play an important role in what follows ([9], [31]).

The next theorem gives some characterizations of the definitions given in Definition 3.

Theorem 3 ([3], [23]): Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k, $R \in D^{q\times p}$ and the finitely presented D-modules:

$$M = D^{1\times p}/(D^{1\times q}\,R), \quad N = D^{1\times q}/(D^{1\times p}\,R^T).$$

Module M	Homological algebra
With torsion	$t(M) \cong \mathrm{ext}^1_D(N,D) \neq 0$
Torsion-free	$\mathrm{ext}^1_D(N,D) = 0$
Reflexive	$\mathrm{ext}^i_D(N,D) = 0,$ $1 \leq i \leq 2$
...	...
Projective	$\mathrm{ext}^i_D(N,D) = 0,$ $1 \leq i \leq n$

Fig. 1.

We then have the equivalences presented in Fig. 1.

Combining the results of Theorem 3 and the Quillen-Suslin theorem (see 3) of Theorem 2), we then obtain a way to check whether or not a finitely presented D-module M has some torsion elements or is torsion-free, reflexive, projective, stably free or free. We point out that the explicit computation of $\mathrm{ext}^i_D(N,D)$ can always be done using Gröbner or Janet bases. See [3], [21], [22] for more details and for the description of the corresponding algorithms. We refer the reader to [2] for the library OREMODULES in which the different algorithms have been implemented as well as the large library of examples which illustrate them.

In order to explain why the definitions given in Definition 3 extend the concepts of primeness defined in Definition 1, we first need to introduce some more definitions.

Definition 4 ([1]): 1) If M is a non-zero finitely generated D-module, then the *grade* $j_D(M)$ of M is defined by:

$$j_D(M) = \min\{i \geq 0 \mid \mathrm{ext}^i_D(M,D) \neq 0\}.$$

2) If M is a non-zero finitely generated D-module, the *dimension* $\dim_D(M)$ of M is defined by

$$\dim_D(M) = \mathrm{Kdim}(D/\sqrt{\mathrm{ann}_D(M)}),$$

where Kdim denotes the *Krull dimension* and:

$$\sqrt{\mathrm{ann}_D(M)} = \{a \in D \mid \exists\, l \in \mathbb{Z}_+ : \quad a^l\, M = 0\}.$$

We are now in position to state an important result.

Theorem 4 ([1]): If M is a non-zero finitely generated D-module, we then have:

$$j_D(M) + \dim_D(M) = n.$$

Let us suppose that R has full row rank and let us consider $M = D^{1\times p}/(D^{1\times q}\, R)$ the D-module presented by the matrix R. Using the notations of Definition 1 and

$$\dim_D(N) = \dim_{\mathbb{C}} V(J),$$

where $N = D^{1\times q}/(D^{1\times p}\, R^T)$ and N is a torsion D-module, i.e., it satisfies $\mathrm{ext}^0_D(M,D) = \hom_D(M,D) = 0$, by Theorem 4, we then obtain:

$$j_D(N) = n - \dim_{\mathbb{C}} V(J) \geq 1.$$

Hence, by Theorems 3 and 4, we obtain that R is minor left-prime (resp., zero left-prime) iff the D-module M is torsion-free (resp., projective, and thus, free by the Quillen-Suslin theorem stated in 3) of Theorem 2). See [23] for more details and the extension of these results to the case of non-commutative rings of differential operators.

We finally obtain the following table given in Fig. 2 summing up some of the results previously stated. We note that the last two columns of the table only hold when the matrix R has full row rank.

To finish, we shall explain what the system interpretations of the definitions given Definition 3 are. In order to do that, we also need to introduce a few more definitions.

Definition 5 ([31]): 1) A D-module $\mathcal{F}$ is called *injective* if, for every D-module M, and, for all $i \geq 1$, we have $\mathrm{ext}^i_D(M,\mathcal{F}) = 0$.

2) A D-module $\mathcal{F}$ is called *cogenerator* if, for every D-module M, we have:

$$\hom_D(M,\mathcal{F}) = 0 \quad \Longrightarrow \quad M = 0.$$

Roughly speaking, an injective cogenerator is a space rich enough for seeking solutions of linear systems of the form $R\,y = 0$, where R is any matrix with entries in D. In particular, using (9), if $\mathcal{F}$ is a cogenerator D-module and $M \neq 0$, then $\hom_D(M,\mathcal{F}) \neq 0$, meaning that the corresponding system $\ker_{\mathcal{F}}(R.)$ is not empty. Finally, if $\mathcal{F}$ is an injective cogenerator D-module, then we can prove that any complex of the form (7) is exact if and only if the corresponding complex (6) is exact.

The following result proves that there always exists an injective cogenerator.

Module M	$\mathbf{ext}_D^i(N,D)$	$\dim_D(N)$	**Primeness**
With torsion	$\mathrm{ext}_D^1(N,D) \cong t(M)$	$n-1$	$\emptyset$
Torsion-free	$\mathrm{ext}_D^1(N,D) = 0$	$n-2$	Minor left-prime
Reflexive	$\mathrm{ext}_D^i(N,D) = 0$, $i=1,2$	$n-3$	
...	...	...	...
...	$\mathrm{ext}_D^i(N,D) = 0$, $1 \leq i \leq n-1$	0	Weakly zero left-prime
Projective	$\mathrm{ext}_D^i(N,D) = 0$, $1 \leq i \leq n$	-1	Zero left-prime

Fig. 2.

Theorem 5 ([31]): An injective cogenerator D-module $\mathcal{F}$ exists for every ring D.

Let us give a few examples of injective cogenerators modules.

Example 1: If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator module over the ring $\mathbb{R}[\partial_1, \ldots, \partial_n]$ of differential operators with coefficients in $\mathbb{R}$, where $\partial_i = \frac{\partial}{\partial x_i}$ [16].

We have the following important corollary of Theorem 3.

Corollary 1 ([3], [21]): Let $D = k[x_1, \ldots, x_n]$, $\mathcal{F}$ an injective cogenerator D-module, $R \in D^{q \times p}$ and the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$. Then, we have the following results:

1) There exists $Q_1 \in D^{q_1 \times q_2}$ such that we have the following exact sequence

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^{q_1} \xleftarrow{Q_1.} \mathcal{F}^{q_2},$$

i.e., $\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{q_2}$, iff the left D-module M is torsion-free, where $p = q_1$.

2) There exist $Q_1 \in D^{q_1 \times q_2}$ and $Q_2 \in D^{q_2 \times q_3}$ such that we have the following exact sequence

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^{q_1} \xleftarrow{Q_1.} \mathcal{F}^{q_2} \xleftarrow{Q_2.} \mathcal{F}^{q_3},$$

iff the left D-module M is reflexive, where $p = q_1$.

3) There exists a chain of n successive parametrizations, namely, there exist $Q_i \in D^{q_i \times q_{i+1}}$, with $i = 1, \ldots, n$, such that we have the following exact sequence

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^{q_1} \xleftarrow{Q_1.} \ldots \xleftarrow{Q_{n-1}.} \mathcal{F}^{q_n} \xleftarrow{Q_n.} \mathcal{F}^{q_{n+1}},$$

iff the left D-module M is projective, where $p = q_1$.

4) There exists an injective parametrization, namely, two matrices $Q \in D^{p \times m}$ and $T \in D^{m \times p}$ such that we have the following exact sequence

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m \longleftarrow 0,$$

and $T\, Q = I_m$ iff M is a free D-module.

The matrices Q_i defined in Corollary 1 are called *parametrizations* ([3], [21], [22], [23]). Indeed, from 1) of Corollary 1, if M is torsion-free, then there exists a matrix of operators $Q_1 \in D^{q_1 \times q_2}$ which satisfies $\ker_{\mathcal{F}}(R.) = Q_1\, \mathcal{F}^{q_2}$. This means that any solution $\eta \in \mathcal{F}^p$ satisfying $R\, \eta = 0$ is of the form $\eta = Q_1\, \xi$, where $\xi \in \mathcal{F}^{q_2}$. In the behaviour approach, the parametrization is also called an *image representation* ([18], [37], [38]). We point out that the parametrizations Q_i are obtained in the computation of the $\mathrm{ext}_D^i(N,D)$ (see Theorem 3). Hence, checking wether or not a D-module is torsion-free, reflexive or projective gives the corresponding successive parametrizations. We refer to [3], [21], [22], [23] for more details and for the extension of the previous results to non-commutative of functional operators.

Explicit examples of computation of parametrizations can be found in [3], [21], [22], [23] as well in the large library of examples of OREMODULES ([2]). We refer the reader to these references or to Section IV. However, let us give a small example in order to illustrate the previous main results.

Example 2: Let us consider the ring $D = \mathbb{Q}[\partial_1, \partial_2, \partial_3]$ of differential operators with rational coefficients, where $\partial_i = \frac{\partial}{\partial x_i}$, the matrix $R = (\partial_1 \quad \partial_2 \quad \partial_3)$ defining the so-called *divergent operator* in $\mathbb{R}^3$ and the finitely presented D-module $M = D^{1\times 3}/(D\,R)$. Let us check whether or not the D-module M has some torsion elements or is torsion-free, reflexive or projective (i.e., free by the Quillen-Suslin theorem). In order to do that, we define the D-module $N = D/(D^{1\times 3}\,R^T)$. A finite free resolution of N can be easily computed by means of Gröbner or Janet bases. We obtain the following exact sequence

$$0 \longrightarrow D \xrightarrow{.P_3} D^{1\times 3} \xrightarrow{.P_2} D^{1\times 3} \xrightarrow{.R^T} D \xrightarrow{\sigma} N \longrightarrow 0,$$

where σ denotes the canonical projection and:

$$P_2 = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}, \quad P_3 = R.$$

We note that P_2 corresponds to the so-called *curl operator* whereas P_3 is the *gradient operator*. The defect of exactness of the following complex

$$0 \longleftarrow D \xleftarrow{.P_3^T} D^{1\times 3} \xleftarrow{.P_2^T} D^{1\times 3} \xleftarrow{.R} D \longleftarrow 0 \qquad (10)$$

are defined by:

$$\begin{cases} \mathrm{ext}^0_D(N, D) \cong \ker_D(.R), \\ \mathrm{ext}^1_D(N, D) \cong \ker_D(.P_2^T)/(D\,R), \\ \mathrm{ext}^2_D(N, D) \cong \ker_D(.P_3^T)/(D^{1\times 3}\,P_2^T), \\ \mathrm{ext}^3_D(N, D) \cong D/(D^{1\times 3}\,P_3^T). \end{cases}$$

Using the fact that R has full row rank, we obtain that $\mathrm{ext}^0_D(N, D) = 0$, which is equivalent to say that N is a torsion D-module. Now computing the syzygy modules $\ker_D(.P_2^T)$ and $\ker_D(.P_3^T)$ by means of Gröbner or Janet bases, we obtain that

$$\ker_D(.P_2^T) = (D\,R), \quad \ker_D(.P_3^T) = (D^{1\times 3}\,P_2^T),$$

which shows that $\mathrm{ext}^1_D(N, D) = \mathrm{ext}^2_D(N, D) = 0$. Finally, we can easily check that 1 does not belong to the ideal $I = D\,\partial_1 + D\,\partial_2 + D\,\partial_3$, and thus, we have:

$$\mathrm{ext}^3_D(N, D) \cong D/I \neq 0.$$

Hence, using Theorem 3, we obtain that M is a reflexive but not a projective, and thus, not a free D-module. This last fact can also be checked as R has full row rank and the dimension $\dim_D(N)$ is 0 as the corresponding system is defined by the gradient operator

$$\begin{cases} \partial_1\,y = 0, \\ \partial_2\,y = 0, \\ \partial_3\,y = 0, \end{cases}$$

whose solution only depends on constant, i.e., on non independent variables. Hence, by Theorem 4, we obtain that $j_D(N) = 3$, meaning that the first non-zero $\mathrm{ext}^i_D(N, D)$ has index 3. By Theorem 3, we then get that M is a reflexive D-module.

Finally, if we consider the D-module $\mathcal{F} = C^\infty(\Omega)$, where Ω is an open convex subset of $\mathbb{R}^3$, using Example 1, we obtain that $\mathcal{F}$ is an injective cogenerator D-module. Hence, if we apply the functor $\hom_D(\cdot, \mathcal{F})$ to the complex (10), we then obtain the following exact sequence:

$$\mathcal{F} \xrightarrow{P_3^T\,\cdot} \mathcal{F}^3 \xrightarrow{P_2^T\,\cdot} \mathcal{F}^3 \xrightarrow{R.} \mathcal{F} \longrightarrow 0.$$

We find again the classical result in mathematical physics that the smooth solutions on a open convex subset of $\mathbb{R}^3$ of the divergence operator are parametrized by the curl operator and the solutions of the curl operator are parametrized by the gradient operator.

