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Constructive Computation of Bases of Free Modules over the Weyl Algebras

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Abstract: A well-known result due to J. T. Stafford asserts that a stably free left module M over the Weyl algebras $D = A_n(k)$ or $B_n(k)$ – where k is a field of characteristic 0 – with $\mathrm{rank}_D(M) \geq 2$ is free. The purpose of this paper is to present a new constructive proof of this result as well as an effective algorithm for the computation of bases of M. This algorithm, based on the new constructive proofs [11, 14] of J. T. Stafford's result on the number of generators of left ideals of D, performs Gaussian elimination on the formal adjoint of the presentation matrix of M. We show that J. T. Stafford's result is a particular case of a more general one asserting that a stably free left D-module M with $\mathrm{rank}_D(M) \geq \mathrm{sr}(D)$ is free, where $\mathrm{sr}(D)$ denotes the stable range of a ring D. This result is constructive if the stability of unimodular vectors with entries in D can be tested. Finally, an algorithm which computes the left projective dimension of a general left D-module M defined by means of a finite free resolution is presented. It allows us to check whether or not the left D-module M is stably free.

Key-words: Stably free modules, free modules, constructive computation of bases, projective dimension, Stafford's results, Weyl algebras, flat multidimensional linear systems.

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Calcul effectif de bases de modules libres sur des algèbres de Weyl

Résumé : Un résultat célèbre dû à J. T. Stafford montre qu'un module à gauche M stablement libre sur les algèbres de Weyl $D = A_n(k)$ ou $B_n(k)$ (k est un corps de caractéristique 0) vérifiant $\mathrm{rank}_D(M) \geq 2$ est libre. Le but de ce papier est de donner une nouvelle preuve constructive de ce résultat ainsi qu'un algorithme effectif pour le calcul de bases de M. Cet algorithme, basé sur de nouvelles preuves constructives [11, 14] d'un résultat de J. T. Stafford sur le nombre de générateurs d'un idéal à gauche de D, est une sorte de méthode de pivot de Gauss appliquée à l'adjoint formel de la matrice de présentation de M. Nous montrons que le résultat de J. T. Stafford est un cas particulier d'un résultat plus général montrant qu'un D-module à gauche M stablement libre satisfaisant $\mathrm{rank}_D(M) \geq \mathrm{sr}(D)$ est libre, où $\mathrm{sr}(D)$ désigne le rang stable de l'anneau D. Ce résultat est constructif dès lors que l'on peut tester la stabilité des vecteurs unimodulaires à coefficients dans D. Finalement, nous donnons un algorithme calculant la dimension projective d'un D-module à gauche défini par une résolution libre de type fini. Ce dernier résultat nous permet de vérifier si un D-module à gauche est stablement libre.

Mots-clés : Modules stablement libres, modules libres, calcul effectif de bases, dimension projective, résultats de Stafford, algèbres de Weyl, systèmes linéaires multidimensionnels plats.

1 Introduction

A famous result in non-commutative algebra, due to J. T. Stafford, states that any left ideal of the Weyl algebras $D = A_n(k)$ or $B_n(k)$, where k is a field of characteristic 0, can be generated by means of two elements of D. See [34] for more details. Two constructive proofs of this result recently appeared in the literature of symbolic computation [11, 14]. A well-known consequence of J. T. Stafford's result is that every stably free left D-module M with $\mathrm{rank}_D(M) \geq 2$ is free [34]. As noticed in [7], the recent results of [11, 14] now allow us to pay more attention to constructive versions of this last result, i.e., to constructive computations of bases of stably free left D-modules which are not isomorphic to left ideals of D. In particular, following the non-constructive proof given by J. T. Stafford, an algorithm has been obtained in [7]. However, we feel that this algorithm is rather involved and the purpose of this paper is to give a simple algorithm which is essentially nothing else than Gaussian elimination performed on the formal adjoint of a minimal presentation matrix of the stably free left D-module M. By minimal presentation matrix of a stably free left D-module M we mean a matrix $R \in D^{q\times p}$ which admits a right-inverse $S \in D^{p\times q}$, i.e. $R\,S = I_q$, and satisfies that $M \cong D^{1\times p}/(D^{1\times q}\,R)$, where $D^{1\times p}$ denotes the left D-module formed by the row vectors of length p with entries in D. Simplifying a result in [7], we give an algorithm which computes such a minimal presentation matrix of a left D-module M defined by means of a finite free resolution. In particular, this algorithm allows us to compute the left projective dimension of any left D-module M defined by a finite free resolution. Implementations of all these algorithms have recently been realized in the package Stafford [31] based on the OreModules library [2]. See also [3] for more details and examples. In particular, this allows us to compute some bases of free left D-modules which are not isomorphic to left ideals of D.

More generally, it is known that a stably free left module M over a ring D with $\mathrm{rank}_D(M) \geq \mathrm{sr}(D)$ is free, where $\mathrm{sr}(D)$ denotes the *stable range* of D [20]. We present a general algorithm which computes bases of free left D-modules. This algorithm was inspired by a result in [15] obtained for commutative rings. If the stability of unimodular vectors with entries in D can be checked effectively, then the algorithm becomes constructive. We note that J. T. Stafford's result on the number of generators of left ideals of the Weyl algebras [34] shows that $\mathrm{sr}(A_n(k)) = 2$ and $\mathrm{sr}(B_n(k)) = 2$, where k is a field of characteristic 0. In the same vein, it is also known that stably free left D-modules M with $\mathrm{rank}_D(M) \geq \mathrm{Kdim}(D) + 1$ are free, where $\mathrm{Kdim}(D)$ denotes the *Krull dimension* of D [20]. Using different results existing in the literature about Krull dimension [20], we finally give some upper bounds on the rank of stably free left modules over some classes of *Ore algebras* of functional operators [1] ensuring that they are free.

We have recently given in [3] some constructive algorithms which check whether or not finitely presented left modules over some classes of Ore algebras have some torsion elements or are torsion-free, reflexive or projective. These algorithms have been implemented in OreModules [2]. In systems theory, this previous classification of modules allows us to check whether or not an underdetermined linear system over an Ore algebra of functional operators is parametrizable, admits a parametrization which is also parametrizable or admits a chain of n successive parametrizations. These results have some applications in mathematical physics where it is sometimes interesting to know if some field equations derive from some potentials, and in control theory where this problem is also called the image representation problem of behaviours [23, 24, 36]. We refer the reader to [2, 3, 24, 25, 27] for more details and illustrating examples such as Einstein equations, Maxwell equations, linear elasticity, wind tunnel model, electric transmission lines, stirred tank models, Lie-Poisson structures...

However, apart from some special situations, we were not able to give in [3, 25, 27] constructive algorithms which check whether or not a finitely presented left module over an Ore algebra is stably free or free. Hence, the results obtained in this paper allow us to extend the previous classification of linear systems over Ore algebras developed in [3, 25, 27] in terms of the algebraic properties of the associated module. In particular, we shall illustrate the interpretation of stably freeness in the system theoretic language. This result, obtained in [30], was motivated by the problem of "blowing-up" the singularities which can appear in some injective parametrizations of analytic time-varying controllable linear control systems [16]. The problem of generalizing the results of [16] to the case of control systems with multi-inputs was asked in [5]. Based on J. T. Stafford's result, we showed in [30] that this problem was theoretically solved in the case of polynomial coefficients as we can prove that such a system with at least two inputs is *flat* [6], i.e., admits injective parametrizations without singularities. These results are recalled and illustrated in Section 4. Indeed, the concept of a flat linear system over an Ore algebra developed in the literature [6, 16, 21, 24, 25] corresponds to the fact that the module associated with the system is free [3]. A basis of the module then corresponds to a so-called *flat output* of the system. Hence, the algorithms presented in this paper give constructive ways to compute flat outputs of some classes of linear systems over Ore algebras.

The problem of recognizing whether or not an underdetermined (linear) system of partial differential equations (PDEs) can be (injectively) parametrized by means of arbitrary functions constitutes the so-called *Monge problem*, which was particularly studied by J. Hadamard and E. Goursat. We refer the reader to [12, 9, 10, 37] for more historical details and for the main contributions of G. Darboux, D. Hilbert and E. Cartan in the case of nonlinear systems of ordinary differential equations (ODEs). Hence, combining the results developed in this paper with the ones in [3, 25] gives constructive solutions to the Monge problem in the case of linear systems of PDEs with polynomial or rational coefficients. To finish, we quote the last paragraph of E. Goursat's introduction of his paper [9]: "Ces résultats sont encore bien particuliers. J'espère cependant qu'ils pourront contribuer à appeler l'attention de quelques jeunes mathématiciens sur un sujet difficile et bien peu étudié." We hope that this paper will contribute to pay more attention to this challenging problem.

The plan of the paper is the following: In Section 2, we recall some useful notations, definitions and results on the duality between systems and modules. In particular, we give some general characterizations of stably free and free modules which will be useful in the rest of the paper. In Section 3, we give a general algorithm which computes the left projective dimension of a left D-module. This algorithm is then used to compute a minimal presentation matrix of a stably free module. Using this minimal presentation matrix, we then explain the concept of stably freeness in the system theoretic language. We give some applications of this interpretation in control theory and illustrate it on two examples. Finally, the problem of the constructive computation of bases of free modules is studied in Section 5 and a general algorithm is presented. We show how this algorithm can be turned effective using the recent results of [11, 14].

2 A module-theoretic classification of linear systems

Let us consider a non-commutative ring D, a left D-module $\mathcal{F}$ and a $q \times p$ matrix R with entries in D, i.e., $R \in D^{q \times p}$. Then, we can define the *system* or *behaviour* [22, 23, 28, 36]

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$$

which is naturally associated with the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ [3, 32]. Indeed, we recall that if we apply the left-exact functor $\hom_D(\cdot, \mathcal{F})$ [3, 32] to the following finite presentation of M, namely, the *exact sequence* [3, 32]

$$\begin{array}{ccccl} D^{1\times q} & \xrightarrow{.R} & D^{1\times p} & \xrightarrow{\pi} M \longrightarrow 0, \\ \lambda & \longmapsto & \lambda\, R & \end{array} \tag{1}$$

we then obtain the following exact sequence

$$\begin{array}{ccccl} \mathcal{F}^q & \xleftarrow{R.} & \mathcal{F}^p & \longleftarrow \hom_D(M, \mathcal{F}) \longleftarrow 0, \\ R\,\eta & \longleftarrow\!\shortmid & \eta & \end{array}$$

where $\hom_D(M, \mathcal{F})$ denotes the abelian group of left D-morphisms from M to $\mathcal{F}$. This implies the following important isomorphism [17, 22]:

$$\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\} \cong \hom_D(M, \mathcal{F}). \tag{2}$$

See [3, 17, 22, 28, 36] and the references therein for more details. In particular, (2) gives an intrinsic characterization of the $\mathcal{F}$-solutions of a linear system over D. It only depends on two objects:

1. The finitely presented left D-module M which represents the equations of the linear system.
2. The left D-module $\mathcal{F}$ which is the functional space in which we seek the solutions.

If D is now a ring of functional operators (e.g., differential operators, time-delay operators, difference operators), then the issue of understanding which $\mathcal{F}$ is suitable for a particular linear system has been studied for a long time in functional analysis and is still a very active subject. It does not seem that constructive algebra and symbolic computation can propose new methods to handle this functional analysis problem. However, they are very useful for classifying $\hom_D(M, \mathcal{F})$ by means of the algebraic properties of the left D-module M. Indeed, a large classification of the properties of modules is developed in homological algebra. See [32] for more information. Before recalling a part of the standard classification, let us introduce the concept of an *Ore ring*.

Definition 1. *[20]* A ring D is said to be a *left Ore ring* if, for every pair $(a_1,\ a_2) \in D^2$, there exists a non-trivial pair $(b_1,\ b_2) \in D^2$ such that $b_1\,a_1 = b_2\,a_2$.

We now recall a few definitions. See [3, 32] for more details.

Definition 2. Let D be a domain which is a left Ore ring and M a finitely generated left D-module. Then, we have:

1. M is *free* if it is isomorphic to $D^{1\times r}$ for a certain $r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$.
2. M is *stably free* if there exist $r, s \in \mathbb{Z}_+$ such that we have $M \oplus D^{1\times s} \cong D^{1\times r}$.
3. M is *projective* if there exist a left D-module N and $r \in \mathbb{Z}_+$ such that $M \oplus N \cong D^{1\times r}$.
4. M is *reflexive* if the canonical map defined by

$$\varepsilon_M : M \longrightarrow \hom_D(\hom_D(M, D), D), \quad \varepsilon_M(m)(f) = f(m),$$

   for all $m \in M, f \in \hom_D(M, D)$, is an isomorphism, where $\hom_D(M, D)$ denotes the right D-module of all D-morphisms from M to D.
5. M is *torsion-free* if the left submodule of M defined by

$$t(M) = \{m \in M \mid \exists\, 0 \neq P \in D :\ P\,m = 0\}$$

   is the zero module. $t(M)$ is called the *torsion submodule* of M and the elements of $t(M)$ are the *torsion elements* of M.
6. M is *torsion* if $t(M) = M$, i.e., every element of M is a torsion element.