We note that the problem of recognizing whether or not an under-determined systems of partial differential equations is parametrizable is called the *Monge problem* ([39]). Hence, we see that Corollary 1 mainly solves this problem in the case of linear systems of partial differential equations with constant coefficients. We refer the reader to [3] for the solutions of the Monge problem for different classes of functional systems with variables coefficients such as differential time-delay systems or difference systems. The only point let opened is to constructively compute the injective parametrization in the case of a free D-module. Indeed, checking the vanishing of the $\mathrm{ext}^i_D(N, D)$, we generally obtain a successive chain of n parametrizations but not an injective one. In the case of linear systems of partial differential equations with polynomial or rational coefficients, we have recently solved this problem in [28], [29], [30] using a constructive proof of a famous result in non-commutative algebra due to Stafford. However, the same technique cannot be used if we want the injective parametrization Q of $\ker_{\mathcal{F}}(R.)$ to have only constant coefficients. The main purpose of this paper is to solve this problem using constructive proof of the Quillen-Suslin theorem and to show some applications of this result or techniques in control theory. Finally, we point out that a multidimensional linear system which admits an injective parametrization was called *flat* in the control theory litterature. See [3], [5], [6], [15], [21] and the references therein. As it was shown in [5], [6], [15] this class of control systems have natural applications in the motion and tracking problems. Hence, the constructive computation of the injective parametrizations of flat multidimensional linear systems and, in particular, of differential time-delay linear systems is an open and interesting issue in control theory. We shall solve this problem in the paper as well as some related ones already discussed in the introduction. We point out that the different algorithms developed here have been recently implemented in OREMODULES.

To finish this section, we recall two classical results of homological algebra which will be useful in what follows.

Proposition 1 ([31]): 1) Let us consider the following short exact sequence of D-modules:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

If M'' is a projective D-module, then the previous exact sequence splits (see 4) of Definition 2).

2) Let $\mathcal{F}$ be a D-module. Then, the functor $\hom_D(\cdot, \mathcal{F})$ transforms split exact sequences of D-modules into split exact sequences of D-modules.

III. THE QUILLEN-SUSLIN THEOREM

Since Quillen and Suslin independently proved in 1976 the Serre's Conjecture stating that projective modules over polynomial rings are free, some algorithmic versions of the proof have been proposed in the literature in order to constructively compute bases of free modules ([8], [11], [13], [17], [32], [33], [34]). We refer the interested reader to Lam's nice book [9] concerning Serre's conjecture.

A. Projective and stably free modules

In module theory, it is well-known that if k is a computable field, $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring and $R \in D^{q\times p}$, then we can always compute a finite free resolution of the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$. See [3] and the references therein. A result due to Serre then proves that any projective D-module is stably free (a stably free module being always a projective D-module). In [28], [30] a constructive proof of this result was given and the corresponding algorithm was implemented in OREMODULES.

Let us recall this useful result. We refer the reader to [28], [30] for a proof.

Proposition 2 ([28], [30]): Let us consider a finite free resolution of a left D-module M of the form:

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} \ldots \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\pi} M \longrightarrow 0. \quad (11)$$

1) If $m \geq 3$ and there exists $S_m \in D^{p_{m-1}\times p_m}$ such that $R_m\, S_m = I_{p_m}$, then we have the finite free resolution of M

$$0 \longrightarrow D^{1\times p_{m-1}} \xrightarrow{.T_{m-1}} D^{1\times (p_{m-2}+p_m)} \xrightarrow{.T_{m-2}} D^{1\times p_{m-3}} \xrightarrow{.R_{m-3}} \ldots \xrightarrow{\pi} M \longrightarrow 0, \quad (12)$$

with the following notations:

$$T_{m-1} = (R_{m-1} \quad S_m), \quad T_{m-2} = \begin{pmatrix} R_{m-2} \\ 0 \end{pmatrix}.$$

2) If $m = 2$ and there exists $S_2 \in D^{p_1\times p_2}$ such that $R_2\, S_2 = I_{p_2}$, then we have the finite free resolution

$$0 \longrightarrow D^{1\times p_1} \xrightarrow{.T_1} D^{1\times (p_0+p_2)} \xrightarrow{\tau} M \longrightarrow 0, \quad (13)$$

with the notations $T_1 = (R_1 \quad S_2)$ and:

$$\begin{array}{rcl} \tau = \pi \oplus 0 : D^{1\times (p_0+p_2)} & \longrightarrow & M \\ \lambda = (\lambda_1 \quad \lambda_2) & \longmapsto & \tau(\lambda) = \pi(\lambda_1). \end{array}$$

Let $R \in D^{q\times p}$ and let us suppose that the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is projective (using the results presented in the previous section, we can always check this result). Combining 1) of Propositions 1 and 2, we obtain that we can always compute a full row rank matrix $R' \in D^{q'\times p'}$ such that:

$$M \cong D^{1\times p'}/(D^{1\times q'}\, R'). \quad (14)$$

Corollary 2: Let $D = k[x_1, \ldots, x_n]$ is a commutative polynomial ring over a computable field k and $R \in D^{q\times p}$. If the D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is projective, then there exists a full row rank matrix $R' \in D^{q'\times p'}$ satisfying the isomorphism (14).

We refer to Example 10 for an illustration of Proposition 2. See also [28], [29], [30] for more examples.

Finally, the D-module $L = D^{1\times p'}/(D^{1\times q'}\, R')$ is stably free as we have the following short exact sequence

$$0 \longrightarrow D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\kappa} L \longrightarrow 0,$$

and using the fact that $L \cong M$ and M is a projective D-module, by 1) of Proposition 1, we obtain that the previous exact sequence splits and then, we get

$$L \oplus D^{1\times q'} \cong D^{1\times p'}$$

([3], [31]), which by 2) of Definition 3, shows that L is a stably free D-module.

B. Stably free and free modules

Using Proposition 2, we consider the problem of finding a basis for a free module defined as a cokernel of a full rank matrix. Let us suppose that $R \in D^{q\times p}$ has full row rank and $M = D^{1\times p}/(D^{1\times q}\, R)$ is the D-module presented by R.

Definition 6: The *general linear group* $\mathrm{GL}_p(D)$ is defined by by:

$$\mathrm{GL}_p(D) = \{U \in D^{p\times p} \mid \exists\, V \in D^{p\times p} :\ U\, V = V\, U = I_p\}.$$

A matrix $U \in \mathrm{GL}_p(D)$ is called *unimodular*.

If $D = k[x_1, \ldots, x_n]$, we then easily check that $U \in \mathrm{GL}_p(D)$ iff the determinant $\det U$ is a non-zero element of k. We are now in position to state the famous Quillen-Suslin theorem.

Theorem 6 ([9], [31]): (Quillen Suslin theorem) Let A be a principal ideal domain and $D = A[x_1, \ldots, x_n]$ a polynomial ring with coefficients in A. Moreover, let $R \in D^{q \times p}$ be a full row rank matrix which admits a *right-inverse* $S \in D^{p \times q}$, namely, $R\,S = I_q$. Then, there exists a $U \in \mathrm{GL}_p(D)$ satisfying:

$$R\,U = (I_q \quad 0). \tag{15}$$

Let us denote by $J = (I_q \quad 0)$. We easily check that $D^{1\times p}/(D^{1\times q}\,J) = D^{1\times(p-q)}$. Hence, using the same notations as previously, Theorem 6 shows that we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow .U & & & & \\
0 & \longrightarrow & D^{1\times q} & \xrightarrow{.J} & D^{1\times p} & \xrightarrow{\sigma} & D^{1\times(p-q)} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

which easily proves that $M \cong D^{1\times(p-q)}$, i.e., M is a free D-module of rank $p-q$.

Conversely, if we have $M \cong D^{1\times(p-q)}$, then combining the isomorphism $\psi : M \longrightarrow D^{1\times(p-q)}$ and the short exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

we then obtain the following exact sequence:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\psi\circ\pi} D^{1\times(p-q)} \longrightarrow 0.$$

If we consider the matrix which corresponds to the D-morphism $\psi \circ \pi$ in the canonical bases of $D^{1\times p}$ and $D^{1\times(p-q)}$, we obtain a matrix $Q \in D^{p\times(p-q)}$. Moreover, by 1) of Proposition 1, the previous exact sequence splits

$$0 \longrightarrow D^{1\times q} \underset{\xleftarrow{.S}}{\xrightarrow{.R}} D^{1\times p} \underset{\xleftarrow{.T}}{\xrightarrow{.Q}} D^{1\times(p-q)} \longrightarrow 0,$$

i.e., there exists a matrix $T \in D^{(p-q)\times p}$ such that we have the following two Bézout identities:

$$\begin{pmatrix} R \\ T \end{pmatrix} (S \quad Q) = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix},$$

$$(S \quad Q) \begin{pmatrix} R \\ T \end{pmatrix} = I_p.$$

See [3], [21], [25], [26], [31] for more details. In particular, we obtain that there exists $U = (S \quad Q) \in \mathrm{GL}_p(D)$ satisfying:

$$R\,U = (I_q \quad 0).$$

In particular, we obtain that $\{\pi(T_i)\}_{i=1,\ldots,p-q}$, where T_i denotes the i^{th} row of T, forms a basis of the D-module M.

Hence, the problem of finding a basis of a projective finitely generated D-module M can be formulated as a problem of computing a matrix $U \in \mathrm{GL}_p(D)$ satisfying (15) or, equivalently, as the problem of completing the matrix R to a square invertible matrix:

$$\begin{pmatrix} R \\ T \end{pmatrix} = U^{-1} \in D^{p\times p}.$$

We note that, even if there are some differences in the constructive proofs of the Quillen-Suslin theorem developed in [8], [9], [11], [17], [32], [33], [34], the main idea remains the same and they proceed by induction on the number of variables x_i. Each inductive step of the algorithm reduces the problem to the case with one variable less. However, a more global and interesting approach has recently been developed in [13], which needs to be studied with care in the future.

Although the tedious inductive method cannot generally be avoided, there are cases where simpler and faster heuristic methods can be used. The purpose of the next paragraphs is to give a short introduction to the problem, present a version of the algorithm for the general case and show some other strategies for particular cases.

C. A few simple cases

1) Modules over principal ideal domains: We first consider the special case of matrices over a principal ideal domain D (e.g., $D = k[x_1]$). Let $R \in D^{q\times p}$ be a matrix which admits a right-inverse over D. Then, computing the Smith normal form ([19]) of R, we obtain two matrices $F \in \mathrm{GL}_q(D)$ and $G \in \mathrm{GL}_p(D)$ such that:

$$R = F\,(I_q \quad 0)\,G.$$

If we denote by $r = p - q$ and

$$G = (G_1^T \quad G_2^T)^T, \quad G_1 \in D^{q\times p}, \quad G_2 \in D^{r\times p},$$
$$G^{-1} = (F_1 \quad F_2) \in D^{p\times p}, \quad F_1 \in D^{p\times q}, \quad F_2 \in D^{p\times r},$$

then we have $R = F\,G_1$, i.e., $G_1 = F^{-1}\,R$, and thus,

$$\Rightarrow \begin{pmatrix} F^{-1}\,R \\ G_2 \end{pmatrix} (F_1 \quad F_2) = I_p$$

$$\Rightarrow \begin{pmatrix} F^{-1} & 0 \\ 0 & I_r \end{pmatrix} \begin{pmatrix} R \\ G_2 \end{pmatrix} (F_1 \quad F_2) = I_p,$$

$$\Rightarrow \begin{pmatrix} R \\ G_2 \end{pmatrix} (F_1 \quad F_2) \begin{pmatrix} F & 0 \\ 0 & I_r \end{pmatrix} = I_p$$

$$\Rightarrow \begin{pmatrix} R \\ G_2 \end{pmatrix} (F_1\,F \quad F_2) = I_p,$$

which solves the problem as we can take $T = G_2$.

2) $(p-1)\times p$ matrices: We consider the case of a matrix $R \in D^{(p-1)\times p}$ which admits a right-inverse. If we denote by m_i the $(p-1)\times(p-1)$ minor of R obtained by removing the i^{th} column of R, by the last row of Fig. 2, we obtain that the ideal of D generated by $\{m_i\}_{i=1,\dots,p}$ satisfying a Bézout identity, namely, there exists a family $\{n_i\}_{i=1,\dots,p}$ of elements of D such that:

$$\sum_{i=1}^{p} n_i\, m_i = 1.$$

If we denote by

$$V = \begin{pmatrix} & R & \\ (-1)^{p+1}\, n_1 & \dots & (-1)^{2\,p}\, n_p \end{pmatrix} \in D^{p\times p},$$

using the Cauchy-Binet formula, we have $\det(V) = 1$, and thus, if we denote by U the inverse of V, we finally obtain

$$R\,U = (I_{p-1} \quad 0),$$

which solves the problem.

3) $q \times p$ matrices: Let us consider a matrix $R \in D^{q\times p}$ which admits a right-inverse and let us denote by R_i the i^{th} row of R, i.e., $R = (R_1^T \dots R_q^T)^T$. If we can compute a matrix $U_1 \in \mathrm{GL}_q(D)$ satisfying

$$R_1\, U_1 = (1 \quad 0 \quad \dots \quad 0),$$

then we have:

$$R\,U_1 = \begin{pmatrix} 1 & 0 \\ * & R_2 \end{pmatrix}.$$

Hence, we are reduced to consider the new matrix $R_2 \in D^{(q-1)\times(p-1)}$, which can be easily shown to admit a right-invers. Hence, we are always able to reduce the problem of a matrix to the one of a row.

4) $1 \times p$ rows: Let us consider a single row vector $f = (f_1 \quad \dots \quad f_p) \in D^{1\times p}$ which admits a right-inverse.

First of all, we consider some particular cases where the matrix U can be easily computed.

a) easyf: We note that if one of the component of f is invertible over k, then we can transform the row f into $(1 \quad 0 \quad \dots \quad 0)$ by means of trivial elementary transformations. For instance, if we have $f_1^{-1} \in k$, then the matrix defined by

$$W = \begin{pmatrix} f_1^{-1} & 0 \\ 0 & I_{p-1} \end{pmatrix},$$

satisfies $\det(W) = f_1^{-1} \in k$ and:

$$f\,W = (1 \quad f_2 \quad \dots \quad f_p).$$

Then, simple elementary operations transforms $f\,W$ into the first vector $(1 \quad 0 \quad \dots \quad 0)$ of the canonical basis of $D^{1\times p}$.

Another simple case is when two components of f generate D. Let us suppose that there exist h_1 and $h_2 \in D$ such that we have the following Bézout identity

$$f_1 h_1 + f_2 h_2 = 1,$$

and let us define the following matrix:

$$W = \begin{pmatrix} h_1 & -f_2 & 0 \\ h_2 & f_1 & 0 \\ 0 & 0 & I_{p-2} \end{pmatrix}.$$

We easily check that $\det W = 1$ and:

$$f\,W = (1 \quad 0 \quad f_3 \quad \dots \quad f_p).$$

Then, we can reduce $f\,W$ to $(1 \quad 0 \quad \dots \quad 0)$ by means of elementary operations.

b) easysr: If the i^{th} component of f is 0 or the ideal generated by the elements $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_p$ generates D, then, following [28], [30], we can easily construct $U \in \mathrm{GL}_p(D)$ satisfying $R\;U = (I_q \quad 0)$. We refer the reader to [28], [30] for more details.

c) easyg: Let us suppose that one of the entries of a right inverse g of f is invertible over k and let suppose that it is g_1. Then, the following matrix

$$W = \begin{pmatrix} g_1 & & & \\ g_2 & 1 & & \\ \vdots & & \ddots & \\ g_p & & & 1 \end{pmatrix}$$

satisfies $\det(T) = g_1$ and $f\,W = (1 \quad f_2 \quad \dots \quad f_p)$.

Example 3: Let us consider the commutative polynomial ring $D = \mathbb{Q}[z_1, z_2, z_3]$ and the row vector:

$$R = (z_1^2\, z_2^2 + 1 \quad z_1^2\, z_3 + 1 \quad z_1\, z_2^2\, z_3).$$

We can check that R admits the following right-inverse S:

$$S = (-z_1^2\, z_3 \quad 1 \quad z_1^3)^T.$$

As the second component of S is invertible over D, we can apply the previous remark in order to find a unimodular matrix U over D which satisfies $R\,U = (1 \quad 0 \quad 0)$. Let us define the following elementary matrices:

$$U_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 0 \\ -z_1^2\, z_3 & 1 & 0 \\ z_1^3 & 0 & 1 \end{pmatrix}.$$

We then have $R\,(U_1\, U_2) = (1 \quad z_1^2\, z_2^2 + 1 \quad z_1\, z_2^2\, z_3)$. Finally, if we denote by

$$U_3 = \begin{pmatrix} 1 & -z_1^2\, z_2^2 - 1 & -z_1\, z_2^2\, z_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we then have $R\,U = (1 \quad 0 \quad 0)$, where the unimodular $U = U_1\,U_2\,U_3$ is defined by:

$$U = \begin{pmatrix} -z_1^2\,z_3 & z_1^4\,z_2^2\,z_3 + z_1^2\,z_3 + 1 & z_1^3\,z_2^2\,z_3^2 \\ 1 & -z_1^2\,z_2^2 - 1 & -z_1\,z_2^2\,z_3 \\ z_1^3 & -z_1^3\,(z_1^2\,z_2^2 + 1) & -z_1^4\,z_2^2\,z_+1 \end{pmatrix}. \tag{16}$$

Another easy case is when two components of g satisfies a Bézout identity. Let us suppose that there exist h_1 and $h_2 \in D$ satisfying $g_1 h_1 + g_2 h_2 = 1$. Then, the matrix

$$W = \begin{pmatrix} g_1 & -h_2 & & & \\ g_2 & h_1 & & & \\ g_3 & & 1 & & \\ \vdots & & & \ddots & \\ g_p & & & & 1 \end{pmatrix},$$

satisfies $\det W = 1$ and $f\,W = (1 \quad * \quad f_3 \quad \dots \quad f_p)$.

D. The general algorithm

Let $D = k[x_1, \dots, x_n]$ be a polynomial ring with coefficients in a field k and $f \in D^{1\times p}$ a row vector which admits a right-inverse.

The general algorithm proceeds by induction on the number n of variables x_i. Each inductive step, which simplifies the problem to the case of one variable less, consists of three main parts:

1) Finding a component normalized in the last variable.
2) Computing finitely many local solutions over local rings (local loop).
3) Patching ("glueing") local solutions together in order to obtain a global one.

1) Normalisation Step: The next lemma is useful for Horrocks' theorem.

Lemma 1 ([31], [33]): Let us consider a polynomial $a \in D$ and let us set $m = \deg(a) + 1$, where $\deg(a)$ denotes the total degree of a. Using the following transformation

$$\begin{cases} y_n = x_n, \\ y_i = x_i - x_n^{m^{n-i}}, \end{cases}$$

we then obtain $a(x_1, \dots, x_n) = r\,b(y_1, \dots, y_n)$, where $0 \neq r \in k$ and b is a *monic polynomial* in y_n with coefficients in the ring $k[y_1, \dots, y_{n-1}]$, namely, the leading coefficient of b is 1.

In the case when k is an infinite field, we can achieve the same result by using only a linear change of variables. The normalisation step can be also generalised to the case $D = A[x_1, \dots, x_n]$, where A is is a principal ideal domain ([32]).

2) Local Loop: In the second step, we compute a finite number of local solutions using the so-called Horrocks' theorem.

Theorem 7 ([33]): Let B be a local ring, namely, a ring with only one maximal ideal, and f a row vector which admits a rigt-inverse over $B[y]$. If f_i is monic, then f is the first row of a unimodular matrix over $B[y]$.

Horrocks' theorem can be easily implemented using, for instance, the approaches developed in [11], [31], [34]. If $\mathcal{M}$ is a maximal ideal of D, we denote by $D_{\mathcal{M}}$ the standard localization of the ring D with respect to the multiplicative set $D\backslash\mathcal{M}$. We can now give the first main part of general algorithm.

Algorithm 1:
- **Input:** Let $f \in k[x_1, \dots, x_n]^{1\times p}$ be a row vector which admits a right-inverse and with a monic component in the last variable x_n.
- **Output:** A finite number of maximal ideals $\{\mathcal{M}_i\}_{i\in I}$ and unimodular matrices $\{U_i\}_{i\in I}$ over the ring $(k[x_1, \dots, x_{n-1}]_{\mathcal{M}_i})[x_n]$ which satisfy

  $$f\,U_i = (1 \quad 0 \quad \dots \quad 0),$$

  and such that the ideal of the denominators of the U_i generate the ring $k[x_1, \dots, x_{n-1}]$.

1) Take an arbitrary maximal ideal $\mathcal{M}_1$ of the ring $k[x_1, \dots, x_{n-1}]$ and using Horrocks' theorem, compute a unimodular matrix U_1 over $(k[x_1, \dots, x_{n-1}]_{\mathcal{M}_1})[x_n]$ such that:

   $$f\,U_1 = (1 \quad 0 \quad \dots \quad 0).$$

2) Let $d_1 \in k[x_1, \dots, x_{n-1}]_{\mathcal{M}_1}$ be the denominator of U_1 and J the ideal in $k[x_1, \dots, x_{n-1}]$ generated by d_1. Set $i = 1$.
3) While $J \neq k[x_1, \dots, x_{n-1}]$, do:
   a) $i := i + 1$.
   b) Compute a maximal ideal $\mathcal{M}_i$ of $k[x_1, \dots, x_{n-1}]$ such that $J \subset \mathcal{M}_i$.
   c) Using Horrocks' theorem, compute a matrix U_i over the ring $(k[x_1, \dots, x_{n-1}]_{\mathcal{M}_i})[x_n]$ such that $\det(U_i)$ is invertible in $k[x_1, \dots, x_{n-1}]_{\mathcal{M}_i}[x_n]$ and such that:

      $$f\,U_i = (1 \quad 0 \quad \dots \quad 0).$$

   d) Let d_i be the denominator of the matrix U_i and consider the ideal $J = (d_1, \dots, d_i)$.

The local loop stops when all the denominators d_i generate $k[x_1, \ldots, x_{n-1}]$. As the ring $k[x_1, \ldots, x_{n-1}]$ is noetherian, the number of the local solutions is finite.

3) Patching: We have the following lemma.

Lemma 2 ([11]): Let f be a row vector which admits a right-inverse over $D = k[x, y]$ and U a unimodular over $(k[x]_{\mathcal{M}})[y]$, for some maximal ideal $\mathcal{M}$ of $k[x]$, which satisfies $f\, U = (1 \quad 0 \quad \ldots \quad 0)$. Let d be the denominator of U. Then, there exists a matrix Δ such that

$$\forall\, z \in D: \quad f(x, y)\, \Delta(y, z) = f(x, y + z),$$

where the common denominator of Δ is d^α and $0 \leq \alpha \leq p$.

We can now state the main result ([11], [31], [33], [34]).

Theorem 8: Let $f \in D^{1\times p}$ be a row vector which admits a right-inverse over the ring $D = k[x_1, \ldots, x_n]$. Then, for every $a \in k$, there exists a matrix $U \in \mathrm{GL}_p(D)$ such that:

$$f(x_1, \ldots, x_n)\, U = f(x_1, \ldots, x_{n-1}, a).$$

Applying Theorem 8, we then find a matrix $U \in \mathrm{GL}_p(D)$ such that $f\, U = (1 \quad 0 \quad \ldots \quad 0)$, which solves the problem and allows us to compute bases of free D-modules. All the previous algorithms have been implemented in OREMODULES [2].

IV. APPLICATIONS TO FLAT MULTIDIMENSIONAL LINEAR SYSTEMS

We obtain the following corollary Theorem 8.

Corollary 3: A flat multidimensional linear system with constant coefficients is algebraically equivalent to a controllable 1-D linear system with constant coefficients obtained by setting all but one functional operators to 0.

Let us illustrate Corollary 3 on two examples.

Example 4: Let us consider the following differential time-delay linear system [15].:

$$\left\{ \begin{array}{l} \dot{y}_1(t) - y_1(t-h) + 2\, y_1(t) + 2\, y_2(t) - 2\, u(t-h) = 0, \\ \dot{y}_1(t) + \dot{y}_2(t) - \dot{u}(t-h) - u(t) = 0. \end{array} \right. \tag{17}$$

Let us define the ring $D = \mathbb{Q}\left[\frac{d}{dt}, \delta\right]$ of differential time-delay operators with constant coefficients and the matrix of operators which defines (17):

$$R = \left(\begin{array}{ccc} \frac{d}{dt} - \delta + 2 & 2 & -2\,\delta \\ \frac{d}{dt} & \frac{d}{dt} & -\frac{d}{dt}\,\delta - 1 \end{array} \right) \in D^{2\times 3}.$$

Using the algorithms developed in [3], [21], we can easily check that R admits a right-inverse defined by

$$S = \left(\begin{array}{cc} 0 & 0 \\ \frac{1}{2}\left(\frac{d}{dt}\,\delta + 1\right) & -\delta \\ \frac{1}{2}\,\frac{d}{dt} & -1 \end{array} \right),$$

a fact which proves that $M = D^{1\times 3}/(D^{1\times 2}\, R)$ is a projective D-module, and thus, free.