With a little abuse of language, we say that a behaviour $\mathcal{B} = \ker_{\mathcal{F}}(R.)$ is *torsion-free* (resp., *reflexive, projective, stably free, free*) if the finitely presented left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is torsion-free (resp., reflexive, projective, stably free, free).

Let us recall some results about the notions given in Definition 2.

Theorem 1. *1. [32] Let D be a domain which is a left Ore ring and M a finitely generated left D-module. Then, we have the following implications among the above concepts:*

$$\text{free} \Rightarrow \text{stably free} \Rightarrow \text{projective} \Rightarrow \text{reflexive} \Rightarrow \text{torsion-free}.$$

2. *[20, 32] If D is a* left hereditary ring *− namely, every left ideal of D is a projective left D-module − then every finitely generated torsion-free left D-module is projective.*

3. *If D is a* left principal ideal domain *− namely, every left ideal of D is principal − then every finitely generated torsion-free left D-module is free.*

4. *[20, 32] (Quillen-Suslin theorem) Every projective module over a commutative polynomial ring with coefficients in a field is free.*

Constructive algorithms which check whether or not a finitely presented left module M over certain classes of *Ore algebras* is respectively torsion-free, reflexive or projective were given in [3, 27]. Moreover, these algorithms have been implemented in the library OreModules [2].

Before recalling these results, we define the concept of an Ore algebra.

Definition 3. 1. [1, 20] Let A be a domain with a unit 1 which is also a k-algebra, where k is a field. The *skew polynomial ring* $A[\partial; \sigma, \delta]$ is the non-commutative ring consisting of all polynomials in ∂ with coefficients in A obeying the commutation rule

$$\forall\, a \in A, \quad \partial\, a = \sigma(a)\, \partial + \delta(a), \tag{3}$$

where σ is a *k-algebra endomorphism of A*, namely, $\sigma : A \rightarrow A$ satisfies

$$\forall\, a,\, b \in A, \quad \begin{cases} \sigma(1) = 1, \\ \sigma(a+b) = \sigma(a) + \sigma(b), \\ \sigma(a\, b) = \sigma(a)\, \sigma(b), \end{cases}$$

and δ is a *σ-derivation of A*, namely, $\delta : A \rightarrow A$ satisfies:

$$\forall\, a,\, b \in A, \quad \begin{cases} \delta(a+b) = \delta(a) + \delta(b), \\ \delta(a\, b) = \sigma(a)\, \delta(b) + \delta(a)\, b. \end{cases}$$

2. [1] Let $A = k[x_1, \ldots, x_n]$ be a commutative polynomial ring over a field k (if $n = 0$ then $A = k$). Then, the iterated skew polynomial ring $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is called *Ore algebra* if the σ_i's and δ_j's commute for $1 \leq i, j \leq m$ and satisfy the following conditions:

$$\forall\, j < i, \quad \sigma_i(\partial_j) = \partial_j, \quad \delta_i(\partial_j) = 0.$$

Let us give some examples of Ore algebras and related algebras.

Example 1. 1. Let k be a field. The *Weyl algebra* $A_n(k)$ is the Ore algebra defined by:

$$A_n(k) = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n], \quad \sigma_i = id_{k[x_1,\ldots,x_n]}, \quad \delta_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n.$$

Equivalently, $A_n(k)$ can be defined as the non-commutative polynomial ring in the $2\, n$ variables x_i and ∂_j, $1 \leq i, j \leq n$, with coefficients in k, satisfying the following commutation relations:

$$x_i\, x_j = x_j\, x_i, \quad \partial_i\, \partial_j = \partial_j\, \partial_i, \quad \partial_i\, x_j = x_j\, \partial_i + \delta_{ij}, \quad 1 \leq i, j \leq n,$$

where δ_{ij} is defined by $\delta_{ij} = 1$ if $i = j$ and 0 else.

In what follows, we shall use the notation $A_1(k) = k[t]\left[\frac{d}{dt}; id_{k[t]}, \frac{d}{dt}\right]$. We can prove that $A_1(k)$ is a hereditary ring [20] (see Definition 1).

By extension, we can define the k-algebra $B_n(k) = k(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ of differential operators with rational coefficients, where σ_i and δ_i are defined as previously. $B_1(k)$ is a left principal ideal domain [20].

2. The Ore algebra of differential time-delay operators with polynomial coefficients is defined by $A_1(k)[\partial_2; \sigma_2, \delta_2]$, where $\delta_2 = 0$ and $\sigma_2(a(t)) = a(t-1)$ for all $a \in k[t]$ and $\sigma_2\left(\frac{d}{dt}\right) = \frac{d}{dt}$.

   Similarly, we can define the k-algebra $B_1(k)[\partial_2; \sigma_2, \delta_2]$ with the same σ_2 and δ_2.

3. The Ore algebra of shift operators with polynomial coefficients is defined by

$$k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n], \quad \delta_i = 0, \quad i = 1, \ldots, n,$$

   and $\sigma_i(a(x_1, \ldots, x_n)) = a(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n)$ for all $a \in k[x_1, \ldots, x_n]$.

   Similarly, we can define the k-algebra $k(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ with the same σ_i and δ_i as defined before.

The next proposition allows us to work effectively in certain classes of Ore algebras.

Proposition 1. [1] Let k be a computable field (e.g., $k = \mathbb{Q}$, $\mathbb{F}_p$), $A = k[x_1, \ldots, x_n]$ the polynomial ring with n indeterminates over k and $A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ an Ore algebra satisfying the conditions

$$\sigma_i(x_j) = a_{ij}\, x_j + b_{ij}, \quad \delta_i(x_j) = c_{ij}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \tag{4}$$

for certain $a_{ij} \in k \setminus \{0\}$, $b_{ij} \in k$, $c_{ij} \in A$. If the c_{ij} are of total degree at most 1 in the x_i's, then a non-commutative version of Buchberger's algorithm terminates for any monomial order on $x_1, \ldots, x_n$, $\partial_1, \ldots, \partial_m$, and its result is a Gröbner basis with respect to the given monomial order.

Proposition 1 holds for the Ore algebras defined in Example 1. Moreover, we can prove that the Ore algebras satisfying the hypotheses of Proposition 1 are *left noetherian rings*, namely, rings over which every left ideal is finitely generated as a left module. In particular, this condition implies that D is a left Ore domain and has the so-called *left invariant basis number*, namely, the property that two bases of a finitely generated free left D-module F have the same cardinality [20, 32]. We call this cardinal the *rank* of the free left D-module F and it is denoted by $\mathrm{rank}_D(F)$.

We now recall the concept of an *involution* of a k-algebra D where k is a field [3].

Definition 4. 1. An *involution* of D is a k-linear map $\theta : D \longrightarrow D$ satisfying:

   (a) $\forall\, P_1,\, P_2 \in D$: $\quad \theta(P_1\, P_2) = \theta(P_2)\, \theta(P_1)$.

   (b) $\theta \circ \theta = id_D$.

2. If $R \in D^{q \times p}$, then we define $\theta(R) = (\theta(R_{ij}))^T \in D^{p \times q}$.

Let us give some examples of involutions.

Example 2. 1. If D is a commutative k-algebra, then $\theta = id_D$ is a trivial involution of D.

2. If $D = A_n(k)$ or $B_n(k)$, then we can define the following involution:

$$\theta(\partial_i) = -\partial_i,\ i = 1, \ldots, n, \quad \theta(x_i) = x_i,\ i = 1, \ldots, n, \quad \forall\, a \in k, \quad \theta(a) = a.$$

We note that if $D = A_n(k)$ or $B_n(k)$, then θ is the involution defined in 2 of Example 2 and if $R \in D^{q\times p}$, then $\theta(R) \in D^{p\times q}$ is usually called the *formal adjoint* of R. See [25] for more details. In what follows, when the involution θ of D is clearly defined, we shall also denote $\theta(R)$ by $\widetilde{R}$.

We are now in position to state some results obtained in [3, 27]. Let us consider an Ore algebra D which satisfies the hypotheses of Proposition 1 and admits an involution θ. Let n be the right global dimension $\mathrm{rgld}(D)$ of D (see Section 3 for more details), $M = D^{1\times p}/(D^{1\times q}\, R)$ a finitely presented left D-module and $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\theta(R))$ its *formal adjoint*. Then, we have the following results [3]:

Module M	**Homological algebra**
with torsion	$t(M) \cong \mathrm{ext}^1_D(\widetilde{N}, D) \neq 0$
torsion-free	$\mathrm{ext}^1_D(\widetilde{N}, D) = 0$
reflexive	$\mathrm{ext}^i_D(\widetilde{N}, D) = 0$, $1 \leq i \leq 2$
...	...
projective	$\mathrm{ext}^i_D(\widetilde{N}, D) = 0$, $1 \leq i \leq n$

We refer to [3, 32] for the definition of the extension modules $\mathrm{ext}^i_D(\widetilde{N}, D)$. Algorithms for computing $\mathrm{ext}^i_D(\widetilde{N}, D)$ are given in [3] and they have been implemented in OreModules [2]. Hence, we can constructively check whether or not the left D-module M admits torsion elements or is torsion-free, reflexive or projective. See [2, 3] for explicit examples.

Moreover, we have the following interpretation of the classification of modules given in Definition 2 in terms of parametrizability of the behaviour $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$.

Theorem 2. *[3, 25] Let D be an Ore algebra which satisfies the hypotheses of Proposition 1 and admits an involution θ. If $\mathcal{F}$ is an injective cogenerator left D-module [3, 22, 32], then we have the following results:*

1. *There exists $Q_1 \in D^{q_1\times q_2}$ such that we have the following exact sequence*

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^{q_1} \xleftarrow{Q_1.} \mathcal{F}^{q_2},$$

i.e., $\ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^{q_2}$, iff the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is torsion-free, where $p = q_1$.

2. *There exist $Q_1 \in D^{q_1\times q_2}$ and $Q_2 \in D^{q_2\times q_3}$ such that we have the following exact sequence*

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^{q_1} \xleftarrow{Q_1.} \mathcal{F}^{q_2} \xleftarrow{Q_2.} \mathcal{F}^{q_3},$$

iff the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ is reflexive, where $p = q_1$.

3. *There exists a chain of n successive parametrizations of* $\ker_{\mathcal{F}}(R.)$, *where n is the right global dimension* $\mathrm{rgld}(D)$ *of D, i.e., there exist $Q_i \in D^{q_i \times q_{i+1}}$, for $i = 1, \ldots, n$, such that we have the following exact sequence*

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^{q_1} \xleftarrow{Q_1.} \mathcal{F}^{q_2} \xleftarrow{Q_2.} \mathcal{F}^{q_3} \xleftarrow{Q_3.} \ldots \xleftarrow{Q_{n-1}.} \mathcal{F}^{q_n} \xleftarrow{Q_n.} \mathcal{F}^{q_{n+1}}, \tag{5}$$

iff the left D-module $M = D^{1\times p}/(D^{1\times q}\, R)$ is projective, where $p = q_1$.

The concept of an injective cogenerator left D-module $\mathcal{F}$ corresponds to a sufficiently rich space of functions. We refer to [3, 22, 32] for a precise definition as we shall not use it in what follows. We can prove that an injective cogenerator left D-module $\mathcal{F}$ exists for every ring D [32]. The reader only needs to keep in mind the following explicit examples.

Example 3. 1. If Ω is an open convex subset of $\mathbb{R}^n$, then the space $C^\infty(\Omega)$ (resp., $\mathcal{D}'(\Omega)$) of smooth functions (resp., distributions) on Ω is an injective cogenerator module over the ring $k[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ of differential operators with coefficients in $k = \mathbb{R}$ or $\mathbb{C}$ [17, 22].

2. [38] If $\mathcal{F}$ denotes the set of all functions that are smooth on $\mathbb{R}$ expect for a finite number of points, then $\mathcal{F}$ is an injective cogenerator left $B_1(\mathbb{R})$-module.

We note that the constructive verification of the vanishing of the $\mathrm{ext}^i_D(\widetilde{N}, D)$ allows us to explicitly compute the matrices Q_i as it is shown in [3, 27]. Therefore, over a large class of algebras of functional operators, which are useful in engineering sciences, the previous results give a constructive way to compute parametrizations of underdetermined linear systems [3, 25]. See [2, 3, 25, 26, 27, 28, 30] for applications of these results in control theory and mathematical physics.