By Corollary 3, we know that the flat differential time-delay system (17) is algebraically equivalent to the controllable ordinary differential linear system defined by

$$\left\{ \begin{array}{l} \dot{z}_1(t) + 2\, z_1(t) + 2\, z_2(t) = 0, \\ \dot{z}_1(t) + \dot{z}_2(t) - v(t) = 0, \end{array} \right. \tag{18}$$

obtained by setting δ to 0 in the matrix R. A few computations show that an invertible transformation which bijectively maps the trajectories of (17) onto the ones of (18) is defined by:

$$\left\{ \begin{array}{rl} y_1(t) = & z_1(t), \\ y_2(t) = & \frac{1}{2}\left(\dot{z}_1(t-2\,h) + z_1(t-h)\right) + z_2(t) \\ & +v(t-h), \\ u(t) = & \frac{1}{2}\,\dot{z}_1(t-h) + v(t). \end{array} \right. \tag{19}$$

$$\Leftrightarrow \left\{ \begin{array}{l} z_1(t) = y_1(t), \\ z_2(t) = -\frac{1}{2}\, y_1(t-h) + y_2(t) - u(t-h), \\ v(t) = -\frac{1}{2}\,\dot{y}_1(t-h) + u(t), \end{array} \right.$$

The ordinary differential system (18) is now equivalent to the following purely algebraic system

$$\left\{ \begin{array}{l} 2\, x_1(t) + 2\, x_2(t) = 0, \\ -w(t) = 0, \end{array} \right. \tag{20}$$

obtained by setting to δ and $\frac{d}{dt}$ to 0 in R. We can check that an invertible transformation which bijectively maps the trajectories of (18) onto the ones of (20) is defined by:

$$\left\{ \begin{array}{l} z_1(t) = x_1(t), \\ z_2(t) = x_2(t) - \frac{1}{2}\,\dot{x}_1(t), \\ v(t) = w(t) - \frac{1}{2}\,\ddot{x}_1(t) + \dot{x}_1(t) + \dot{x}_2(t), \end{array} \right. \tag{21}$$

$$\Leftrightarrow \left\{ \begin{array}{l} x_1(t) = z_1(t), \\ x_2(t) = z_2(t) + \frac{1}{2}\,\dot{z}_1(t), \\ w(t) = v(t) + \dot{z}_1(t) + \dot{z}_2(t), \end{array} \right.$$

Combining (19) and (21), we obtain a one-to-one correspondence between the solutions of (17) and (20).

We note that the solutions of (17) (resp., (18)) are parametrized by means of (19) (resp., (21)), where z_1, z_2 and v (resp., x_1, x_2 and w) are not arbitrary functions as they must satisfy (18) (resp., (20)). However, solving the algebraic system (20), we obtain that $x_2 = -x_1$ and $w = 0$. Substituting these values in (21) and substituting the result into (19), we finally find the following injective parametrization of (17):

$$\left\{\begin{array}{ll} y_1(t) = & x_1(t), \\ y_2(t) = & \frac{1}{2}\,(-\ddot{x}_1(t-h) + \dot{x}_1(t-2\,h) - \dot{x}_1(t) \\ & +x_1(t-h) - 2\,x_1(t)), \\ u(t) = & \frac{1}{2}\,(\dot{x}_1(t-h) - \ddot{x}_1(t)). \end{array}\right. \tag{22}$$

The previous results prove that we have the following split exact sequence of D-modules

$$0 \longrightarrow \quad D^{1\times 2} \quad \underset{\xleftarrow{.S}}{\xrightarrow{.R}} \quad D^{1\times 3} \quad \underset{\xleftarrow{.T}}{\xrightarrow{.Q}} \quad D \longrightarrow 0 \tag{23}$$

with the notations:

$$\left\{\begin{array}{l} Q = \frac{1}{2}\left(\begin{array}{c} 2 \\ -\frac{d^2}{dt^2}\,\delta + \frac{d}{dt}\,\delta^2 - \frac{d}{dt} + \delta - 2 \\ \frac{d}{dt}\,\delta - \frac{d^2}{dt^2} \end{array}\right) \in D^{3\times 1}, \\ T = (1 \quad 0 \quad 0) \in D^{1\times 3}. \end{array}\right.$$

From (23), we easily check that we have

$$M = D^{1\times 3}/(D^{1\times 2}\,R) \cong (D^{1\times 3}\,Q) = D,$$

i.e., we find again that M is a free D-module of rank 1. Now, if $\mathcal{F}$ is a D-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$) and if we apply the functor $\hom_D(\cdot, \mathcal{F})$ to the split exact sequence (23), we then obtain the following split exact sequence of D-modules [31]:

$$0 \longleftarrow \quad \mathcal{F}^2 \quad \underset{\xrightarrow{S.}}{\xleftarrow{R.}} \quad \mathcal{F}^3 \quad \underset{\xrightarrow{T.}}{\xleftarrow{Q.}} \quad \mathcal{F} \longleftarrow 0.$$

Hence, for any D-module $\mathcal{F}$ (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$), we get that all $\mathcal{F}$-solutions of (17) are parametrized by means of the injective parametrization (22), where $x_1 \in \mathcal{F}$.

We can check that an injective parametrization of (18) is obtained by setting $\delta = 0$ in the matrix of operators defining (22), i.e.:

$$\left\{\begin{array}{l} z_1(t) = \psi(t), \\ z_2(t) = -\frac{1}{2}\,(\ddot{\psi}(t) + 2\,\psi(t)), \qquad \forall\,\psi \in \mathcal{F}. \\ v(t) = -\frac{1}{2}\,\ddot{\psi}(t), \end{array}\right.$$

Similarly, if we set δ and $\frac{d}{dt}$ to 0 in the matrix of operators defining (22), we obtain the following injective parametrization

$$\left\{\begin{array}{l} x_1(t) = \varphi(t), \\ x_2(t) = -\varphi(t), \qquad \forall\,\varphi \in \mathcal{F}. \\ w(t) = 0, \end{array}\right.$$

These last results can be easily explained by applying the functor $(D/(D\,\delta)) \otimes_D \cdot$ (resp., $(D/\left(D\,\delta + D\,\frac{d}{dt}\right)) \otimes_D \cdot$) to the split exact sequence (23).

Finally, we note that transformations (19) and (21) respectively define the following isomorphisms:

$$\begin{array}{rcl} M & = & D^{1\times 3}/\left(D^{1\times 2}\,R\left(\frac{d}{dt}, \delta\right)\right) \\ & & \cong D^{1\times 3}/\left(D^{1\times 2}\,R\left(\frac{d}{dt}, 0\right)\right) \\ & & \cong D^{1\times 3}/(D^{1\times 2}\,R(0,0)). \end{array}$$

Let us consider another differential time-delay example.

Example 5: Let us consider the differential time-delay system of neutral type studied in [12] (a is a real constant):

$$\left\{\begin{array}{l} \dot{x}_1(t) + x_1(t) - u(t) = 0, \\ \dot{x}_2(t) - \dot{x}_2(t-h) - x_1(t) + a\,x_2(t) = 0. \end{array}\right. \tag{24}$$

Let us consider the ring $D = \mathbb{Q}(a)\left[\frac{d}{dt}, \delta\right]$, the system matrix of operators which defines (24)

$$R = \left(\begin{array}{ccc} \frac{d}{dt} + 1 & 0 & -1 \\ -1 & \frac{d}{dt} - \frac{d}{dt}\,\delta + a & 0 \end{array}\right) \in D^{2\times 3},$$

and the D-module $M = D^{1\times 3}/(D^{1\times 2}\,R)$. We can check that R admits a right-inverse defined by

$$S = \left(\begin{array}{cc} 0 & -1 \\ 0 & 0 \\ -1 & -\frac{d}{dt} - 1 \end{array}\right) \in D^{2\times 3},$$

a fact which proves that M is a projective, and thus, a free D-module. By Corollary 4, we know that (24) is equivalent to the following ordinary differential system

$$\left\{\begin{array}{l} \dot{z}_1(t) + z_1(t) - v(t) = 0, \\ \dot{z}_2(t) + a\,z_2(t) - z_1(t) = 0, \end{array}\right. \tag{25}$$

obtained by setting δ to 0 in the matrix R, under the corresponding invertible transformations:

$$\left\{\begin{array}{l} x_1(t) = z_1(t) - \dot{z}_2(t-h), \\ x_2(t) = z_2(t), \\ u(t) = v(t) - \ddot{z}_2(t-h) - \dot{z}_2(t-h), \end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{l} z_1(t) = x_1(t) + \dot{x}_2(t-h), \\ z_2(t) = x_2(t), \\ v(t) = u(t) + \ddot{x}_2(t-h) + \dot{x}_2(t-h). \end{array}\right.$$

Hence, the smooth solutions of the differential time-delay system (24) are in one-to-one correspondence with the one of the ordinary differential system (25).

We can see that we can also set the different functional operators appearing in the system matrix of a flat multidimensional linear system to any particular values. Applying this particular result to the class of flat differential time-delay linear systems, we show that any flat differential time-delay linear system is equivalent to the controllable ordinary differential linear system obtained by setting all the time-delay amplitudes to 0, i.e., to the corresponding ordinary differential system without delays.

Corollary 4: A time-invariant flat differential time-delay linear system is equivalent to the controllable ordinary differential linear system obtained by setting the amplitudes of all the delays to 0, i.e., it is equivalent to the linear system without delays.

Let us illustrate Corollary 4.

Example 6: Let us consider again the flat differential time-delay linear system defined by (17). Applying Corollary 4 on (17), we obtain that (17) is equivalent to the ordinary differential linear system obtained by substituting $h = 0$ into (17), i.e., by setting $\delta = 1$ in the matrix Q defined in Example 4, namely:

$$\left\{ \begin{array}{l} \dot{z}_1(t) + z_1(t) + 2\, z_2(t) - 2\, v(t) = 0, \\ \dot{z}_1(t) + \dot{z}_2(t) - \dot{v}(t) - v(t) = 0. \end{array} \right. \tag{26}$$

We check that the invertible transformation defined by

$$\left\{ \begin{array}{lll} z_1(t) & = & y_1(t), \\ z_2(t) & = & \frac{1}{2}\,(\dot{y}_1(t) - \dot{y}_1(t-h) + y_1(t) - y_1(t-h)) \\ & & +y_2(t) + u(t) - u(t-h), \\ v(t) & = & \frac{1}{2}\,(\dot{y}_1(t) - \dot{y}_1(t-h)) + u(t), \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lll} y_1(t) & = & z_1(t), \\ y_2(t) & = & -\frac{1}{2}\,(\dot{z}_1(t-h) - \dot{z}_1(t-2\,h) \\ & & +z_1(t-h) - z_1(t)) + z_2(t) \\ & & +v(t-h) - v(t), \\ u(t) & = & \frac{1}{2}\,(\dot{z}_1(t-h) - \dot{z}_1(t)) + v(t), \end{array} \right. \tag{27}$$

bijectively maps the trajectories of (17) onto the ones of (26). An injective parametrization of (26) can then be obtained by taking $h = 0$ in (22), i.e.:

$$\left\{ \begin{array}{l} z_1(t) = \psi(t), \\ z_2(t) = -\frac{1}{2}\,(\ddot{\psi}(t) + \psi(t)), \qquad \forall\, \psi \in \mathcal{F}. \\ v(t) = \frac{1}{2}\,(-\ddot{\psi}(t) + \dot{\psi}(t)), \end{array} \right.$$

Example 7: We consider again the differential time-delay system of neutral type defined by (24). As we have already proved that (24) is a flat system, by Corollary 4, we know that (24) is equivalent to the following ordinary differential linear system

$$\left\{ \begin{array}{l} \dot{z}_1(t) + z_1(t) - v(t) = 0, \\ -z_1(t) + a\, z_2(t) = 0, \end{array} \right.$$

obtained by setting $h = 0$ in (24) . A few computations lead to the corresponding invertible transformations:

$$\left\{ \begin{array}{l} x_1(t) = z_1(t) + \dot{z}_2(t) - \dot{z}_2(t-h), \\ x_2(t) = z_2(t), \\ u(t) = v(t) + \ddot{z}_2(t) - \ddot{z}_2(t-h) + \dot{z}_2(t) - \dot{z}_2(t-h), \end{array} \right.$$

$$\Leftrightarrow$$

$$\left\{ \begin{array}{l} z_1(t) = x_1(t) - \dot{x}_2(t) + \dot{x}_2(t-h), \\ z_2(t) = x_2(t), \\ v(t) = u(t) - \ddot{x}_2(t) + \ddot{x}_2(t-h) - \dot{x}_2(t) + \dot{x}_2(t-h). \end{array} \right.$$

In the previous examples, the invertible transformations were easily computed by hand but it is generally not the case for more complicated examples. Hence, we need to use an implementation of constructive versions of the Quillen-Suslin theorem for computing the invertible transformations and the injective parametrizations. A such an implementation has recently been done in Maple which, coupled with the library OREMODULES [2], allows us to effectively handle these difficult computations.

As for the non-linear ordinary differential systems, using the fact that there is a one-to-one correspondence between the trajectories of both systems, we can use stabilizing controllers of the controllable linear ordinary differential system without delays in order to stabilize the flat differential time-delay linear system. This approach echoes of the Smith predictor method. Let us illustrate the main idea on an explicit example (more general ones can be handled in a similar way).