We point out that if $\mathcal{F}$ is any left D-module, then the exact sequences given in Theorem 2 will only be complexes. For instance, if $\mathcal{F}$ is not an injective left D-module but M is a torsion-free left D-module, then we have that $Q_1\, \mathcal{F}^{q_2} \subseteq \ker_{\mathcal{F}}(R.)$, i.e., we can generate a family of $\mathcal{F}$-solutions of the linear system $R\, \eta = 0$, which is sometimes enough for the applications in engineering sciences.

Finally, if $M = D^{1\times p}/(D^{1\times q}\, R)$ is a projective left D-module, where D satisfies the hypotheses of Theorem 2, then (5) is always an exact sequence without any assumption about the left D-module $\mathcal{F}$. This result follows from the fact that the functor $\hom_D(\cdot, \mathcal{F})$ transforms long *split exact sequences* of left D-modules into long split exact sequences of abelian groups [3, 25, 32].

The papers [3, 25, 27] have mainly left open the question of recognizing whether a finitely presented left module M over an Ore algebra is stably free or free. The purpose of this paper is to give some general answers to these questions. In particular, an algorithm for the computation of bases of free modules over some algebras will be presented in Section 5.

Let us state some useful results concerning the relationship between projective and stably free modules.

Proposition 2. *(Proposition 11.1.6 in [20]) A finitely generated projective left D-module M is stably free iff M admits a finite free resolution.*

This result is due to J.-P. Serre. We also have the following interesting proposition.

Proposition 3. *[20] If $D = A[\partial_1; \sigma_1, \delta_1] \ldots [\partial_m; \sigma_m, \delta_m]$ is an Ore algebra where σ_i is an automorphism for $i = 1, \ldots, m$, then every projective left D-module is stably free.*

In particular, Proposition 3 holds for the class of Ore algebras defined in Proposition 1. Hence, the verification of the vanishing of the $\mathrm{ext}^i_D(\widetilde{N}, D)$, for $i = 1, \ldots, n$, checks whether or not a finitely presented left D-module M is stably free when D satisfies the hypotheses given in Proposition 1.

Let us give a characterization of free modules in terms of matrices.

Lemma 1. *Let D be a left noetherian domain and $M = D^{1\times p}/(D^{1\times q}\, R)$ a finitely presented left D-module. Then, M is a free left D-module iff there exist $Q \in D^{p\times m}$ and $T \in D^{m\times p}$ satisfying:*

1. $M \cong D^{1\times p}\, Q$.

2. $T\, Q = I_m$.

Proof. $\Rightarrow$ The fact that D is a left noetherian domain implies that the concept of rank of a free left D-module is well-defined. Hence, using the fact that M is a finitely generated module over a left noetherian domain, there exists an isomorphism $\phi : M \longrightarrow D^{1\times m}$, where $\mathrm{rank}_D(M) = m$. Therefore, we get the following exact sequence

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times m} \longrightarrow 0, \tag{6}$$

where Q is the matrix which represents the D-morphism $\phi \circ \pi$ with respect to the canonical bases of $D^{1\times p}$ and $D^{1\times m}$, where $\pi : D^{1\times p} \longrightarrow M$ denotes the canonical projection onto M (see (1)). Finally, the exact sequence (6) ends with the free left D-module $D^{1\times m}$, and thus, it splits [3, 32]. Therefore, there exists $T \in D^{m\times p}$ such that $T\, Q = I_m$.

$\Leftarrow$ If Q satisfies 1 and 2, then we obtain $M \cong D^{1\times p}\, Q = D^{1\times m}$ as $D^{1\times p}\, Q \subseteq D^{1\times m}$ and, for all $\lambda \in D^{1\times m}$, we have $\lambda = (\lambda\, T)\, Q \in D^{1\times p}\, Q$, which shows $D^{1\times m} \subseteq D^{1\times p}\, Q$. □

Let us give an interpretation of free modules in systems theory. If M is a free left module over a left noetherian domain D, then by Lemma 1, we get the split exact sequence (6). If $\mathcal{F}$ is any left D-module, then, by applying the functor $\hom_D(\cdot, \mathcal{F})$ to (6) and using the fact that $\hom_D(\cdot, \mathcal{F})$ transforms split exact sequences of left D-modules into split exact sequences of abelian groups [3, 32], we obtain the following split exact sequence:

$$\mathcal{F}^q \xleftarrow{R.} \mathcal{F}^p \xleftarrow{Q.} \mathcal{F}^m \longleftarrow 0.$$

Therefore, for every $\eta \in \mathcal{F}^p$ satisfying $R\,\eta = 0$, there exists a unique $\xi \in \mathcal{F}^m$ such that $\eta = Q\,\xi$. In particular, ξ is given by $\xi = T\,\eta$ where $T \in D^{m\times p}$ is a left-inverse of Q, i.e., $T\, Q = I_m$. Such a system/behaviour $\ker_{\mathcal{F}}(R.)$ is said to be *flat* in the control theory literature [6, 16] and ξ is then called a *flat output* of $\ker_{\mathcal{F}}(R.)$. The class of flat systems has been shown to have important applications in control theory and, in particular, for the motion planning, tracking and optimal control problems. We refer the reader to [3, 6, 16, 29] and the references therein for more details and illustrations. Finally, an important issue in the theory of flat systems is to be able to recognize whether a system is flat and, if so, to compute a flat output. In a module-theoretic language, it means to be able to check whether or not a finitely presented left D-module M is free and, if so, to compute a basis of M. The results that we shall present in the next sections will give some constructive answers for some Ore algebras.

3 Minimal free resolutions and projective dimensions

The purpose of this section is to give a constructive algorithm which computes the left projective dimension $\mathrm{lpd}_D(M)$ of a left D-module M defined by means of a finite free resolution. In particular, this algorithm can be used for the Ore algebras D defined in Proposition 3 (see Proposition 6), and thus, for the class of Ore algebras defined in Proposition 1. This result simplifies one obtained in [7]. Finally, we shall use this algorithm in order to test if M is stably free and to compute a minimal free resolution of M which will be of crucial importance in Section 5 for the computation of bases of free left D-modules.

Let us start by recalling the concept of a *projective* and *free resolution* of a left D-module M.

Definition 5. [32] A *projective resolution* of a left D-module M is an exact sequence of the form

$$\ldots \xrightarrow{\delta_{m+1}} P_m \xrightarrow{\delta_m} P_{m-1} \xrightarrow{\delta_{m-1}} P_{m-2} \xrightarrow{\delta_{m-2}} P_{m-3} \xrightarrow{\delta_{m-3}} \ldots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0, \tag{7}$$

where the left D-modules P_i are projective. If, moreover, the P_i are free, then (7) is called a *free resolution* of M. Finally, if the P_i are finitely generated free left D-modules and $P_{m+1} = 0$, then (7) is said to be a *finite free resolution* of M.

As a free left D-module is projective (see Theorem 1), we obtain that a free resolution is also a projective one. The next proposition will play an important role in what follows.

Proposition 4. *Let us consider a projective resolution of a left D-module M:*

$$0 \longrightarrow P_m \xrightarrow{\delta_m} P_{m-1} \xrightarrow{\delta_{m-1}} P_{m-2} \xrightarrow{\delta_{m-2}} P_{m-3} \xrightarrow{\delta_{m-3}} \dots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0. \tag{8}$$

If $m \geq 2$ and there exists a D-morphism $\sigma_m : P_{m-1} \longrightarrow P_m$ such that $\sigma_m \circ \delta_m = id_{P_m}$, then we have the following projective resolution of M:

$$0 \longrightarrow P_{m-1} \xrightarrow{\tau_{m-1}} P_{m-2} \oplus P_m \xrightarrow{\tau_{m-2}} P_{m-3} \xrightarrow{\delta_{m-3}} P_{m-4} \xrightarrow{\delta_{m-4}} \dots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0, \tag{9}$$

with the notations:

$$\tau_{m-1} = \begin{pmatrix} \delta_{m-1} \\ \sigma_m \end{pmatrix}, \quad \tau_{m-2} = (\delta_{m-2} \quad 0).$$

Proof. Using the fact that (8) is a complex at P_{m-2}, i.e., $\delta_{m-2} \circ \delta_{m-1} = 0$, we obtain $\tau_{m-2} \circ \tau_{m-1} = \delta_{m-2} \circ \delta_{m-1} = 0$, which proves that $\operatorname{im} \tau_{m-1} \subseteq \ker \tau_{m-2}$.

Let us now prove $\ker \tau_{m-2} \subseteq \operatorname{im} \tau_{m-1}$. We consider $(a \quad b)^T \in \ker \tau_{m-2}$. Then, we have $a \in P_{m-2}$, $b \in P_m$ and $\tau_{m-2}((a \quad b)^T) = \delta_{m-2}(a) = 0$. Since (8) is exact at P_{m-2}, there exists $c \in P_{m-1}$ such that $a = \delta_{m-1}(c)$. Now, let us define:

$$d = (id_{P_{m-1}} - \delta_m \circ \sigma_m \quad \delta_m)\,(c \quad b)^T = c - (\delta_m \circ \sigma_m)(c) + \delta_m(b) \in P_{m-1}.$$

Then we have

$$\tau_{m-1}(d) = \begin{pmatrix} \delta_{m-1}(c) - \delta_{m-1}(\delta_m(\sigma_m(c))) + \delta_{m-1}(\delta_m(b)) \\ \sigma_m(c) - ((\sigma_m \circ \delta_m) \circ \sigma_m)(c) + (\sigma_m \circ \delta_m)(b) \end{pmatrix} = \begin{pmatrix} \delta_{m-1}(c) \\ \sigma_m(c) - \sigma_m(c) + b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

which shows that $(a \quad b)^T \in \operatorname{im} \tau_{m-1}$, and thus, we have $\ker \tau_{m-2} \subseteq \operatorname{im} \tau_{m-1}$, which proves the exactness of (9) at $P_{m-2} \oplus P_m$.

Let us compute $\ker \tau_{m-1}$. If $d \in \ker \tau_{m-1}$, then we have $\tau_{m-1}(d) = 0$, i.e., $\delta_{m-1}(d) = 0$ and $\sigma_m(d) = 0$. Now, let us consider the short exact sequence:

$$0 \longrightarrow P_m \xrightarrow{\delta_m} P_{m-1} \xrightarrow{\delta_{m-1}} \operatorname{im} \delta_{m-1} \longrightarrow 0. \tag{10}$$

Using the existence of $\sigma_m : P_{m-1} \longrightarrow P_m$ satisfying $\sigma_m \circ \delta_m = id_{P_m}$, we obtain that (10) splits, i.e., there exists a D-morphism $\kappa_{m-1} : \operatorname{im} \delta_{m-1} \longrightarrow P_{m-1}$ such that $id_{P_{m-1}} = \delta_m \circ \sigma_m + \kappa_{m-1} \circ \delta_{m-1}$. Hence, we have

$$d = \delta_m(\sigma_m(d)) + \kappa_{m-1}(\delta_{m-1}(d)) = 0,$$

which proves that τ_{m-1} is an injective D-morphism.

Finally, we have $\operatorname{im} \tau_{m-2} = \tau_{m-2}(P_{m-2} \oplus P_m) = \delta_{m-2}(P_{m-2}) = \operatorname{im} \delta_{m-2} = \ker \delta_{m-3}$ as (8) is exact at P_{m-3}. Hence, we obtain that (9) is exact at P_{m-3}, and thus, (9) is an exact sequence. □

We note that Proposition 4 simplifies a result obtained in [7] by, on the one hand, expliciting the morphisms in (8) and, on the other hand, giving a simple and direct proof. We have the following straightforward corollary of Proposition 4.