Example 8: The differential time-delay system

$$\dot{x}(t) + x(t-h) = u(t) \tag{28}$$

is flat as we have the injective parametrization of (28):

$$\left\{ \begin{array}{l} x(t) = y(t), \\ u(t) = \dot{y}(t) + y(t-h). \end{array} \right.$$

We easily check that (28) is equivalent to the controllable ordinary differential system obtained by setting $h = 0$

$$\dot{z}(t) + z(t) = v(t) \tag{29}$$

under the following invertible transformation:

$$\left\{ \begin{array}{l} x(t) = z(t), \\ u(t) = v(t) - (z(t) - z(t-h)), \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{l} z(t) = x(t), \\ v(t) = u(t) + (x(t) - x(t-h)). \end{array} \right. \tag{30}$$

The transfer functions of (28) and (29) are defined by:

$$p_1 = \frac{1}{(s + e^{-h\, s})}, \quad p_2 = 1/(s+1).$$

Let us show how to use the unimodular transformation (30) in order to parametrize all the stabilizing controllers of p_1 by means of the ones of p_2. Let us we consider the algebra $A = RH_\infty$ of the proper and stable real rational transfer functions and the Hardy algebra $B = H_\infty(\mathbb{C}_+)$ in the right half plane. See [4], [25], [26], [27], [33] for more details. We recall that A is a $\mathbb{R}$-sub-algebra of B. As $p_2 \in A$, Zames' parametrization of all stabilizing controllers of p_2 then has the form [27], [33]:

$$c_2(q) = \frac{q}{1 + q\, p_2}, \qquad q \in RH_\infty.$$

Now, taking the Laplace transforms of (30), we get:

$$\left\{ \begin{array}{l} \hat{z} = \hat{x}, \\ \hat{v} = \hat{u} + (1 - e^{-h\, s})\, \hat{x}, \end{array} \right.$$

Using the fact that $\hat{v} = c_2(q)\,\hat{z}$, we finally obtain the following stabilizing controllers of p_1:

$$\hat{u} = -(1 - e^{-h\,s} - c_2(q))\,\hat{x}, \quad q \in A.$$

Let us prove that the controller

$$c_1(q) = -(1 - e^{-h\,s} - c_2(q))$$

internally stabilizes p_1. We can check that we have

$$\begin{aligned}\frac{1}{1 - p_1\,c_1(q)} &= \frac{s + e^{-h\,s}}{s + 1 - c_2(q)}\\ &= \frac{(s + e^{-h\,s})}{(s+1)}\,\frac{1}{\left(1 - \frac{c_2(q)}{(s+1)}\right)},\end{aligned}$$

$$\frac{p_1}{1 - p_1\,c_1(q)} = \frac{1}{s + 1 - c_2(q)} = \frac{1}{(s+1)}\,\frac{1}{\left(1 - \frac{c_2(q)}{(s+1)}\right)},$$

and:

$$\begin{aligned}\frac{c_1(q)}{1 - p_1\,c_1(q)} &= -\frac{(s + e^{-h\,s})}{(s+1)}\,\frac{1}{\left(1 - \frac{c_2(q)}{(s+1)}\right)}\\ &\qquad (1 - e^{-h\,s} - c_2(q)),\\ &= -\frac{(s + e^{-h\,s})}{(s+1)}\left(\frac{1 - e^{-h\,s}}{\left(1 - \frac{c_2(q)}{(s+1)}\right)} - \frac{c_2(q)}{\left(1 - \frac{c_2(q)}{(s+1)}\right)}\right).\end{aligned}$$

Then, using the fact that for all $q \in A$, we have

$$\frac{1}{1 - \dfrac{c_2(q)}{(s+1)}}, \quad \frac{c_2(q)}{1 - \dfrac{c_2(q)}{(s+1)}} \in A,$$

as $c_2(q)$ internally stabilizes p_2 and the facts that

$$\frac{(s + e^{-h\,s})}{(s+1)}, \quad 1 - e^{-h\,s} \in B,$$

we finally obtain

$$\frac{1}{(1 - p_1\,c_1(q))}, \quad \frac{p_1}{(1 - p_1\,c_1(q))}, \quad \frac{c_1(q)}{(1 - p_1\,c_1(q))} \in B,$$

which shows $c_1(q)$ internally stabilizes p_1 for all $q \in A$. For more details, see [4], [25], [26], [27], [33]. We note that following [27], we can then find the general Q-parametrization of all stabilizing controllers of p_1.

Now, if we take $q = 0$, then the particular internal stabilizing controller $c_1(0) = -(1 - e^{-h\,s})$ of p_1, i.e.,

$$u(t) = -x(t) + x(t-h), \tag{31}$$

$L_2(\mathbb{R}_+) - L_2(\mathbb{R}_+)$-stabilizes system (28). See [4] for more details. We note that a similar result holds if we consider the Wiener algebra $\hat{\mathcal{A}}$ of bounded input bounded output transfer functions [4], [27], [33] instead of $B = H_\infty(\mathbb{C}_+)$. Hence, we also get that the controller defined by (31) $L_\infty(\mathbb{R}_+) - L_\infty(\mathbb{R}_+)$-stabilizes system (28).

Finally, using some results of [27] and the fact that $c_1(0) \in B$, we obtain that p admits the following coprime factorization $p = n/d$:

$$\begin{cases} n = \dfrac{p_1(0)}{(1 - p_1\,c_1(0))} = \dfrac{1}{(s+1)} \in B,\\[2mm] d = \dfrac{1}{(1 - p_1\,c_1(0))} = \dfrac{(s + e^{-h\,s})}{(s+1)} \in B.\end{cases}$$

We easily check that we have the Bézout identity:

$$\frac{(s + e^{-h\,s})}{(s+1)} - (e^{-h\,s} - 1)\,\frac{1}{(s+1)} = 1.$$

We find that the stable controller $c_1(0) = -(1 - e^{-h\,s})$ strongly stabilizes p_1 [27], [33].

V. Pommaret's theorem of the Lin-Bose conjecture

Let $D = k[x_1, \ldots, x_n]$ be a commutative polynomial ring with coefficients in a field, $R \in D^{q\times p}$ a full row rank and $M = D^{1\times p}/(D^{1\times q}\,R)$ the D-module finitely presented by R. Let us suppose that $M/t(M)$ is a free D-module.

Problem 1: Does it exist a full row rank matrix $R' \in D^{q\times p}$ satisfying $M/t(M) = D^{1\times p}/(D^{1\times q}\,R')$? If so, compute such matrices R'.

If we can solve Problem 1, we then have

$$t(M) = (D^{1\times q}\,R')/(D^{1\times q}\,R),$$

and using the fact that $(D^{1\times q}\,R) \subseteq (D^{1\times q'}\,R')$, there exists a matrix $R'' \in D^{q\times q}$ such that:

$$R = R''\,R'. \tag{32}$$

Let us denote by $r = p!/((p-q)!\,q!)$. The fact that $M/t(M)$ is a projective D-module implies that the greatest common divisor of the $q\times q$ minors $\{m'_i\}_{i=1,\ldots,r}$ of R' is 1, and thus, there exists a family $\{p_i\}_{i=1,\ldots,r}$ of D satisfying:

$$\sum_{i=1}^{r} p_i\,m'_i = 1. \tag{33}$$

Now, using the fact that we have $m_i = (\det\,R'')\,m'_i$, for $i = 1, \ldots, r$, where the m_i denote the $q\times q$-minors of R, we obtain that the following inclusion of ideals of D:

$$\sum_{i=1}^{r} D\,m_i \subseteq D\,(\det\,R'').$$

Multiplying relation (33) by $\det\,R''$, we obtain

$$\det\,R'' = \sum_{i=1}^{r} p_i\,(\det\,R'')\,m'_i = \sum_{i=1}^{r} p_i\,m_i,$$

which shows that $D\,(\det\,R'') \subseteq \sum_{i=1}^{r} D\,m_i$ and:

$$\sum_{i=1}^{r} D\,m_i = D\,(\det\,R'').$$

The greatest common divisor of the $q \times q$ minors m_i is then equal to $\det R''$.

Hence, solving Problem 1 gives a way to factorize R under the form $R = R'' R'$, where $R' \in D^{q\times p}$ has full row rank and $\det R''$ is the greatest common divisor of the $q \times q$ minors of R. The question of the possibility to achieve this factorization was first asked by Lin and Bose in [10] and solved by Pommaret in [20]. See also [35] for another proof. It was proved in [20] that this factorization problem is equivalent to Problem 1. The purpose of this paragraph is to give a general constructive algorithm which solves Problem 1, and thus, performs the corresponding factorization. The corresponding algorithm has recently been implemented in OREMODULES.

Based on the Quillen-Suslin theorem, we first prove that a matrix R' satisfying Problem 1 always exists. We then show how to effectively compute it.

The fact that R has full row rank implies that we have the following exact sequence:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \qquad (34)$$

Using the constructive algorithm based on the *extension functor* developed in [3], [21], [22], there exists a matrix $Q \in D^{q'\times p}$ such that $M/t(M) = D^{1\times p}/(D^{1\times q'}\,Q)$. In particular, using the fact that $(D^{1\times q}\,R) \subseteq (D^{1\times q'}\,Q)$, there exists a matrix $P \in D^{q\times q'}$ satisfying $R = P\,Q$. We refer the reader to [2] for the implementation of the corresponding algorithms in the library OREMODULES as well as the large library of examples which demonstrates these results.

We then have the following commutative exact diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & 0 & & t(M) & & \\
 & & \downarrow & & \downarrow i & & \\
0 \longrightarrow & D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} & M & \longrightarrow 0 \\
 & \downarrow .P & & \| & & \downarrow \rho & \\
 & D^{1\times q'} & \xrightarrow{.Q} & D^{1\times p} & \xrightarrow{\pi'} & M/t(M) & \longrightarrow 0. \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

As, by hypothesis the D-module $M/t(M)$ is projective, we obtain that the following exact sequence

$$0 \longrightarrow (D^{1\times q'}\,Q) \longrightarrow D^{1\times p} \xrightarrow{\pi'} M/t(M) \longrightarrow 0 \qquad (35)$$

splits and we obtain

$$D^{1\times p} \cong M/t(M) \oplus (D^{1\times q'}\,Q),$$

which shows that $(D^{1\times q'}\,Q)$ is a projective D-module. By the Quillen-Suslin theorem, we obtain that $(D^{1\times q'}\,Q)$ is then a free D-module.

Let us compute the rank of the free D-module $(D^{1\times q'}\,Q)$. Applying the exact functor $K \otimes_D \cdot$ to the short exact sequence (35), where $K = Q(D)$ denotes the quotient field of D, we obtain (*Euler characteristic*) that:

$$\mathrm{rank}_D(D^{1\times q'}\,Q) = p - \mathrm{rank}_D((M/t(M)).$$

See [31] for more details. Similarly with the two short exact sequences (34) and

$$0 \longrightarrow t(M) \xrightarrow{i} M \xrightarrow{\rho} M/t(M) \longrightarrow 0,$$

and, using the fact that $K \otimes_D t(M) = 0$ because $t(M)$ is a torsion D-module [31], we get:

$$\begin{cases} \mathrm{rank}_D(M) = p - q, \\ \mathrm{rank}_D(M/t(M)) = \mathrm{rank}_D(M). \end{cases}$$

Therefore, we obtain

$$\mathrm{rank}_D(D^{1\times q'}\,Q) = p - (p - q) = q,$$

which shows that $(D^{1\times q'}\,Q)$ is a free D-module of rank q, i.e., $(D^{1\times q'}\,Q) \cong D^{1\times q}$. Computing a basis of this free D-module, we obtain a full row rank matrix $R' \in D^{q\times p}$ satisfying

$$(D^{1\times q'}\,Q) = (D^{1\times q}\,R'), \qquad (36)$$

which implies that $M/t(M) = D^{1\times p}/(D^{1\times q}\,R')$ and we have the following finite free short resolution of $M/t(M)$:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R'} D^{1\times p} \xrightarrow{\pi'} M/t(M) \longrightarrow 0. \qquad (37)$$

We note that if Q has full row rank, we then have $q' = q$ and we can take $R' = Q$.

In order to compute the matrix $R' \in D^{q\times p}$ which satisfies (36), we need to compute a basis of the free D-module $(D^{1\times q'}\,Q)$. If we denote by $Q_2 \in D^{q'_2\times q'}$ a matrix satisfying $\ker_D(.Q) = (D^{1\times q'_2}\,Q_2)$, we then obtain

$$(D^{1\times q'}\,Q) \cong D^{1\times q'}/(D^{1\times q'_2}\,Q_2),$$

and we have the following exact sequence:

$$D^{1\times q'_2} \xrightarrow{.Q_2} D^{1\times q'} \xrightarrow{.Q} D^{1\times p} \xrightarrow{\pi'} M/t(M) \longrightarrow 0. \quad (38)$$

Hence, we are now in position to use the previous results in order to compute a basis of the cokernel D-module $L = D^{1\times q'}/(D^{1\times q'_2}\,Q_2)$, and thus, a basis of $(D^{1\times q'}\,Q)$.

Algorithm 2:
- **Input:** A commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over a computable field k, a full row rank matrix $R \in D^{q\times p}$ and the D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ such that $M/t(M)$ is a free D-module.
- **Output:** A full row rank matrix $R' \in D^{q\times p}$ such that:

$$M/t(M) = D^{1\times p}/(D^{1\times q}\,R').$$

1) Transpose the matrix R and define the D-module:

$$N = D^{1\times q}/(D^{1\times p}\,R^T).$$

2) Compute the D-module $\mathrm{ext}^1_D(N, D)$. We obtain a matrix $Q \in D^{q'\times p}$ such that:

$$M/t(M) = D^{1\times p}/(D^{1\times q'}\,Q).$$

3) Compute the first syzygy module $\ker_D(.Q)$ of $(D^{1\times q'}\,Q)$.

4) If $\ker_D(.Q) = 0$, then Q has full row rank and exit the algorithm with $R' = Q$ and $q' = q$.

5) Denote by $Q_2 \in D^{q'_2 \times q'}$ a matrix satisfying:
$$\ker_D(.Q) = (D^{1\times q'_2}\, Q_2).$$

6) Compute a basis of the free D-module
$$L = D^{1\times q'}/(D^{1\times q'_2}\, Q_2).$$
In particular, we obtain a full row rank matrix $B \in D^{q\times q'}$ such that $L = \pi_2(D^{1\times q}\, B)$, where $\pi_2 : D^{1\times q'} \longrightarrow L$ denotes the canonical projection.

7) Return the full row rank matrix $R' = B\, Q \in D^{q\times p}$.

Remark 1: The computation of a basis of L gives two matrices $P_2 \in D^{q'\times q}$ and $B \in D^{q\times q'}$ such that we have the following split exact sequence

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
D^{1\times q'_2} & \xrightarrow{.Q_2} & D^{1\times q'} & \xrightarrow{\pi_2} & L & \longrightarrow 0 & \\
 & & \| & & \uparrow \phi & & \\
 & & D^{1\times q'} & \underset{\xleftarrow{.B}}{\xrightarrow{.P_2}} & D^{1\times q} & \longrightarrow 0, & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array}$$

where $\phi : D^{1\times q} \longrightarrow L$ denotes the corresponding isomorphism. We can now check that the matrix $R' = B\, Q$ has full row rank. Let $\lambda \in D^{1\times q}$ be such that $\lambda\, R' = 0$. Then, we get $(\lambda\, B)\, Q = 0$, i.e., $\lambda\, B \in \ker_D(.Q) = (D^{1\times q'_2}\, Q_2)$, and thus, there exists $\mu \in D^{1\times q'_2}$ such that $\lambda\, B = \mu\, Q_2$. Using the identity $B\, P_2 = I_q$, we then obtain:
$$\lambda = (\lambda\, B)\, P_2 = \mu\, (Q_2\, P_2) = 0.$$

Example 9: Let us consider the differential time-delay model of a flexible rod with a torque developed in [15]:
$$\left\{ \begin{array}{l} \dot y_1(t) - \dot y_2(t-1) - u(t) = 0, \\ 2\, \dot y_1(t-1) - \dot y_2(t) - \dot y_2(t-2) = 0. \end{array} \right. \tag{39}$$
Let us define the Ore algebra $D = \mathbb{Q}\left[\frac{d}{dt}, \delta\right]$ of differential time-delay operators with rational constant coefficients and the matrix of operators which defines the system (39):
$$R = \left(\begin{array}{ccc} \frac{d}{dt} & -\frac{d}{dt}\,\delta & -1 \\ 2\,\frac{d}{dt}\,\delta & -\frac{d}{dt}\,\delta^2 - \frac{d}{dt} & 0 \end{array} \right).$$
Let $M = D^{1\times 3}/(D^{1\times 2}\, R)$ be the left D-module associated with (39) and $N = D^{1\times 2}/(D^{1\times 3}\, R^T)$. We can check that N admits the following finite free resolution
$$0 \longleftarrow N \xleftarrow{\sigma} D^{1\times 2} \xleftarrow{.R^T} D^{1\times 3} \xleftarrow{.R_2^T} D \longleftarrow 0,$$
where $R_2^T = \left(-\delta^2 - 1 \quad -2\,\delta \quad \frac{d}{dt}\,\delta^2 - \frac{d}{dt}\right)$. The defects of exactness of the following complex
$$0 \longrightarrow D^{1\times 2} \xrightarrow{.R} D^{1\times 3} \xrightarrow{.R_2} D \longrightarrow 0$$
are then given by:
$$\left\{ \begin{array}{l} \mathrm{ext}^0_D(N, D) = \ker_D(.R) = 0, \\ \mathrm{ext}^1_D(N, D) = \ker_D(.R_2)/(D^{1\times 2}\, R), \\ \mathrm{ext}^2_D(N, D) = D/(D^{1\times 3}\, R_2). \end{array} \right.$$
Computing the first syzygy module $\ker_D(.R_2)$ of $(D^{1\times 2}\, R)$, we obtain $\ker_D(.R_2) = (D^{1\times 3}\, Q)$, where the matrix Q is defined by:
$$Q = \left(\begin{array}{ccc} -2\,\delta & \delta^2 + 1 & 0 \\ -\frac{d}{dt} & \frac{d}{dt}\,\delta & 1 \\ \frac{d}{dt}\,\delta & -\frac{d}{dt} & \delta \end{array} \right) \in D^{3\times 3}. \tag{40}$$
We get $t(M) \cong (D^{1\times 3}\, Q)/(D^{1\times 2}\, R)$ and reducing the rows of Q with respect to $(D^{1\times 2}\, R)$, we obtain that the only non-trivial torsion element of M is defined by
$$\left\{ \begin{array}{l} m = -2\,\delta\, y_1 + (\delta^2 + 1)\, y_2, \\ \frac{d}{dt}\, m = 0, \end{array} \right.$$
where y_1, y_2 and y_3 denote the residue classes of the standard basis of $D^{1\times 3}$ in M.

Following Algorithm 2, we compute the first syzygy module $\ker_D(.Q)$ and obtain $\ker_D(.Q) = (D\, Q_2)$, where:
$$Q_2 = \left(\begin{array}{ccc} \frac{d}{dt} & -\,\delta & 1 \end{array} \right). \tag{41}$$
We now have to compute a basis of the free D-module $L = D^{1\times 3}/(D\, Q_2)$. Using a constructive version of the Quillen-Suslin theorem, we obtain the split exact sequence
$$0 \longrightarrow D \underset{\xleftarrow{.S_2}}{\xrightarrow{.Q_2}} D^{1\times 2} \underset{\xleftarrow{.B}}{\xrightarrow{.P_2}} D \longrightarrow 0$$
where:
$$\left\{ \begin{array}{l} S_2 = (0 \quad 0 \quad 1)^T, \\ P_2 = \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \\ \frac{d}{dt} & \delta \end{array} \right), \\ B = \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right). \end{array} \right.$$
Computing $R' = B\, Q$, we obtain that the full row rank matrix
$$R' = \left(\begin{array}{ccc} 2\,\delta & -\delta^2 - 1 & 0 \\ -\frac{d}{dt} & \frac{d}{dt}\,\delta & 1 \end{array} \right)$$
satisfies $(D^{1\times 3}\, Q) = (D^{1\times 2}\, R')$. Finally, we have the factorization $R = R''\, R'$, where the R'' is defined by
$$R'' = \left(\begin{array}{cc} 0 & -1 \\ \frac{d}{dt} & 0 \end{array} \right),$$
and satisfies $\det(R'') = \frac{d}{dt}$, where $\frac{d}{dt}$ is the greatest common divisor of the 2×2 minors of R (or is the operator which annihilates the torsion element m).

Using the fact that $M/t(M)$ is a free D-module of rank $p-q$, i.e., there exists an isomorphism

$$\psi : M/t(M) \longrightarrow D^{1\times(p-q)},$$

and the exact sequence (37), we then obtain the following exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R'} D^{1\times p} \xrightarrow{.P} D^{1\times(p-q)} \longrightarrow 0, \quad (42)$$

where $P \in D^{p\times(p-q)}$ is the matrix defining the morphism $\pi' \circ \psi$ in the standard bases of $D^{1\times p}$ and $D^{1\times(p-q)}$. The exact sequence (42) ends with a free D-module, and thus, it splits, i.e., there exist $S \in D^{p\times q}$ and $T \in D^{(p-q)\times p}$ such that we have the following two Bézout identities:

$$\begin{pmatrix} R' \\ T \end{pmatrix} (S \quad P) = \begin{pmatrix} I_q & 0 \\ 0 & I_{p-q} \end{pmatrix}, \quad (43)$$

$$(S \quad P) \begin{pmatrix} R' \\ T \end{pmatrix} = I_p. \quad (44)$$

Now, we have

$$\begin{pmatrix} R \\ T \end{pmatrix} = \begin{pmatrix} R''\,R' \\ T \end{pmatrix} = \begin{pmatrix} R'' & 0 \\ 0 & I_{p-q} \end{pmatrix} \begin{pmatrix} R' \\ T \end{pmatrix}$$

and using (43), we obtain that $\det((R'^T \quad T^T)^T) = 1$ and:

$$\det \begin{pmatrix} R \\ T \end{pmatrix} = \det \begin{pmatrix} R''\,R' \\ T \end{pmatrix} = \det R''.$$

Finally, using the fact that we have proved that $\det R''$ is the greatest common divisor of the $q \times q$ minors of the matrix R, we then have solved the following problem.

Problem 2: Let $R \in D^{q\times p}$ be a full row rank matrix such that the ideal $\sum_{i=1}^r D\,m_i$ of D generated by the $q\times q$ minors $\{m_i\}_{i=1,\dots,r}$ of the matrix R satisfies

$$\sum_{i=1}^r D\,m_i = D\,d,$$

where d denotes the greatest common divisor of the $q \times q$ minors of the matrix R. Find a matrix $T \in D^{(p-q)\times p}$ such that we have:

$$\det \begin{pmatrix} R \\ T \end{pmatrix} = d.$$

Such a problem was proved to be equivalent to Problem 1. See [10] for more details.

Algorithm 3:
- **Input:** A commutative polynomial ring $D = k[x_1, \dots, x_n]$ over a computable field k, a full row rank matrix $R \in D^{q\times p}$ such that the ideal of D generated by the $q \times q$ minors $\{m_i\}_{i=1,\dots,r}$ of R satisfies $\sum_{i=1}^r D\,m_i = D\,d$, where d denotes the greatest common divisor of the $q\times q$ minors of R.
- **Output:** A matrix $T \in D^{(p-q)\times p}$ such that:

$$\det \begin{pmatrix} R \\ T \end{pmatrix} = d.$$

1) Transpose the matrix R and define the D-module:

$$N = D^{1\times q}/(D^{1\times p}\,R^T).$$

2) Compute the D-module $\mathrm{ext}^1_D(N, D)$. We obtain a matrix $Q \in D^{q'\times p}$ such that:

$$M/t(M) = D^{1\times p}/(D^{1\times q'}\,Q).$$

3) Compute a basis of the free D-module $M/t(M) = D^{1\times p}/(D^{1\times q'}\,Q)$. In particular, we obtain a full row rank matrix $T \in D^{(p-q)\times p}$ such that $M/t(M) = \pi'(D^{1\times(p-q)}\,T)$, where $\pi' : D^{1\times p} \longrightarrow M/t(M)$ denotes the canonical projection.

4) Return the matrix $U = (R^T \quad T^T)^T$ which satisfies:

$$\det(U) = d.$$

Example 10: We consider again the model of a flexible rod with a torque defined in (39). In Example 9, we have proved that $M/t(M) = D^{1\times 3}/(D^{1\times 3}\,Q)$, where the matrix Q is defined by (40). Let us compute a basis of the free D-module $M/t(M)$. The D-module $M/t(M)$ admits the following free resolution

$$0 \longrightarrow D \xrightarrow{.Q_2} D^{1\times 3} \xrightarrow{.Q} D^{1\times 3} \xrightarrow{\pi'} M/t(M) \longrightarrow 0,$$

where Q_2 is defined by (41). Using the fact that Q_2 admits the right-inverse S_2 defined by (9), we obtain the following minimal free resolution of $M/t(M)$

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.\overline{Q}} D^{1\times 4} \xrightarrow{\pi'\oplus 0} M/t(M) \longrightarrow 0,$$

where the full row rank matrix $\overline{Q}$ is defined by:

$$\overline{Q} = (Q^T \quad S_2^T)^T.$$

Applying a constructive version of the Quillen-Suslin theorem to $\overline{Q}$, we then find that a basis of $M/t(M)$ is given by $(\pi' \oplus 0)(\overline{T})$, where $\overline{T}$ denotes the matrix:

$$\overline{T} = \begin{pmatrix} 1 & \frac{1}{2}\,\delta & 0 & 0 \end{pmatrix}.$$

If we denote by T the matrix defined by the three first entries of $\overline{T}$, we then obtain a square matrix $U = (R^T \quad T^T)^T$ satisfying $\det(U) = \frac{d}{dt}$.