Corollary 1. *Let us consider a finite free resolution of a left D-module M:*

$$0 \longrightarrow D^{1\times p_m} \xrightarrow{.R_m} D^{1\times p_{m-1}} \xrightarrow{.R_{m-1}} D^{1\times p_{m-2}} \xrightarrow{.R_{m-2}} \dots \xrightarrow{.R_2} D^{1\times p_1} \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\delta_0} M \longrightarrow 0. \tag{11}$$

1. *If $m \geq 3$ and there exists $S_m \in D^{p_{m-1} \times p_m}$ such that $R_m\, S_m = I_{p_m}$, then we have the following finite free resolution of M:*

$$0 \longrightarrow D^{1\times p_{m-1}} \xrightarrow{.T_{m-1}} D^{1\times (p_{m-2}+p_m)} \xrightarrow{.T_{m-2}} D^{1\times p_{m-3}} \xrightarrow{.R_{m-3}} \dots \xrightarrow{.R_1} D^{1\times p_0} \xrightarrow{\delta_0} M \longrightarrow 0, \tag{12}$$

with the notations:

$$T_{m-1} = (R_{m-1} \quad S_m) \in D^{p_{m-1}\times (p_{m-2}+p_m)}, \quad T_{m-2} = \left(\begin{array}{c} R_{m-2} \\ 0 \end{array}\right) \in D^{(p_{m-2}+p_m)\times p_{m-3}}.$$

2. *If $m = 2$ and there exists $S_2 \in D^{p_1 \times p_2}$ such that $R_2\, S_2 = I_{p_2}$, then we have the following finite presentation of M:*

$$0 \longrightarrow D^{1\times p_1} \xrightarrow{.T_1} D^{1\times (p_0+p_2)} \xrightarrow{\tau_0} M \longrightarrow 0, \tag{13}$$

with the notations:

$$T_1 = (R_1 \quad S_2) \in D^{p_1\times (p_0+p_2)}, \quad \tau_0 = \left(\begin{array}{c} \delta_0 \\ 0 \end{array}\right).$$

Remark 1. In case 2 of Corollary 1, we obtain the following isomorphism:

$$M = D^{1\times p_0}/(D^{1\times p_1}\, R_1) \cong \operatorname{coker}_D(.T_1) = D^{1\times (p_0+p_2)}/(D^{1\times p_1}\, T_1).$$

In terms of equations, the left D-module M is defined by $R_1\, z = 0$, whereas $\operatorname{coker}_D(.T_1)$ is defined by $R_1\, y_1 + S_2\, y_2 = 0$. Applying R_2 on the left of the last system, we then have $(R_2\, R_1)\, y_1 + (R_2\, S_2)\, y_2 = 0$ and using the facts that $R_2\, R_1 = 0$ and $R_2\, S_2 = I_{p_2}$, we finally obtain $y_2 = 0$, and thus, $R_1\, y_1 = 0$. Hence, we obtain the isomorphism $\phi : M \longrightarrow \operatorname{coker}_D(.T_1)$ defined by $\phi(z_i) = y_{1i}$, $i = 1, \dots, p_0$, whose inverse $\psi : \operatorname{coker}_D(.T_1) \longrightarrow M$ is induced by $\psi(y_{1i}) = z_i$, $i = 1, \dots, p_0$ and $\psi(y_{2j}) = 0$, $j = 1, \dots, p_2$.

Let us illustrate Corollary 1.

Example 4. We consider the following linear ordinary differential system

$$\left\{ \begin{array}{l} t^2\, y(t) = 0, \\ t\, \dot{y}(t) + 2\, y(t) = 0, \end{array} \right.$$

whose solution in $\mathcal{D}'(\mathbb{R})$ is $y = \dot{\delta}$, namely, the derivative of the Dirac distribution δ at $t = 0$. If we consider the ring $D = A_1(\mathbb{Q})$ of differential operators in $\frac{d}{dt}$ with polynomial coefficients in t over $\mathbb{Q}$, the matrix $R_1 = \left(t^2 \quad t\, \frac{d}{dt} + 2\right)^T \in D^2$ and the left D-module $M = D/(D^{1\times 2}\, R_1) = D/\left(D\, t^2 + D\, \left(t\, \frac{d}{dt} + 2\right)\right)$, then a finite free resolution of M is defined by $0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R_1} D \xrightarrow{\delta_0} M \longrightarrow 0$, where $R_2 = \left(\frac{d}{dt} \quad -t\right) \in D^{1\times 2}$ (see [2, 30] for more details). We easily check that $S_2 = \left(t \quad \frac{d}{dt}\right)^T \in D^2$ is a right-inverse of R_2. Hence, using Corollary 1, we obtain the following finite free resolution of M:

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.T_1} D^{1\times 2} \xrightarrow{\tau_0} M \longrightarrow 0, \quad T_1 = \left(\begin{array}{cc} t^2 & t \\ t\, \frac{d}{dt} + 2 & \frac{d}{dt} \end{array}\right) \in D^{2\times 2}, \quad \tau_0 = (\delta_0 \quad 0)^T. \tag{14}$$

Example 5. Let us consider the Weyl algebra $D = A_3(\mathbb{Q})$ and the matrix

$$R_1 = \left(\begin{array}{ccc} \frac{1}{2}\, x_2\, \partial_1 & x_2\, \partial_2 + 1 & x_2\, \partial_3 + \frac{1}{2}\, \partial_1 \\ -\frac{1}{2}\, x_2\, \partial_2 - \frac{3}{2} & 0 & \frac{1}{2}\, \partial_2 \\ -\partial_1 - \frac{1}{2}\, x_2\, \partial_3 & -\partial_2 & -\frac{1}{2}\, \partial_3 \end{array}\right) \in D^{3\times 3},$$

which defines the system $R_1\,\xi = 0$ of the infinitesimal transformations of the Lie pseudogroup defined by the contact transformations (see Example V.1.84 in [24]). Using OreModules [2], we obtain the following free resolution of the left D-module $M = D^{1\times 3}/(D^{1\times 3}\,R_1)$

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 3} \xrightarrow{.R_1} D^{1\times 3} \xrightarrow{\delta_0} M \longrightarrow 0,$$

where $R_2 = (\partial_2 \quad -(\partial_1 + x_2\,\partial_3) \quad x_2\,\partial_2 + 2) \in D^{1\times 3}$. We easily check that $S_2 = (-x_2 \quad 0 \quad 1)^T$ is a right-inverse of R_2, and thus, by Corollary 1, we obtain the following finite free resolution of M:

$$0 \longrightarrow D^{1\times 3} \xrightarrow{.T_1} D^{1\times 4} \xrightarrow{\tau_0} M \longrightarrow 0, \quad T_1 = \begin{pmatrix} \frac{1}{2}\,x_2\,\partial_1 & x_2\,\partial_2 + 1 & x_2\,\partial_3 + \frac{1}{2}\,\partial_1 & -x_2 \\ -\frac{1}{2}\,x_2\,\partial_2 - \frac{3}{2} & 0 & \frac{1}{2}\,\partial_2 & 0 \\ -\partial_1 - \frac{1}{2}\,x_2\,\partial_3 & -\partial_2 & -\frac{1}{2}\,\partial_3 & 1 \end{pmatrix}. \tag{15}$$

More detailed examples can be found in [2].

We recall the definitions of the *left projective dimension* of a left D-module M and the *left global dimension* of a ring D [32].

Definition 6. 1. Let M be a left D-module. Then, we call *left projective dimension* of M, denoted by $\mathrm{lpd}_D(M)$, the smallest n such that there exists a projective resolution of M of the form

$$0 \longrightarrow P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} M \longrightarrow 0. \tag{16}$$

If no such finite projective resolution exists, then we set $\mathrm{lpd}_D(M) = +\infty$.

2. The *left global dimension* of D, denoted by $\mathrm{lgld}(D)$, is the supremum of $\mathrm{lpd}_D(M)$ over all the left D-modules M.

The right projective dimension of a right D-module M and the right global dimension $\mathrm{rgld}(D)$ of D are defined similarly. If D is a left and right noetherian ring, then we have $\mathrm{lgld}(D) = \mathrm{rgld}(D)$ [32].

Example 6. [20] We have the following left and right global dimensions:

1. If k is a field of characteristic 0, then $\mathrm{lgld}(A_n(k)) = \mathrm{rgld}(A_n(k)) = n$.
2. If k is a field of characteristic 0, then $\mathrm{lgld}(B_n(k)) = \mathrm{rgld}(B_n(k)) = n$.

See [3] for more examples.

Proposition 5. *(Proposition 5.11 in [13]). Let M be a left D-module. If $n \geq 1$, then we have $\mathrm{lpd}_D(M) = n$ iff there exists a finite projective resolution of M as (16) where δ_n is* nonsplit, *namely, there exists no D-morphism $\tau_n : P_{n-1} \longrightarrow P_n$ such that $\tau_n \circ \delta_n = id_{P_n}$.*

Following [7], we obtain Algorithm 1 for the computation of the left projective dimension of a left D-module $M = D^{1\times p_0}/(D^{1\times p_1}\,R_1)$.

Algorithm 1.
- **Input:** A left D-module M defined by a finite free resolution (11).
- **Output:** The left projective dimension $\mathrm{lpd}_D(M)$ of M.

1. Set $j = m$ and $T_j = R_m$.
2. Check whether or not T_j admits a right-inverse S_j over D.
   (a) If no right-inverse of T_j exists, then we have $\mathrm{lpd}_D(M) = j$ and stop the algorithm.
   (b) If there exists a right-inverse S_j of T_j and

   i. if $j = 1$, then we have $\text{lpd}_D(M) = 0$ and stop the algorithm.
   ii. if $j = 2$, then compute (13).
   iii. if $j \geq 3$, then compute (12).

3. Return to step 2 with $j \leftarrow j - 1$.

Remark 2. We refer to [3] for the description of a constructive algorithm which checks whether or not a matrix over certain classes of Ore algebras admits a right-inverse and to [2] for an implementation in OreModules. Algorithm 1 has recently been implemented in OreModules and it can be applied by means of the command ProjectiveDimension(Rat). See [2] for more details.

Example 7. We consider again Example 4 (resp., Example 5). We check that the matrix T_1 defined in (14) (resp., (15)) does not admit a right-inverse. Hence, we obtain that $\text{lpd}_D(M) = 1$.

We are now in position to define the concept of a *minimal free resolution* of a left D-module.

Definition 7. We call *minimal free resolution* of M the last free resolution obtained by Algorithm 1, namely, a finite free resolution of M which satisfies that either $m = 1$ and R_1 admits a right-inverse or the last matrix R_m of the free resolution does not admit a right-inverse.

Lemma 2. *A left D-module M is stably free iff there exists a matrix $R \in D^{q\times p}$ which admits a right-inverse $S \in D^{p\times q}$, i.e., $R\,S = I_q$, and satisfies $M \cong D^{1\times p}/(D^{1\times q}\,R)$.*

Proof. If M is a stably free left D-module, then there exist $p,\, q \in \mathbb{Z}_+$ such that $M \oplus D^{1\times q} \cong D^{1\times p}$. Let us denote by $\psi : D^{1\times p} \longrightarrow M \oplus D^{1\times q}$ the above D-isomorphism and by $\pi_1 : M \oplus D^{1\times q} \longrightarrow M$ the canonical projection onto M. Hence, we obtain the following commutative exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \ker_D(\pi_1\circ\psi) & & D^{1\times q} & & \\
 & & \downarrow & & \downarrow {\scriptstyle i_1} & & \\
0 & \longrightarrow & D^{1\times p} & \xrightarrow{\psi} & M\oplus D^{1\times q} & \longrightarrow & 0\\
 & & \downarrow {\scriptstyle \pi_1\circ\psi} & & \downarrow {\scriptstyle \pi_1} & & \\
0 & \longrightarrow & M & \xrightarrow{id} & M & \longrightarrow & 0,\\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

which shows that $\psi(\ker_D(\pi_1 \circ \psi)) = 0 \oplus D^{1\times q} = i_1(D^{1\times q})$. Thus, we obtain the following exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi_1\circ\psi} M \longrightarrow 0, \tag{17}$$

where $R \in D^{q\times p}$ is the matrix representing the D-morphism $\psi^{-1} \circ i_1 : D^{1\times q} \longrightarrow D^{1\times p}$ with respect to the standard bases of $D^{1\times q}$ and $D^{1\times p}$. Now, if we denote by $\pi_2 : M \oplus D^{1\times q} \longrightarrow D^{1\times q}$ the canonical projection onto $D^{1\times q}$, we then have $\pi_2 \circ i_1 = id_{D^{1\times q}}$. Hence, the D-morphism defined by $\pi_2 \circ \psi : D^{1\times p} \longrightarrow D^{1\times q}$, represented by $S \in D^{p\times q}$ with respect to the standard bases of $D^{1\times p}$ and $D^{1\times q}$, satisfies that $(\pi_2 \circ \psi) \circ (\psi^{-1} \circ i_1) = id_{D^{1\times q}}$, i.e., $R\,S = I_q$, which proves the result.

Conversely, if the left D-module M is the cokernel of the D-morphism $.R : D^{1\times q} \longrightarrow D^{1\times p}$, where the matrix R admits a right-inverse S, then we obtain $\ker_D(.R) = \{\lambda \in D^{1\times q} \mid \lambda\,R = 0\} = 0$ as $\lambda = (\lambda\,R)\,S = 0$. Therefore, the exact sequence (17) splits and we finally obtain $M \oplus D^{1\times q} \cong D^{1\times p}$, which shows that M is a stably free left D-module. □

We recall the following interesting result.

Proposition 6. *(Proposition 8 in [3]) If $D = A[\partial_1; \sigma_1, \delta_1] \dots [\partial_m; \sigma_m, \delta_m]$ is an Ore algebra where σ_i is an automorphism for $i = 1, \dots, m$, then every finitely generated left D-module admits a finite free resolution of length less than or equal to* $\mathrm{lgld}(D) + 1$.

Proposition 6 shows that every finitely generated left module over the Ore algebra D defined previously admits a finite free resolution. In particular, if we can compute Gröbner bases over D, then we can obtain finite free resolutions [3]. We then arrive at the following important remark.

Remark 3. We note that the proof of Lemma 2 is a non-constructive one. However, if D satisfies the hypothesis of Proposition 1, then, using the fact that any finite free resolution (11) of a stably free left D-module $M = D^{1\times p_0}/(D^{1\times p_1} R_1)$ splits [32], Algorithm 1 gives us a constructive way to compute a matrix $R \in D^{q\times p}$ which admits a right-inverse $S \in D^{p\times q}$ and satisfies $M \cong D^{1\times p}/(D^{1\times q} R)$. Such a matrix R can be obtained in OreModules by using the command ShorterFreeResolution for certain classes of Ore algebras. See [2] for more details and examples.

Let us illustrate Remark 3 by means of an explicit example.

Example 8. Let us consider $D = A_1(\mathbb{Q})$ and the left D-module $M = D^{1\times 2}/(D^{1\times 2} R)$, where the matrix R is defined by:

$$R = \begin{pmatrix} -t^2 & t\,\frac{d}{dt} - 1 \\ -t\,\frac{d}{dt} - 2 & \frac{d^2}{dt^2} \end{pmatrix} \in D^{2\times 2}.$$

We can check that M has the following free resolution:

$$0 \longrightarrow D \xrightarrow{.R_2} D^{1\times 2} \xrightarrow{.R} D^{1\times 2} \xrightarrow{\delta_0} M \longrightarrow 0, \quad R_2 = \left(\frac{d}{dt}, \quad -t\right) \in D^{1\times 2}.$$

Moreover, the matrix $S_2 = \left(t \quad \frac{d}{dt}\right)^T$ is a right-inverse of R_2. Hence, if we denote by $T_1 = (R_1 \quad S_2)$, then, by Corollary 1, we obtain the finite free resolution of M:

$$0 \longrightarrow D^{1\times 2} \xrightarrow{.T_1} D^{1\times 3} \xrightarrow{\tau_0} M \longrightarrow 0. \tag{18}$$

We finally check that T_1 admits the following right-inverse S_1 defined by:

$$S_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ \frac{d}{dt} & -t \end{pmatrix} \in D^{3\times 2}.$$

Therefore, the exact sequence (18) splits, and thus, M is a stably free left D-module of rank 1 and (18) is a minimal free resolution of M.

4 Blowing-up of stably free behaviours

The purpose of this section is to show that we can always find a flat behaviour which trivially projects onto a given stably free behaviour. An explicit formula for such a flat behaviour is obtained. These results are essentially the same as understanding the concept of a stably free left D-module M in the system theoretic language. However, we shall see some interesting applications in control theory [6, 16]. We refer the reader to [30] for more details and motivations.

Proposition 7. *Let us consider the following projective resolution of a left D-module M:*

$$0 \longrightarrow P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\pi} M \longrightarrow 0.$$

If there exists a left D-morphism $\sigma_1 : P_0 \longrightarrow P_1$ satisfying $\sigma_1 \circ \delta_1 = id_{P_1}$, then we have the following split exact sequence:

$$0 \longrightarrow P_1 \underset{\xleftarrow{k}}{\xrightarrow{f}} P_0 \oplus P_1 \underset{\xleftarrow{h}}{\xrightarrow{g}} P_0 \longrightarrow 0, \tag{19}$$

with the following notations:

$$f = \begin{pmatrix} \delta_1 \\ 0 \end{pmatrix}, \quad g = (id_{P_0} - \delta_1 \circ \sigma_1 \quad \delta_1), \quad h = \begin{pmatrix} id_{P_0} \\ \sigma_1 \end{pmatrix}, \quad k = (\sigma_1 \quad - id_{P_1}). \tag{20}$$

Proof. We easily check the following identities

$$\begin{cases} g \circ f = \delta_1 - \delta_1 \circ \sigma_1 \circ \delta_1 = \delta_1 - \delta_1 = 0, \\ g \circ h = id_{P_0} - \delta_1 \circ \sigma_1 + \delta_1 \circ \sigma_1 = id_{P_0}, \\ k \circ h = \sigma_1 - \sigma_1 = 0, \\ k \circ f = \sigma_1 \circ \delta_1 = id_{P_1}, \\ f \circ k + h \circ g = id_{P_0 \oplus P_1}, \end{cases}$$

which prove that (19) is a split exact sequence. See [3] for more details. □

Let us consider a stably free left D-module M and the stably free behaviour $\hom_D(M, \mathcal{F})$. Using Remark 3, we can always suppose that M is defined by a matrix $R \in D^{q \times p}$ which admits a right-inverse $S \in D^{p \times q}$, i.e., $M = D^{1 \times p}/(D^{1 \times q} R)$ and $R S = I_q$. We then have the following straightforward corollary of Proposition 7.

Corollary 2. *[30] Let $R \in D^{q \times p}$ be a matrix which admits a right-inverse $S \in D^{p \times q}$, i.e., $R S = I_q$.*

1. *If we denote by $R' = (R \quad 0) \in D^{q \times (p+q)}$, then we have the following split exact sequence:*

$$0 \longrightarrow D^{1 \times q} \underset{\xleftarrow{.S'}}{\xrightarrow{.R'}} D^{1 \times (p+q)} \underset{\xleftarrow{.T'}}{\xrightarrow{.Q'}} D^{1 \times p} \longrightarrow 0, \tag{21}$$

   with the following notations:

$$S' = \begin{pmatrix} S \\ -I_q \end{pmatrix} \in D^{(p+q) \times q}, \quad T' = (I_p \quad S) \in D^{p \times (p+q)}, \quad Q' = \begin{pmatrix} I_p - S R \\ R \end{pmatrix} \in D^{(p+q) \times p}. \tag{22}$$

   Equivalently, we have the following Bézout identities:

$$\begin{pmatrix} R' \\ T' \end{pmatrix} (S' \quad Q') = I_{p+q}, \quad (S' \quad Q') \begin{pmatrix} R' \\ T' \end{pmatrix} = I_{p+q}.$$

2. *Let us consider the left D-module $M = D^{1 \times p}/(D^{1 \times q} R)$, the canonical projection $\pi : D^{1 \times p} \longrightarrow M$ and the left D-morphism $\kappa : D^{1 \times (p+q)} \longrightarrow D^{1 \times (p+q)}/(D^{1 \times q} R')$ defined by:*

$$\kappa((\lambda_1, \ldots, \lambda_{p+q})) = (\pi(\lambda_1, \ldots, \lambda_p), \lambda_{p+1}, \ldots, \lambda_{p+q}).$$

   (a) *We have $M \oplus D^{1 \times q} \cong D^{1 \times p}$, i.e., $M \oplus D^{1 \times q}$ is a free left D-module with a basis defined by $\{\kappa(T'_i)\}_{1 \le i \le p}$, where T'_i denotes the i^{th} row of $T' \in D^{p \times (p+q)}$.*

(b) If $\mathcal{F}$ is a left D-module, then we have the following equality:

$$\mathcal{B}' = \left\{ \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \in \mathcal{F}^{p+q} \mid R\,\eta = 0 \right\} = Q'\,\mathcal{F}^p.$$

In particular, for all $\zeta \in \mathcal{F}^q$ and $\eta \in \mathcal{F}^p$ satisfying the system $R\,\eta = 0$, there exists a unique $\xi \in \mathcal{F}^p$ such that

$$\begin{cases} \eta = (I_p - S\,R)\,\xi, \\ \zeta = R\,\xi, \end{cases}$$

and $\xi = T'\,(\eta^T \quad \zeta^T)^T = \eta + S\,\zeta$. Hence, the free behaviour $\mathcal{B}' \cong \mathcal{B} \oplus \mathcal{F}^q$ projects onto the stably free behaviour $\mathcal{B}$ under the projection $\mathcal{F}^{p+q} \longrightarrow \mathcal{F}^p : (\eta^T \quad \zeta^T)^T \longmapsto \eta^T$.

If $\mathcal{B} = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$ is a stably free/flat behaviour, then Corollary 2 shows that we can always find a free behaviour $\mathcal{B}'$ by adding q new degrees of freedom $\zeta \in \mathcal{F}^q$ obtained by embedding $\mathcal{F}^p$ into $\mathcal{F}^{p+q}$. Corollary 2 was used in [30] in order to "desingularize" time-varying controllable linear control systems and stably free behaviours. Let us illustrate Corollary 2.

Example 9. Let us consider $D = A_1(\mathbb{Q})$, $R = \left(\frac{d}{dt} \quad -t\right)$ and the left D-module $M = D^{1\times 2}/(D\,R)$. We easily check that $S = \left(t \quad \frac{d}{dt}\right)^T$ is a right-inverse of R, i.e., $R\,S = 1$. Therefore, the exact sequence

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{\pi} M \longrightarrow 0 \tag{23}$$

splits and we obtain $M \oplus D \cong D^{1\times 2}$, which shows that M is a stably free left D-module of rank 1. Moreover, we shall see in Example 17 that M is not a free left D-module. Therefore, if $\mathcal{F}$ is any left D-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$), we get the stably free but not free behaviour:

$$\mathcal{B} = \{(x \quad u)^T \in \mathcal{F}^2 \mid \dot{x}(t) - t\,u(t) = 0\}.$$

In particular, we know that there exists no injective parametrization of $\mathcal{B}$. But, using the algorithms developed in [3, 25], we can obtain the following parametrization of $\mathcal{B}$

$$\begin{cases} x(t) = -t\,\dot{y}_1(t) + y_1(t) + t^2\,y_2(t), \\ u(t) = -\ddot{y}_1(t) + t\,\dot{y}_2(t) + 2\,y_2(t), \end{cases}$$

where $y_1,\ y_2 \in \mathcal{F}$. However, we cannot express y_1 and y_2 in terms of x, u and their derivatives as it would imply that $\mathrm{rank}_D(M)$ is 2 whereas it is clearly 1 with regard to (23).

If we consider the $B_1(\mathbb{Q})$-module $B_1(\mathbb{Q}) \otimes_{A_1(\mathbb{Q})} M \cong B_1(\mathbb{Q})^{1\times 2}/(B_1(\mathbb{Q})\,R)$, then we obtain the following singular injective parametrization of $\mathcal{B}$ over any left $B_1(\mathbb{Q})$-module $\mathcal{F}$ (e.g., $\mathcal{F} = \mathbb{R}(t)$):

$$\begin{cases} x(t) = y(t), \\ u(t) = \dfrac{1}{t}\,\dot{y}(t). \end{cases}$$

The fact that M is not a free left D-module means that we cannot remove the singularity at $t = 0$. However, Corollary 2 allows us to "blow up" the singularity at $t = 0$ as we have:

$$\begin{cases} \dot{x}(t) - t\,u(t) = 0, \\ v \in \mathcal{F}, \end{cases} \Leftrightarrow \begin{cases} x(t) = -t\,\dot{y}_1(t) + y_1(t) + t^2\,y_2(t), \\ u(t) = -\ddot{y}_1(t) + t\,\dot{y}_2(t) + 2\,y_2(t), \\ v(t) = \dot{y}_1(t) - t\,y_2(t), \end{cases}$$

where $y_1(t) = x(t) + t\,v(t)$ and $y_2(t) = u(t) + \dot{v}(t)$. It means that $\mathcal{B}$ is the image of the flat behaviour $\mathcal{B}' = \mathcal{B} \oplus \mathcal{F}$ under the projection $\mathcal{F}^3 \longrightarrow \mathcal{F}^2 : (x,\ u,\ v)^T \longmapsto (x,\ u)^T$.

In [16], it was shown that a dynamic compensator of the form $\dot{v}(t) = -u(t)$ can be used in order to obtain the following flat system:

$$\begin{cases} \dot{x}(t) - t\,u(t) = 0, \\ \dot{v}(t) + u(t) = 0, \end{cases} \Leftrightarrow \begin{cases} x(t) = -t\,\dot{y}(t) + y(t), \\ u(t) = -\ddot{y}(t), \\ v(t) = \dot{y}(t), \end{cases} \qquad y(t) = x(t) + t\,v(t).$$

More generally, for analytic time-varying single-input controllable linear systems, a general algorithm is given in [16] in order to construct the dynamic compensator which allows to obtain a flat system. As we have shown in [30], Corollary 2 gives an alternative way to desingularize a stably free behaviour by means of a flat one. In particular, it shows that an analytic time-varying controllable linear system is the projection of a flat system and Corollary 2 gives an explicit form for it. This result clarifies an idea developed in [18]. We also note that Corollary 2 can be applied to multi-input systems or multidimensional linear systems such as differential time-delay linear systems as we now illustrate it.