Finally, the computation of $\mathrm{ext}^1_D(N, D)$ gives a matrix $R_{-1} \in D^{p\times m}$ which satisfies $\ker_D(.R_{-1}) = (D^{1\times q'}\,Q)$, i.e., such that we have the following exact sequence:

$$D^{1\times q'} \xrightarrow{.Q} D^{1\times p} \xrightarrow{.R_{-1}} D^{1\times m}.$$

A direct way to solve Problem 2 exists when the matrix R_{-1} admits a left-inverse $S_{-1} \in D^{m\times p}$. Then, we have $M/t(M) \cong (D^{1\times p}\,R_{-1}) = D^{1\times m}$ and using the fact that $\mathrm{rank}_D(M/t(M)) = p - q$, we get $m = p - q$. Moreover, from the fact that $(D^{1\times q'}\,Q)$ is a free D-module of rank q, there exists $R' \in D^{q\times p}$ satisfying

$(D^{1\times q'}\,Q) = (D^{1\times q}\,R')$. Combining this result with the previous exact sequence, we obtain the split exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R'} D^{1\times p} \xrightarrow{.R_{-1}} D^{1\times (p-q)} \longrightarrow 0,$$

which shows that $P = R_{-1}$ and $T = S_{-1}$ solve Problem 2.

Example 11: Let us consider again the model of a flexible rod with a torque defined in (39) and let us compute $T \in D^{1\times 3}$ such that the determinant of the matrix $(R^T \quad T^T)^T$ equals $\frac{d}{dt}$. In Example 9, we proved that we have the following exact sequence

$$D^{1\times 3} \xrightarrow{.Q} D^{1\times 3} \xrightarrow{.R_2} D,$$

where $R_2 = \left(-\delta^2 - 1 \quad -2\,\delta \quad \frac{d}{dt}\,\delta^2 - \frac{d}{dt}\right)^T$. The matrix R_2 admits a left-inverse T defined by

$$T = \left(1 \quad -\frac{1}{2}\,\delta \quad 0\right),$$

which proves that $M/t(M)$ is a free D-module of rank 1 as we have the following isomorphisms:

$$M/t(M) = D^{1\times 3}/(D^{1\times 3}\,Q) \cong (D^{1\times 3}\,R_2) \cong D.$$

We finally obtain that the matrix defined by

$$U = \left(\begin{array}{c} R \\ T \end{array}\right) = \left(\begin{array}{ccc} \frac{d}{dt} & -\frac{d}{dt}\,\delta & -1 \\ 2\,\frac{d}{dt}\,\delta & -\frac{d}{dt}\,\delta^2 - \frac{d}{dt} & 0 \\ 1 & -\frac{1}{2}\,\delta & 0 \end{array}\right)$$

satisfies $\det U = \frac{d}{dt}$.

Example 12: Let us consider the commutative polynomial ring $D = \mathbb{Q}[z_1, z_2, z_3]$ and the following matrix defined in [36]:

$$R = \left(\begin{array}{ccc} z_1\,z_2^2\,z_3 & 0 & -z_1^2\,z_2^2 - 1 \\ z_1^2\,z_3^2 + z_3 & -z_3 & -z_1^3\,z_3 - z_1 \end{array}\right) \in D^{2\times 3}.$$

Let us define the D-modules $M = D^{1\times 3}/(D^{1\times 2}\,R)$ and $N = D^{1\times 2}/(D^{1\times 3}\,R^T)$. Computing $\mathrm{ext}^1_D(N, D)$, we get

$$\left\{\begin{array}{l} t(M) = (D^{1\times 4}\,Q)/(D^{1\times 2}\,R), \\ M/t(M) = D^{1\times 3}/(D^{1\times 4}\,Q), \\ M/t(M) \cong (D^{1\times 3}\,P), \end{array}\right.$$

with the notations:

$$Q = \left(\begin{array}{ccc} -z_2^2\,z_3 & z_2^2\,z_3 & z_1\,z_2^2 - z_1\,z_3 \\ -z_3 - z_1^2\,z_3^2 & z_3 & z_1 + z_1^3\,z_3 \\ -z_1^2\,z_3 - 1 & z_1^2\,z_2^2 + 1 & 0 \\ 0 & z_1\,z_2^2\,z_3 & -z_1^2\,z_3 - 1 \end{array}\right), \tag{45}$$

$$P = \left(\begin{array}{c} z_1^2\,z_2^2 + 1 \\ z_1^2\,z_3 + 1 \\ z_1\,z_2^2\,z_3 \end{array}\right).$$

Reducing the rows of Q with respect to the rows of R, we obtain that the only torsion element of M is defined by

$$\left\{\begin{array}{l} m = -(z_1^2\,z_3 + 1)\,y_1 + (z_1^2\,z_2^2 + 1)\,y_2, \\ z_3\,m = 0, \end{array}\right.$$

where y_1, y_2 and y_3 denote the residue classes of the standard basis of $D^{1\times 3}$ in M. We refer the reader to [2] for more details concerning the explicit computations.

We easily check that P admits the left-inverse

$$T = (-z_1^2\,z_3 \quad 1 \quad z_1^3),$$

which shows that $M/t(M)$ is a free D-module of rank 2. We obtain that the matrix $U = (R^T \quad T^T)^T$ defined by

$$U = \left(\begin{array}{ccc} z_1\,z_2^2\,z_3 & 0 & -z_1^2\,z_2^2 - 1 \\ z_1^2\,z_3^2 + z_3 & -z_3 & -z_1^3\,z_3 - z_1 \\ -z_1^2\,z_3 & 1 & z_1^3 \end{array}\right)$$

satisfies $\det U = z_3$, which solves Problem 2.

Let us solve Problem 1. From the previous result, we know that $\ker_D(.P) = (D^{1\times 4}\,Q)$ is a free D-module of rank 2. In order to be able to apply a constructive version of the Quillen-Suslin theorem, we first need to compute the first syzygy module of $(D^{1\times 4}\,Q)$. We obtain that $\ker_D(.Q) = (D^{1\times 2}\,Q_2)$, where the matrix $Q_2 \in D^{2\times 4}$ is defined by:

$$Q_2 = \left(\begin{array}{cccc} z_1^2\,z_3 + 1 & z_3 - z_2^2 & -z_3^2 & 0 \\ 0 & 1 & -z_3 & z_1 \end{array}\right).$$

Hence, we have:

$$(D^{1\times 4}\,Q) \cong L = D^{1\times 4}/(D^{1\times 2}\,Q_2).$$

Apply a constructive version of the Quillen-Suslin theorem, we obtain $L = \pi_2(D^{1\times 2}\,B)$, where the full row rank matrix B is defined by

$$B = \left(\begin{array}{cccc} z_1^4 & 0 & -z_1^2\,z_3 + 1 & 0 \\ 0 & z_1^3\,z_3\,(z_2^2 - z_3) & 0 & 1 \end{array}\right)$$

and $\pi_2 : D^{1\times 2} \longrightarrow L$ denotes the canonical projection. Hence, we obtain that the full row rank matrix defined by

$$R' = B\,Q = \left(\begin{array}{ccc} R'_{11} & R'_{12} & R'_{13} \\ R'_{21} & R'_{21} & R'_{23} \end{array}\right) \in D^{2\times 3},$$

where

$$\left\{\begin{array}{l} R'_{11} = -z_1^4\,z_2^2\,z_3 + z_1^4\,z_3^2 - 1, \\ R'_{12} = z_1^2\,z_2^2 - z_1^2\,z_3 + 1, \\ R'_{13} = z_1^5\,(z_2^2 - z_3), \\ R'_{21} = -z_1^3\,z_3^2\,(z_2^2 - z_3)\,(z_1^2\,z_3 + 1), \\ R'_{21} = -z_1^3\,z_3^3 + z_1^3\,z_2^2 + z_1\,z_2^2\,z_3, \\ R'_{23} = -z_1^4\,z_3^2 - z_1^6\,z_3^2 + z_1^4\,z_2^2\,z_3 + z_1^6\,z_2^2\,z_3^2 - z_1^2\,z_3 - 1, \end{array}\right.$$

satisfies $(D^{1\times 4}\,Q) = (D^{1\times 2}\,R')$ and the two independent rows of R' define a basis of $(D^{1\times 4}\,Q)$. Finally, we obtain that $R = R''\,R'$, where the matrix R'' is defined by

$$\left(\begin{array}{cc} -z_1\,z_2^2\,z_3 - z_1^3\,z_2^2\,z_3^2 + z_1^3\,z_3^3 & z_1^2\,z_2^2 - z_1^2\,z_3 + 1 \\ -z_1^2\,z_3^2 - z_3 & z_1 \end{array}\right)$$

and $\det(R'') = z_3$, which solves Problem 1.

Finally, we note that we can use the fact that P has a full column rank in order to also solve Problem 1. Indeed, we can use a constructive version of the Quillen-Suslin theorem to compute a basis of $\ker_D(.P)$. If we transpose the column vector P, we then obtain the row vector defined in Example 3. Hence, if we take the last two rows of U^T, where U is the unimodular matrix defined in (16), we obtain that a full row rank R_2' defined by

$$\left(\begin{array}{ccc} 1+z_1^4\,z_2^2\,z_3+z_1^2\,z_3 & -z_1^2\,z_2^2-1 & -z_1^3\,(z_1^2\,z_2^2+1) \\ z_1^3\,z_3^2\,z_2^2 & -z_1\,z_2^2\,z_3 & -z_1^4\,z_2^2\,z_3+1 \end{array}\right), \tag{46}$$

satisfies $(D^{1\times 4}\,Q) = (D^{1\times 2}\,R_2')$. Finally, we obtain the factorization $R = R_2''\,R_2'$, where:

$$R_2'' = \left(\begin{array}{cc} z_1\,z_2^2\,z_3 & -z_1^2\,z_2^2-1 \\ z_3 & -z_1 \end{array}\right), \quad \det(R_2'') = z_3.$$

Finally, we point out that similar results as the ones developed in this section hold if we replace the (computable) field k by a (computable) integral domain A. This straightforward generalization is let to the interested readers.

VI. (Weakly) doubly coprime factorizations

We now turn out to another application of the constructive proofs of the Quillen-Suslin theorem in multidimensional systems theory, namely, the problem of finding (weakly) left-/right-/doubly coprime factorizations of rational transfer matrices over the commutative polynomial ring $k[x_1,\ldots,x_n]$ with coefficients in a field k. The general problem of the existence of (weakly) left-/right-/doubly coprime factorizations for linear systems was recently studied with care in [25], [26]. Let us recall a few definitions.

Definition 7 ([25]): Let D be a commutative integral domain, $K = Q(D) = \{n/d \mid 0 \neq d,\, n \in D\}$ the quotient field of D and $P \in K^{q\times r}$ a transfer matrix.

1) A *fractional representation* of P is any representation of P of the form
$$P = D_P\,N_P^{-1} = \widetilde{N}_P\,\widetilde{D}_P^{-1},$$
where
$$\left\{\begin{array}{l} R = (D_P \quad -N_P) \in D^{q\times(q+r)}, \\ \widetilde{R} = \left(\begin{array}{c} \widetilde{N}_P \\ \widetilde{D}_P \end{array}\right) \in D^{(q+r)\times r}, \end{array}\right. \tag{47}$$
i.e., the entries of the matrices R and $\widetilde{R}$ belong to the ring D.

2) A fractional representation $P = D_P^{-1}\,N_P$ of P is called a *weakly left-coprime factorization* of P if:
$$\forall\,\lambda \in K^{1\times q}:\ \lambda\,R \in D^{1\times(q+r)} \ \Rightarrow\ \lambda \in D^{1\times q}.$$

3) A fractional representation $P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$ is called a *weakly right-coprime factorization* of P if we have:
$$\forall\,\lambda \in K^{r}:\ \widetilde{R}\,\lambda \in D^{(q+r)} \ \Rightarrow\ \lambda \in D^{r}.$$

4) A fractional representation
$$P = D_P^{-1}\,N_P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$$
is called a *weakly doubly coprime factorization* of P if $P = D_P^{-1}\,N_P$ is a weakly left-coprime factorization of P and $P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$ is a weakly right-coprime factorization of P.