Example 10. Let us consider the following differential time-delay system:

$$\dot{x}(t) = t\,u(t) + u(t-1). \tag{24}$$

We introduce the ring $D = A_1(\mathbb{Q})[\delta; \sigma, 0]$ of differential time-delay operators, where $\sigma(a(t)) = a(t-1)$, $R = (\frac{d}{dt} \quad -(t+\delta)) \in D^{1\times 2}$ and the left D-module $M = D^{1\times 2}/(D\,R)$. We can check that the matrix $S = (\delta + t \quad \frac{d}{dt})^T \in D^2$ is a right-inverse of R. Therefore, the finite free resolution of M defined by $0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \longrightarrow M \longrightarrow 0$ splits and we obtain $M \oplus D \cong D^{1\times 2}$, i.e., M is a stably free left D-module of rank 1. Using an algorithm developed in [3], we obtain the long split exact sequence

$$0 \longrightarrow D \xrightarrow{.R} D^{1\times 2} \xrightarrow{.Q_1} D^{1\times 2} \xrightarrow{.Q_2} D \longrightarrow 0, \tag{25}$$

with the following notations:

$$Q_1 = \begin{pmatrix} -\delta\,\frac{d}{dt} - t\,\frac{d}{dt} + 1 & \delta^2 + (2\,t-1)\,\delta + t^2 \\ -\frac{d^2}{dt^2} & t\,\frac{d}{dt} + \delta\,\frac{d}{dt} + 2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} \delta + t \\ \frac{d}{dt} \end{pmatrix}.$$

Let $\mathcal{F}$ be a left D-module (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$). As (25) is a long split exact sequence, by applying the functor $\hom_D(\cdot, \mathcal{F})$ to (25), we then obtain the following split exact sequence of abelian groups:

$$0 \longleftarrow \mathcal{F} \xleftarrow{R.} \mathcal{F}^2 \xleftarrow{Q_1.} \mathcal{F}^2 \xleftarrow{Q_2.} \mathcal{F} \longleftarrow 0.$$

Hence, we get $\mathcal{B} = \ker_{\mathcal{F}}(R.) = Q_1\,\mathcal{F}^2$, i.e., we have the parametrization of all $\mathcal{F}$-solutions of (24)

$$\begin{cases} x(t) = -t\,\dot{y}_1(t) - \dot{y}_1(t-1) + y_1(t) + y_2(t-2) + (2\,t-1)\,y_2(t-1) + t^2\,y_2(t), \\ u(t) = -\ddot{y}_1(t) + t\,\dot{y}_2(t) + \dot{y}_2(t-1) + 2\,y_2(t), \end{cases} \tag{26}$$

where y_1 and y_2 are two arbitrary functions in $\mathcal{F}$. Parametrization (26) is not injective since $Q_1\,y = 0$ is equivalent to $y = Q_2\,z$ for a certain $z \in \mathcal{F}$. Therefore, it is not possible to obtain y_1 and y_2 as D-linear combinations of x and u.

However, if we embed the stably free behaviour

$$\mathcal{B} = \left\{(x \quad u)^T \in \mathcal{F}^2 \mid \dot{x}(t) = t\,u(t) + u(t-1)\right\}$$

into $\mathcal{F}^3$, then, by Corollary 2, we obtain the injective parametrization of all $\mathcal{F}$-solutions of (24)

$$\begin{cases} x(t) = -t\,\dot{y}_1(t) - \dot{y}_1(t-1) + y_1(t) + y_2(t-2) + (2\,t-1)\,y_2(t-1) + t^2\,y_2(t), \\ u(t) = -\ddot{y}_1(t) + t\,\dot{y}_2(t) + \dot{y}_2(t-1) + 2\,y_2(t), \\ v(t) = \dot{y}_1(t) - t\,y_2(t) - y_2(t-1), \end{cases}$$

where $y_1(t) = x(t) + t\,v(t) + v(t-1)$ and $y_2(t) = u(t) + \dot{v}(t)$. Hence, $\mathcal{B}$ is a projection onto $\mathcal{F}^2$ of the flat behaviour defined by:

$$\mathcal{B}' = \mathcal{B} \oplus \mathcal{F} = \{(x \quad u \quad v)^T \in \mathcal{F}^3 \mid \dot{x}(t) - t\,u(t) - u(t-1) = 0\}.$$

Finally, K. B. Datta proposes in [5] that an interesting problem is to extend the results in [16] to analytic time-varying controllable linear systems with multi-inputs. In the case of polynomial coefficients, we have proved in [30] that this problem is theoretically solved as such linear systems are shown to be flat. The purpose of the next section is to adapt this result to more general situations. Moreover, a constructive algorithm for the computation of bases of free modules over the Weyl algebra $A_n(k)$ is obtained. In particular, this next result will give an effective algorithm for the computation of flat outputs of time-varying controllable linear systems with multi-inputs.

5 Computation of bases of free modules

In what follows, we shall consider a left noetherian ring D. In particular, this condition implies that D is a left Ore domain and has the left invariant basis number. The rank of a free left D-module F is then well-defined (see Section 2). By extension, the rank of a finitely generated left D-module M is defined as the maximal $\operatorname{rank}_D(F)$ such that there exists an exact sequence of the form

$$0 \longrightarrow F \longrightarrow M \longrightarrow M/F \longrightarrow 0,$$

where F is a free left D-module and M/F is a torsion left D-module.

5.1 The general case

The purpose of this section is to give a general algorithm which computes bases of free left D-modules based on the concept of *stable range* [20]. Let us state a few definitions.

Definition 8.
1. A column vector $v \in D^m$ is called *unimodular* if v admits a left-inverse $w = (w_1, \ldots, w_m) \in D^{1\times m}$, i.e., if we have $w\,v = \sum_{i=1}^n w_i\,v_i = 1$. We denote by $U_c(m, D)$ the set of all unimodular columns of length m over D.
2. A unimodular column $v = (v_1, \ldots, v_m)^T \in U_c(m, D)$ is called *stable* (*reductible*) if there exist $a_1, \ldots, a_{m-1} \in D$ such that $v' = (v_1 + a_1\,v_m, \ldots, v_{m-1} + a_{m-1}\,v_m)^T$ is unimodular, i.e., we have $v' \in U_c(m-1, D)$.
3. We say that l *is in the stable range of* ${}_D D$ (i.e., D as a left D-module), if, for every $m \geq l$, every unimodular column $v \in U_c(m, D)$ is then stable.
4. The least positive integer l in the stable range of ${}_D D$ is called *the stable range of* ${}_D D$. It is denoted by $\operatorname{sr}({}_D D)$. If no such integer exists, then we set $\operatorname{sr}({}_D D) = +\infty$.

Similar definitions hold for *unimodular rows.* If we denote by $U_r(m, D)$ the set of unimodular rows of length m with entries in D, then we can similarly define the stable range $\operatorname{sr}(D_D)$ of D_D (i.e., D as a right D-module).

Proposition 8. *(Proposition 11.3.4 in [20]) We have* $\operatorname{sr}({}_D D) = \operatorname{sr}(D_D)$.

Hence, in what follows, we shall only write $\operatorname{sr}(D)$ instead of $\operatorname{sr}({}_D D)$ or $\operatorname{sr}(D_D)$.

Example 11. We have the following results about the stable range:

1. If D is a principal left ideal domain, then $\operatorname{sr}(D) \leq 2$ (e.g., $\operatorname{sr}(\mathbb{Z}) = 2$; if k is a field, then $\operatorname{sr}(k[x]) = 2$; if K is a *differential field* (e.g., $K = \mathbb{Q}(t)$) [24], then $\operatorname{sr}(K\left[\frac{d}{dt}; id, \frac{d}{dt}\right]) \leq 2$).

2. [20] For any field $k \subseteq \mathbb{R}$, we have $\mathrm{sr}(k[x_1, \ldots, x_n]) = n + 1$.
3. [34] If k is a field containing $\mathbb{Q}$, then we have $\mathrm{sr}(A_n(k)) = 2$.
4. [34] Under the same hypothesis as in 3 we have $\mathrm{sr}(B_n(k)) = 2$.

Definition 9. The *elementary group* $E(m, D)$ is the subgroup of

$$\mathrm{GL}(m, D) = \{ U \in D^{m \times m} \mid \exists\, V \in D^{m \times m} : \, U\, V = V\, U = I_m \}$$

which is generated by matrices of the form $I_m + r\, E_{ij}$, where $r \in D$, $i \neq j$ and E_{ij} denotes the matrix defined by 1 in the (i, j)-position and 0 elsewhere.

Example 12. [20] Any triangular matrix of $D^{m \times m}$ having one of the following forms

$$\left(\begin{array}{ccccc} 1 & \star & \ldots & \ldots & \star \\ 0 & 1 & \star & \ldots & \star \\ \vdots & 0 & 1 & \star & \star \\ \vdots & \vdots & 0 & 1 & \star \\ 0 & \ldots & \ldots & 0 & 1 \end{array} \right), \quad \left(\begin{array}{ccccc} 1 & 0 & \ldots & \ldots & 0 \\ \star & 1 & 0 & \ldots & 0 \\ \vdots & \star & 1 & 0 & 0 \\ \vdots & \vdots & \star & 1 & 0 \\ \star & \ldots & \ldots & \star & 1 \end{array} \right),$$

belongs to $E(m, D)$.

We can now state the following useful proposition.

Proposition 9. *If v is a stable element of $U_c(m, D)$, then there exists $E \in E(m, D)$ such that:*

$$E\, v = (1, 0, \ldots, 0)^T.$$

Proof. Let $v = (v_1, \ldots, v_m)^T$ be a stable element of $U_c(m, D)$. Then there exist $a_1, \ldots, a_{m-1} \in D$ such that:

$$v' = (v_1 + a_1\, v_m, \; v_2 + a_2\, v_m, \; v_3 + a_3\, v_m, \ldots, v_{m-1} + a_{m-1}\, v_m)^T \in U_c(m-1, D). \tag{27}$$

Now, let us denote by $v_i' = v_i + a_i\, v_m$, for $i = 1, \ldots, m-1$, and:

$$E_1 = \left(\begin{array}{cccccc} 1 & 0 & 0 & \ldots & 0 & a_1 \\ 0 & 1 & 0 & \ldots & 0 & a_2 \\ 0 & 0 & 1 & \ldots & 0 & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & a_{m-1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{array} \right) \in E(m, D). \tag{28}$$

Then, we easily check that we have $E_1\, v = (v_1', \; v_2', \ldots, v_{m-1}', \; v_m)^T$.

Now, using the fact that $v' \in U_c(m-1, D)$, then there exist $b_1, \ldots, b_{m-1} \in D$ such that we have:

$$\sum_{i=1}^{m-1} b_i\, v_i' = 1.$$

Hence, multiplying both sides of the previous expression by $v_1' - 1 - v_m$, then we get:

$$\sum_{i=1}^{m-1} (v_1' - 1 - v_m)\, (b_i\, v_i') = v_1' - 1 - v_m. \tag{29}$$

If we now denote by $v_i'' = (v_1' - 1 - v_m)\, b_i$, for $i = 1, \ldots, m-1$, and

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ v_1'' & v_2'' & v_3'' & \ldots & v_{m-1}'' & 1 \end{pmatrix} \in E(m, D), \tag{30}$$

then we have $E_2\, (v_1', \ldots, v_{m-1}', v_m)^T = (v_1', \ldots, v_{m-1}', v_1' - 1)^T$. Moreover, if we define

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & -1 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in E(m, D), \tag{31}$$

then we easily check that we have $E_3\, (v_1', \ldots, v_{m-1}', v_1' - 1)^T = (1,\ v_2', \ldots, v_{m-1}',\ v_1' - 1)^T$.

Finally, if we denote by

$$E_4 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -v_2' & 1 & 0 & \ldots & 0 & 0 \\ -v_3' & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -v_{m-1}' & 0 & 0 & \ldots & 1 & 0 \\ -v_1' + 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \in E(m, D), \tag{32}$$

then we obtain $E_4\, (1,\ v_2', \ldots, v_{m-1}',\ v_1' - 1)^T = (1,\ 0, \ldots,\ 0)^T$. Hence, the elementary matrix defined by $E = E_4\, E_3\, E_2\, E_1 \in E(m, D)$ satisfies $E\, (v_1, \ldots, v_m)^T = (1,\ 0, \ldots,\ 0)^T$, which proves the result. $\square$

We sum up the constructive proof of Proposition 9 in the next algorithm.

Algorithm 2.

- **Input:** A stable element $v = (v_1, \ldots, v_m)^T$ of $U_c(m, D)$.
- **Output:** An elementary matrix $E \in D^{m \times m}$ such that $E\, v = (1, 0, \ldots, 0)^T$.

1. Compute $a_1, \ldots, a_{m-1} \in D$ satisfying condition (27).
2. Compute the elementary matrix E_1 defined by (28).
3. Compute $b_1, \ldots, b_{m-1} \in D$ satisfying $\sum_{i=1}^{m-1} b_i\, v_i' = 1$, where v_i' denotes the i^{th} component of the vector $E_1\, v$, $i = 1, \ldots, m-1$, and denote by $v_i'' = (v_1' - 1 - v_m)\, b_i \in D$, $i = 1, \ldots, m-1$.
4. Compute the elementary matrices E_2, E_3 and E_4 respectively defined by (30), (31) and (32).
5. Return the product $E = E_4\, E_3\, E_2\, E_1$.

Let us illustrate Proposition 9 on an example.

Example 13. Let us consider the algebra $D = A_3(\mathbb{Q})$ and the column vector $v = (\partial_1 + x_3 \quad \partial_2 \quad \partial_3)^T$. We easily check that $w = (\partial_3 \quad 0 \quad -(\partial_1 + x_3))$ is a left-inverse of v, i.e., $v \in U_c(3, D)$. Moreover, the vector $v' = (\partial_1 + x_3 \quad \partial_2 + \partial_3)^T$ admits a left-inverse $w' = (\partial_2 + \partial_3 \quad -(\partial_1 + x_3))$, which shows that v' is unimodular, and thus, v is stable. Hence, by Proposition 9, there exists an elementary matrix E such that $E\, v = (1 \quad 0 \quad 0)^T$. Let us compute such a matrix E following Algorithm 2.

The unimodular vector v' shows that we take $a_1 = 0$ and $a_2 = 1$. Hence, if we denote by

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

we then obtain $E_1\, v = (\partial_1 + x_3 \quad \partial_2 + \partial_3 \quad \partial_3)^T$. We check that we have Bézout identity:

$$(\partial_2 + \partial_3)\,(\partial_1 + x_3) - (\partial_1 + x_3)\,(\partial_2 + \partial_3) = 1.$$

Therefore, if we define $v_1'' = (\partial_1 + x_3 - 1 - \partial_3)\,(\partial_2 + \partial_3)$ and $v_2'' = -(\partial_1 + x_3 - 1 - \partial_3)\,(\partial_1 + x_3)$ and the following elementary matrix

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_1'' & v_2'' & 1 \end{pmatrix},$$

we then get $E_2\,(\partial_1 + x_3 \quad \partial_2 + \partial_3 \quad \partial_3)^T = (\partial_1 + x_3 \quad \partial_2 + \partial_3 \quad \partial_1 + x_3 - 1)^T$. Finally, if we define

$$E_3 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ -(\partial_2 + \partial_3) & 1 & 0 \\ -(\partial_1 + x_3 - 1) & 0 & 1 \end{pmatrix},$$

and $E = E_4\, E_3\, E_2\, E_1$, then we have $E\, v = (1 \quad 0 \quad 0)^T$.

We are now in position to state the second main result of this paper.

Theorem 3. *Let k be a field and D a (non-commutative) k-algebra with an involution θ. Then, any stably free left D-module M defined by a finite free resolution of the form*

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0, \tag{33}$$

with $p - q \geq \mathrm{sr}(D)$ is free.

Proof. Using the fact that M is a stably free left D-module, then the exact sequence (33) splits [32], and thus, R admits a right-inverse $S \in D^{p\times q}$. Let us denote by $\widetilde{R} = \theta(R) \in D^{p\times q}$ (see Definition 4) and $\widetilde{S} = \theta(S) \in D^{q\times p}$. Using the fact that $R\,S = I_q$, we then get $\widetilde{S}\,\widetilde{R} = \theta(S)\,\theta(R) = \theta(R\,S) = \theta(I_q) = I_q$, a result showing that we have the following split exact sequence:

$$0 \longleftarrow D^{1\times q} \xleftarrow{.\widetilde{R}} D^{1\times p} \longleftarrow \ker_D(.\widetilde{R}) \longleftarrow 0.$$

Since we have $p > p - q \geq \mathrm{sr}(D)$, the first column $\widetilde{R}_1 \in D^p$ of $\widetilde{R}$ is then stable. Therefore, applying Proposition 9 to $\widetilde{R}_1$, we obtain an elementary matrix $G_1 \in E(p, D)$ which satisfies:

$$G_1\, \widetilde{R}_1 = (1, 0, \ldots, 0)^T.$$

Hence, we obtain

$$G_1\, \widetilde{R} = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \widetilde{R}_2 \\ 0 & \end{pmatrix}, \quad \widetilde{R}_2 \in D^{(p-1)\times(q-1)},$$

where $\star$ denotes an appropriate number of elements in D.

Let us prove that the first column of the matrix $\widetilde{R}_2$ is unimodular. The matrix $G_1\,\widetilde{R}$ admits a left-inverse (e.g., $\widetilde{S}\,G_1^{-1} \in D^{q\times p}$). We then easily check that every left-inverse L of $G_1\,\widetilde{R}$ has the form

$$L = \begin{pmatrix} 1 & \star \\ 0 & L_2 \end{pmatrix}, \quad L_2 \in D^{(q-1)\times(p-1)},$$

which shows that:

$$L_2\,\widetilde{R}_2 = I_{q-1}.$$

As we have $p-1 \geq p-q \geq \mathrm{sr}(D)$, we can apply Proposition 9 to the first column of $\widetilde{R}_2$, we obtain an elementary matrix $F_2 \in E(p-1, D)$ such that:

$$F_2\,\widetilde{R}_2 = \begin{pmatrix} 1 & \star \\ 0 & \\ \vdots & \widetilde{R}_3 \\ 0 & \end{pmatrix}, \quad \widetilde{R}_3 \in D^{(p-2)\times(q-2)}.$$

Hence, if we define $G_2 = \begin{pmatrix} 1 & 0 \\ 0 & F_2 \end{pmatrix}$, then we have:

$$(G_2\,G_1)\,\widetilde{R} = \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ \vdots & 0 & \\ \vdots & \vdots & \widetilde{R}_3 \\ 0 & 0 & \end{pmatrix}.$$

By induction on the number of columns and using the fact that $p-q \geq \mathrm{sr}(D)$, we finally obtain an elementary matrix $G \in E(p, D)$ which satisfies:

$$G\,\widetilde{R} = \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

We now easily check that we have $\ker_D(.(G\,\widetilde{R})) = D^{1\times(p-q)}\,(0 \quad I_{p-q})$. Hence, if we define the matrix $P = (0 \quad I_{p-q}) \in D^{(p-q)\times p}$ and use the fact that G is invertible over D, then we obtain the following commutative exact diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & & \longleftarrow & & D^{1\times q} & \xleftarrow{.\widetilde{R}} & D^{1\times p} & \longleftarrow & \ker_D(.\widetilde{R}) \longleftarrow 0 \\ & & & & \| & & \uparrow .G & & \\ 0 \longleftarrow \mathrm{coker}_D(.(G\,\widetilde{R})) & & \longleftarrow & & D^{1\times q} & \xleftarrow{.(G\,\widetilde{R})} & D^{1\times p} & \xleftarrow{.P} & D^{1\times(p-q)} \longleftarrow 0. \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array} \tag{34}$$

In particular, from (34), we obtain $\ker_D(.\widetilde{R}) = D^{1\times(p-q)}\,(P\,G) \cong D^{1\times(p-q)}$. If we denote by $\widetilde{Q} = P\,G$, then we have the following split exact sequence:

$$0 \longleftarrow D^{1\times q} \stackrel{.\widetilde{R}}{\longleftarrow} D^{1\times p} \stackrel{.\widetilde{Q}}{\longleftarrow} D^{1\times(p-q)} \longleftarrow 0.$$

Using the fact that the adjoint of a split exact sequence is also a split exact sequence [3, 32], we finally obtain the split exact sequence

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{.Q} D^{1\times(p-q)} \longrightarrow 0, \tag{35}$$

with the notation $Q = \theta(\widetilde{Q}) \in D^{p\times(p-q)}$. Therefore, we have

$$M = D^{1\times p}/(D^{1\times q}\,R) \cong D^{1\times p}\,Q = D^{1\times(p-q)},$$

which shows that M is a free left D-module of rank $p-q$ and Q admits a left-inverse. If we denote by $T \in D^{(p-q)\times p}$ a left-inverse of Q, i.e., $T\,Q = I_{p-q}$, then $\{\pi(T_i)\}_{1\leq i\leq p-q}$ is a basis of M, where T_i denotes the i^{th} row of T and $\pi : D^{1\times p} \longrightarrow M$ is the D-morphism which maps any vector in $D^{1\times p}$ to its residue class in M. □

The proof of Theorem 3 was inspired by the one obtained in Corollaire 2.14 in [15] for commutative rings. Hence, Theorem 3 extends Corollaire 2.14 in [15] to non-commutative rings.

Remark 4. Theorem 3 has been stated under the hypothesis that D admits an involution θ. However, using a dual version of Proposition 9, namely, for every $v \in U_r(m, D)$, there exists $E \in E(m, D)$ such that $v\,E = (1, 0, \ldots, 0)$, we can follow the proof of Theorem 3 using, however, right multiplication of R by elementary matrices instead of left multiplication of $\widetilde{R}$. Hence, Theorem 3 is true without this restrictive hypothesis. However, as we are mainly interested in an effective implementation of Theorem 3 in OreModules [2], where only Gröbner bases of left D-modules are computed, we need to impose this condition.

Remark 5. We note that the number $p-q$ only depends on the left D-module M. Indeed, if we have another finite presentation of M of the form

$$0 \longrightarrow D^{1\times q'} \xrightarrow{.R'} D^{1\times r} \xrightarrow{\pi'} M \longrightarrow 0,$$

then, by Schanuel's lemma [32], we obtain that $D^{1\times q'} \oplus D^{1\times p} \cong D^{1\times q} \oplus D^{1\times p'}$. Now, using the fact that D has the left invariant basis number, we then obtain that $q'+p = q+p'$, and thus, $p'-q' = p-q$.

Let us sum up the constructive proof of Theorem 3 in the next algorithm.

Algorithm 3.
- **Input:** A k-algebra D with an involution θ, a matrix $R \in D^{q\times p}$ which admits a right-inverse $S \in D^{p\times q}$ and satisfies that $p-q \geq \mathrm{sr}(D)$ and $\pi : D^{1\times p} \longrightarrow M = D^{1\times p}/(D^{1\times q}\,R)$ the canonical projection.
- **Output:** Two matrices $Q \in D^{p\times(p-q)}$ and $T \in D^{(p-q)\times p}$ satisfying $T\,Q = I_{p-q}$ and such that $\{\pi(T_i)\}_{1\leq i\leq p-q}$ is a basis of the free left D-module M, where T_i denotes the i^{th} row of T.

1. Compute $\widetilde{R} = \theta(R) \in D^{p\times q}$ and set $i = 1$, $V = \widetilde{R}$, $U = I_p$.
2. Denote by $V_i \in D^{p-i+1}$ the column vector formed by taking the last $p-i+1$ elements of the i^{th} column of V.
3. Applying Algorithm 2 to V_i, compute an elementary matrix $F_i \in D^{(p-i+1)\times(p-i+1)}$ such that:

$$F_i\,V_i = (1, 0, \ldots, 0)^T.$$

4. Define the elementary matrix $G_i = \begin{pmatrix} I_{i-1} & 0 \\ 0 & F_i \end{pmatrix} \in D^{p \times p}$ with the convention $G_1 = F_1$.
5. If $i < q$ then return to step 2 with $V \leftarrow G_i\, V$, $U \leftarrow G_i\, U$ and $i \leftarrow i+1$.
6. Define $G = G_q\, U$ and denote by $\widetilde{Q}$ the matrix formed by selecting the last $p-q$ rows of G.
7. Define $Q = \theta(\widetilde{Q}) \in D^{p \times (p-q)}$ and compute a left-inverse $T \in D^{(p-q) \times p}$ of Q.

Let us illustrate Algorithm 3 on an example.