5) A fractional representation $P = D_P^{-1}\,N_P$ of P is called a *left-coprime factorization* of P if R admits a right-inverse over D, i.e., if there exists a matrix $S = (X^T \quad Y^T)^T \in D^{(q+r)\times q}$ such that:
$$R\,S = D_P\,X - N_P\,Y = I_q.$$

6) A fractional representation $P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$ of P is called a *right-coprime factorization* of P if $\widetilde{R}$ admits a left-inverse over D, namely, if there exists a matrix $\widetilde{S} = (-\widetilde{Y} \quad \widetilde{X}) \in D^{r\times(q+r)}$ such that:
$$\widetilde{S}\,\widetilde{R} = -\widetilde{Y}\,\widetilde{N}_P + \widetilde{X}\,\widetilde{D}_P = I_r.$$

7) A fractional representation
$$P = D_P^{-1}\,N_P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$$
is called a *doubly coprime factorization* of P if $P = D_P^{-1}\,N_P$ is a left-coprime factorization of P and $P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$ is a right-coprime factorization of P.

The next definition will play an important role in what follows.

Definition 8 ([25]): Let $R \in D^{q\times p}$ be a full row rank matrix. We call *D-closure* $\overline{(D^{1\times q}\,R)}$ of the D-module $(D^{1\times q}\,R)$ in $D^{1\times p}$ the D-module defined by:
$$\overline{(D^{1\times q}\,R)} = \{\lambda \in D^{1\times p} \mid \exists\, 0 \neq d \in D:\ d\,\lambda \in (D^{1\times q}\,R)\}.$$

We have the following important results.

Proposition 3 ([25]): Let $R \in D^{q\times p}$ be a full row rank matrix and $M = D^{1\times p}/(D^{1\times q}\,R)$ the D-module finitely presented by R. We then have:

1) $\overline{(D^{1\times q}\,R)} = (K^{1\times q}\,R) \cap D^{1\times p}$, where K denotes the quotient field of D.

2) The following equalities hold:
$$\left\{\begin{array}{l} t(M) = ((K^{1\times q}\,R) \cap D^{1\times p})/(D^{1\times q}\,R), \\ M/t(M) = D^{1\times p}/((K^{1\times q}\,R) \cap D^{1\times p}). \end{array}\right.$$

The next theorem gives necessary and sufficient conditions for the existence of a (weakly) left-/right-/doubly coprime factorization of a transfer matrix.

Theorem 9 ([25]): Let $P \in K^{q\times(q+r)}$ and $P = D_P^{-1} N_P = \widetilde{N}_P \widetilde{D}_P^{-1}$ be any fractional representation of P, where the matrices R and $\widetilde{R}$ are defined by (47).

1) P admits a weakly left-coprime factorization iff the D-module $\overline{(D^{1\times q} R)}$ is free of rank q.

2) P admits a weakly right-coprime factorization iff the D-module $\overline{(D^{1\times r} \widetilde{R}^T)}$ is free of rank r.

3) P admits a left-coprime factorization iff $\overline{(D^{1\times q} R)}$ is a free D-module of rank q and $D^{1\times(q+r)}/(\overline{(D^{1\times q} R)})$ is a stably free D-module of rank r.

4) P admits a right-coprime factorization iff $\overline{(D^{1\times r} \widetilde{R}^T)}$ is a free D-module of rank r and $D^{1\times(q+r)}/(\overline{(D^{1\times r} \widetilde{R}^T)})$ is a stably free D-module of rank q.

5) P admits a left-coprime factorization iff $D^{1\times(q+r)}/(\overline{(D^{1\times r} \widetilde{R}^T)})$ is a free D-module of rank q.

6) P admits a right-coprime factorization iff $D^{1\times(q+r)}/(\overline{(D^{1\times q} R)})$ is a free D-module of rank r.

Testing freeness of modules is a very difficult issue in (non-commutative) algebra. Hence, using Theorem 9, we deduce that it is generally difficult to check whether or not a transfer matrix $P \in K^{q\times r}$ admits a (weakly) left-/right-/doubly coprime factorization and if so, to compute them. See [25], [26] for results for $D = H_\infty(\mathbb{C}_+)$ or the ring of structural stable multidimensional systems.

However, if we consider the commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over a field k and $K = k(x_1, \ldots, x_n)$ its quotient field, then we can use constructive versions of the Quillen-Suslin theorem in order to effectively compute (weakly) left-/right-/doubly coprime factorizations of a rational transfer matrix. We first note that using Proposition 3 and the computation of extension functors, we can test whether the necessary and sufficient conditions of Theorem 9 are fulfilled. The next algorithm gives a constructive way to compute the corresponding factorizations.

Algorithm 4:
- **Input:** A commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over a computable field k, a fractional representation $P = D_P^{-1} N_P$ of a transfer matrix $P \in K^{q\times p}$ which admits a weakly left-coprime factorization.
- **Output:** A weakly left-coprime factorization of P.

1) Define the matrix $R = (D_P \quad -N_P) \in D^{q\times(q+r)}$ and the D-module:
$$M = D^{1\times(q+r)}/(D^{1\times q} R).$$
2) Transpose the matrix R and define the D-module:
$$N = D^{1\times q}/(D^{1\times(q+r)} R^T).$$
3) Compute the D-module $\mathrm{ext}^1_D(N, D)$. We obtain a matrix $Q \in D^{q'\times(q+r)}$ such that:
$$M/t(M) = D^{1\times(q+r)}/(D^{1\times q'} Q).$$
4) Compute a basis of the free D-module:
$$\overline{(D^{1\times q} R)} = (D^{1\times q'} Q).$$
We obtain a full row rank matrix $R' \in D^{q\times(q+r)}$ such that $(D^{1\times q'} Q) = (D^{1\times q} R')$.
5) Write $R' = (D'_P \quad -N'_P)$ where $D'_P \in D^{q\times q}$ and $N'_P \in D^{q\times r}$. If $\det D'_P \neq 0$, then $P = (D'_P)^{-1} N'_P$ is a weakly left-coprime factorization of P.

Weakly right-coprime factorizations can be similarly obtained by transposition. Let us illustrate Algorithm 4 by means of an example.

Example 13: Let us consider the commutative polynomial ring $D = k[z_1, z_2, z_3]$, $K = k(z_1, z_2, z_3)$ the quotient field of D and the following rational transfer matrix:
$$P = \begin{pmatrix} \frac{z_1^2 z_2^2+1}{z_1 z_2^2 z_3} \\ \frac{z_1^2 z_3+1}{z_1 z_2^2 z_3} \end{pmatrix} \in K^2. \tag{48}$$
Let us check whether or not P admits a weakly left-coprime factorization and if so, let us compute one. We consider the fractional representation $P = D_P^{-1} N_P$ of P obtained by cleaning the denominators of P, where the matrices D_P and $N_P \in D^2$ are defined by:
$$\begin{cases} D_P = \begin{pmatrix} z_1 z_2^2 z_3 & 0 \\ 0 & z_1 z_2^2 z_3 \end{pmatrix} \in D^{2\times 2}, \\ N_P = \begin{pmatrix} z_1^2 z_2^2 + 1 \\ z_1^2 z_3 + 1 \end{pmatrix} \in D^2. \end{cases}$$
We denote by $R = (D_P \quad -N_P) \in D^{2\times 3}$ and define the finitely presented D-modules:
$$M = D^{1\times 3}/(D^{1\times 2} R), \quad N = D^{1\times 2}/(D^{1\times 3} R^T).$$
Computing $\mathrm{ext}^1_D(N, D)$, we then obtain:
$$\begin{cases} t(M) = (D^{1\times 4} Q)/(D^{1\times 2} R), \\ M/t(M) = D^{1\times 3}/(D^{1\times 4} Q), \end{cases}$$
where the matrix Q is defined by (45) in Example 12. Using the results obtained in Example 12, we obtain that

the full row rank matrix $R_2' \in D^{2\times 3}$ defined by (46) satisfies $(D^{1\times 4}\,Q) = (D^{1\times 2}\,R_2')$. Thus, if we denote by

$$\left\{\begin{array}{l} D_P' = \begin{pmatrix} 1+z_1^4\,z_2^2\,z_3+z_1^2\,z_3 & -z_1^2\,z_2^2-1 \\ z_1^3\,z_3^2\,z_2^2 & -z_1\,z_2^2\,z_3 \end{pmatrix}, \\ N_P' = \begin{pmatrix} z_1^3\,(z_1^2\,z_2^2+1) \\ z_1^4\,z_2^2\,z_3-1 \end{pmatrix}, \end{array}\right. \tag{49}$$

$P = (D_P')^{-1}\,N_P'$ is then a weakly left-coprime factorization of P.

Finally, by construction, the D-module

$$M/t(M) = D^{1\times 3}/(D^{1\times 4}\,Q) = D^{1\times 3}/(D^{1\times 2}\,R_2')$$

is torsion-free. We can easily check that $\mathrm{ext}_D^2(N', D) = 0$ and $\mathrm{ext}_D^3(N', D) = 0$, where $N' = D^{1\times 2}/(D^{1\times 3}\,(R_2')^T)$, which shows that $M/t(M)$ is a projective, and thus, a free D-module. Hence, by 3 of Theorem 9, we obtain that $P = (D_P')^{-1}\,N_P'$ is a left-coprime factorization of P. We find that the matrix R_2' admits the right-inverse:

$$\begin{pmatrix} 1 & 0 \\ z_1^2\,z_3 & -z_1^3 \\ 0 & 1 \end{pmatrix}.$$

Therefore, we have $D\,X - N\,Y = I_2$, where:

$$X = \begin{pmatrix} 1 & 0 \\ z_1^2\,z_3 & -z_1^3 \end{pmatrix}, \quad Y = (0 \quad 1).$$

The next algorithm gives a way to compute left-coprime factorization of a transfer matrix. Right-coprime factorizations can be similarly obtained by transposition.

Algorithm 5: • **Input:** A commutative polynomial ring $D = k[x_1, \ldots, x_n]$ over a computable field k, a fractional representation $P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$ of a transfer matrix $P \in K^{q\times p}$ which admits a left-coprime factorization.

- **Output:** A left-coprime factorization of P.

1) Define the matrix $\widetilde{R} = (\widetilde{N}_P^T \quad \widetilde{D}_P^T)^T \in D^{(q+r)\times r}$ and define the D-module:
$$\widetilde{M} = D^{1\times (q+r)}/(D^{1\times r}\,\widetilde{R}^T).$$
2) Define the D-module:
$$\widetilde{N} = D^{1\times r}/(D^{1\times (q+r)}\,\widetilde{R}).$$
3) Compute the D-module $\mathrm{ext}_D^1(\widetilde{N}, D)$. We obtain a matrix $\widetilde{Q}^T \in D^{r'\times (q+r)}$ such that:
$$\widetilde{M}/t(\widetilde{M}) = D^{1\times (q+r)}/(D^{1\times r'}\,\widetilde{Q}^T).$$
4) Compute a basis of the free D-module $\widetilde{M}/t(\widetilde{M})$. We obtain a full column rank matrix
$$\widetilde{L}^T = (D_P' \quad -N_P')^T \in D^{(q+r)\times q},$$
where $D_P' \in D^{q\times q}$ and $N_P' \in D^{q\times r}$, such that we have the following split exact sequence:
$$0 \longleftarrow D^{1\times q} \xleftarrow{.\widetilde{L}^T} D^{1\times (q+r)} \xleftarrow{.\widetilde{Q}^T} D^{1\times r'}.$$
5) Transpose the matrix $\widetilde{L}^T$ to obtain:
$$\widetilde{L} = (D_P' \quad -N_P') \in D^{q\times (q+r)}.$$
If $\det D_P' \neq 0$, then $P = (D_P')^{-1}\,N_P'$ is a left-coprime factorization of P.

Let us illustrate Algorithm 5 by means of an example.

Example 14: We consider again Example 13 and the rational transfer matrix P defined by (48). We have the following trivial fractional $P = \widetilde{N}_P\,\widetilde{D}_P^{-1}$ of P, where:

$$\left\{\begin{array}{l} \widetilde{N}_P = \begin{pmatrix} z_1^2\,z_2^2+1 \\ z_1^2\,z_3+1 \end{pmatrix} \in D^{2\times 2}, \\ \widetilde{D}_P = z_1^2\,z_2^2\,z_3 \in D. \end{array}\right.$$

Let us define the matrix $\widetilde{R} = (\widetilde{N}_P^T \quad \widetilde{D}_P^T)^T$ and the D-modules:

$$\widetilde{M} = D^{1\times (q+r)}/(D^{1\times r}\,\widetilde{R}^T), \quad \widetilde{N} = D^{1\times r}/(D^{1\times (q+r)}\,\widetilde{R}).$$

The row vector $\widetilde{R}^T$ is exactly the one defined in Example 3. Hence, using the results obtained in Example 3, we obtain that the unimodular matrix U defined by (16) satisfies $\widetilde{R}^T\,U = (1 \quad 0 \quad 0)$. Hence, selecting the last two columns of U and transposing the corresponding matrix, we then find again the matrix R_2' defined by (46). Hence, using Example 13, we obtain that $P = (D_P')^{-1}\,N_P'$ is a weakly left-coprime factorization of P, where the matrices D_P' and N_P' are defined by (49).

Finally, we point out that similar results as the ones developed in this section hold if we replace the (computable) field k by a (computable) integral domain A. This straightforward generalization is let to the interested readers.

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