Example 14. Let us consider the following time-varying ordinary differential linear system:

$$\begin{cases} \dot{x}_2(t) = u_2(t), \\ \dot{x}_1(t) = t\, u_1(t). \end{cases} \tag{36}$$

We define the algebra $D = A_1(\mathbb{Q})$, the matrix

$$R = \begin{pmatrix} 0 & \frac{d}{dt} & 0 & -1 \\ \frac{d}{dt} & 0 & -t & 0 \end{pmatrix} \in D^{2 \times 4},$$

and the left D-module $M = D^{1 \times 4}/(D^{1 \times 2}\, R)$. Using an algorithm developed in [3, 25], we obtain that

$$S = \begin{pmatrix} 0 & t \\ 0 & 0 \\ 0 & \frac{d}{dt} \\ -1 & 0 \end{pmatrix} \in D^{4 \times 2}$$

is a right-inverse of R, i.e., $R\, S = I_2$. Therefore, the left D-module M is stably free with $\mathrm{rank}_D(M) = 2$. Using Theorem 3 and 4 of Example 11, i.e., $\mathrm{sr}(D) = 2$, we then obtain that M is a free left D-module. Let us compute a basis of M following Algorithm 3. We first compute the formal adjoint $\widetilde{R}$ of R:

$$\widetilde{R} = \begin{pmatrix} 0 & -\frac{d}{dt} \\ -\frac{d}{dt} & 0 \\ 0 & -t \\ -1 & 0 \end{pmatrix} \in D^{4 \times 2}.$$

Now, following Algorithm 2 for the first column $v_1 = \begin{pmatrix} 0 & -\frac{d}{dt} & 0 & 1 \end{pmatrix}^T$ of $\widetilde{R}$, we obtain that the vector $v_1' = \begin{pmatrix} 1 & -\frac{d}{dt} & 0 \end{pmatrix}^T$ is trivially unimodular, which shows that we can choose $a_1 = 1$ and $a_2 = 0$ and define the elementary matrix:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We then have $E_1\, v_1 = \begin{pmatrix} 1 & -\frac{d}{dt} & 0 & -1 \end{pmatrix}^T$. Now, using that $w' = (1 \quad 0 \quad 0)$ is a left-inverse of v_1', we can take $b_1 = 1$, $b_2 = 0$ and define the following elementary matrices:

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{d}{dt} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}.$$

We easily check that we have:

$$G_1 = E_4\, E_3\, E_2\, E_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{d}{dt} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad G_1\, \widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -t \\ 0 & -\frac{d}{dt} \end{pmatrix}.$$

Let us now consider the sub-column $v_2 = \begin{pmatrix} 0 & -t & -\frac{d}{dt} \end{pmatrix}^T$ of the matrix $G_1\, \widetilde{R}$. We apply Algorithm 2 to v_2. We easily check that $v_2' = \begin{pmatrix} -\frac{d}{dt} & -t \end{pmatrix}^T$ has a left-inverse defined by $w_2' = \begin{pmatrix} -t & -\frac{d}{dt} \end{pmatrix}$. Therefore, we can take $a_1 = 1$ and $a_2 = 0$ and define the following elementary matrices:

$$E_1' = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -t & \frac{d}{dt} & 1 \end{pmatrix}, \quad E_3' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_4' = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{d}{dt} + 1 & 0 & 1 \end{pmatrix}.$$

We then have

$$F_2 = E_4'\, E_3'\, E_2'\, E_1' = \begin{pmatrix} 1+t & -\frac{d}{dt} & t \\ t\,(t+1) & -t\,\frac{d}{dt} + 1 & t^2 \\ t\,\frac{d}{dt} + \frac{d}{dt} + 2 & -\frac{d^2}{dt^2} & t\,\frac{d}{dt} + 2 \end{pmatrix}, \quad F_2\, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Finally, let us define the following matrices:

$$G_2 = \begin{pmatrix} 1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad G = G_2\, G_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ t & t+1 & -\frac{d}{dt} & -(t+1)\,\frac{d}{dt} \\ t^2 & t\,(t+1) & -t\,\frac{d}{dt} + 1 & -t\,(t+1)\,\frac{d}{dt} \\ t\,\frac{d}{dt} + 2 & t\,\frac{d}{dt} + \frac{d}{dt} + 2 & -\frac{d^2}{dt^2} & -\left(t\,\frac{d}{dt} + t + 2\right)\,\frac{d}{dt} \end{pmatrix}.$$

Then, we have:

$$G\, \widetilde{R} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, if we consider the matrix

$$Q = \begin{pmatrix} t^2 & -t\,\frac{d}{dt} + 1 \\ t^2 + t & -(t+1)\,\frac{d}{dt} + 1 \\ t\,\frac{d}{dt} + 2 & -\frac{d^2}{dt^2} \\ t^2\,\frac{d^2}{dt^2} + t\,\frac{d}{dt} + 2\,t + 1 & -(t+1)\,\frac{d^2}{dt^2} \end{pmatrix} \tag{37}$$

formed by taking the last two columns of the formal adjoint of G, then Q admits a left-inverse $T \in D^{2\times 4}$ defined by:

$$T = \begin{pmatrix} 0 & 0 & t+1 & -1 \\ t+1 & -t & 0 & 0 \end{pmatrix}.$$

Hence, a basis of M is defined by $\{\pi((0,\ 0,\ t+1,\ -1)),\ \pi((t+1,\ -t,\ 0,\ 0))\}$, where $\pi : D^{1\times 4} \longrightarrow M$ denotes the canonical projection onto M.

Let us now consider a left D-module $\mathcal{F}$ (e.g., $\mathcal{F} = C^\infty(\mathbb{R})$, $\mathcal{D}'(\mathbb{R})$) and the $\mathcal{F}$-behaviour $\ker_{\mathcal{F}}(R.)$. Using the matrix Q defined by (37), we obtain the injective parametrization of (36)

$$\left\{ \begin{array}{l} \dot{x}_2(t) = u_2(t), \\ \dot{x}_1(t) = t\, u_1(t), \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} x_1(t) = t^2\, y_1(t) - t\, \dot{y}_2(t) + y_2(t), \\ x_2(t) = (t^2 + t)\, y_1(t) - (t+1)\, \dot{y}_2(t) - y_2(t), \\ u_1(t) = t\, \dot{y}_1(t) + 2\, y_1(t) - \ddot{y}_2(t), \\ u_2(t) = t^2\, \ddot{y}_1(t) + t\, \dot{y}_1(t) + (2\, t + 1)\, y_1(t) - (1+t)\, \ddot{y}_2(t), \end{array} \right. \tag{38}$$

which proves that (36) is a flat system. Finally, a flat output $(y_1,\ y_2)$ of $\ker_{\mathcal{F}}(R.)$ is defined by:

$$\left\{ \begin{array}{l} y_1(t) = (t+1)\, u_1(t) - u_2(t), \\ y_2(t) = (t+1)\, x_1(t) - t\, x_2(t). \end{array} \right.$$

The next corollary is a well-known result in the literature of non-commutative algebra. See for instance [20]. However, we give here a simple and constructive proof based on Algorithm 1 and the kind of Gaussian elimination used in the proof of Theorem 3 (see Algorithm 3).

Corollary 3. *Let k be a field and D a (non-commutative) k-algebra with an involution θ, M a stably free left D-module with $\operatorname{rank}_D(M) \geq \operatorname{sr}(D)$ and (11) a finite free resolution of M. Then, M is a free left D-module.*

Proof. Let us consider a stably free left D-module $M = D^{1\times p_0}/(D^{1\times p_1}\, R_1)$, $R \in D^{p_1\times p_0}$. Using Algorithm 1, we can always suppose that M is defined by $M \cong D^{1\times p}/(D^{1\times q}\, R)$, where the matrix $R \in D^{q\times p}$ admits a right-inverse $S \in D^{p\times q}$. See Remark 3 for more details. Therefore, we have the following finite free resolution of M:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0.$$

In particular, we obtain that $\operatorname{rank}_D(M) = p - q$, and thus, the hypothesis that $\operatorname{rank}_D(M) \geq \operatorname{sr}(D)$ implies $p \geq q + \operatorname{sr}(D)$. Hence, by Theorem 3, we obtain that M is a free left D-module. □

Algorithm 4.
- **Input:** A k-algebra with an involution θ, a matrix $R_1 \in D^{p_1\times p_0}$ such that the left D-module $M = D^{1\times p_0}/(D^{1\times p_1}\, R_1)$ is stably free with $\operatorname{rank}_D(M) \geq \operatorname{sr}(D)$ and a finite free resolution (11) of M.
- **Output:** The matrices $R \in D^{q\times p}$, $Q \in D^{p\times(p-q)}$ and $T \in D^{(p-q)\times p}$ satisfying $M \cong D^{1\times p}/(D^{1\times q}\, R)$, $T\, Q = I_{p-q}$ and $\{\pi(T_i)\}_{1\leq i\leq p-q}$ is a basis of the free left D-module $D^{1\times p}/(D^{1\times q}\, R)$, where T_i denotes the i^{th} row of T and $\pi : D^{1\times p} \longrightarrow D^{1\times p}/(D^{1\times q}\, R)$ the canonical projection.

1. Applying Algorithm 1, we obtain a finite free resolution of M of the form:

$$0 \longrightarrow D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0. \tag{39}$$

2. Applying Algorithm 3 to the matrix $R \in D^{q\times p}$, we finally obtain the matrices $Q \in D^{p\times(p-q)}$ and $T \in D^{(p-q)\times p}$ satisfying $T\, Q = I_{p-q}$ and such that $\{\pi(T_i)\}_{1\leq i\leq p-q}$ is a basis of the free left D-module $D^{1\times p}/(D^{1\times q}\, R) \cong M$ (see Remark 1).

5.2 The Weyl algebra case

We shall now focus on the two particular cases $D = A_n(k)$ and $D = B_n(k)$.

We state another nice result due to J. T. Stafford which will allow us to constructively compute the elements $a_i \in D$ satisfying (27), i.e., to effectively handle step 1 of Algorithm 2.

Theorem 4. *[34] Let k be a field containing $\mathbb{Q}$ and $D = A_n(k)$ or $B_n(k)$. If v_1, v_2 and $v_3 \in D$, then there exist $a_1, a_2 \in D$ such that the left ideal $I = D\, v_1 + D\, v_2 + D\, v_3$ of D satisfies:*

$$I = D\,(v_1 + a_1\, v_3) + D\,(v_2 + a_2\, v_3).$$

We illustrate Theorem 4 on two simple examples.

Example 15. Let us consider $D = A_3(\mathbb{Q})$ and the left ideal $I = D\,\partial_1 + D\,\partial_2 + D\,\partial_3$. We can check that we have $I = D\,\partial_1 + D\,(\partial_2 + x_1\,\partial_3)$ because we have:

$$\begin{cases} \partial_2 = (x_1\,(\partial_2 + x_1)\,\partial_3)\,\partial_1 + (-x_1\,\partial_1 + 1)\,(\partial_2 + x_1\,\partial_3), \\ \partial_3 = -(\partial_2 + x_1\,\partial_3)\,\partial_1 + \partial_1\,(\partial_2 + x_1\,\partial_3). \end{cases}$$

If we now consider the left ideal $J = D\,(\partial_1 + x_3) + D\,\partial_2 + D\,\partial_3$ of D (see Example 13), the we have $J = D\,(\partial_1 + x_3) + D\,(\partial_2 + \partial_3)$ as:

$$\begin{cases} \partial_2 = (\partial_2\,(\partial_2 + \partial_3))\,(\partial_1 + x_3) - (\partial_2\,(\partial_1 + x_3))\,(\partial_2 + \partial_3), \\ \partial_3 = (\partial_3\,(\partial_2 + \partial_3))\,(\partial_1 + x_3) - (\partial_3\,(\partial_1 + x_3))\,(\partial_2 + \partial_3). \end{cases}$$

Two constructive algorithms of Theorem 4 have recently been presented by A. Hillebrand and W. Schmale on the one hand and by A. Leykin on the other hand. We refer the reader to [11, 14] for more details. Both strategies have been implemented in the package Stafford [31]. However, we point out that, due to the large number of Gröbner basis computations used in [11, 14], Theorem 4 only works constructively on relatively small examples.

Let us now consider a unimodular column vector $v = (v_1, \ldots, v_m)^T$ where $m \geq 2$. Using the fact that $\mathrm{sr}(D) = 2$, the vector v is then stable. Therefore, there exist $a_1, \ldots, a_{m-1} \in D$ such that the column vector $v' = (v_1 + a_1\, v_m, \ldots, v_{m-1} + a_{m-1}\, v_m)^T$ is unimodular. A constructive way to compute the a_i is, for instance, to apply a constructive version of Theorem 4 to the left ideal $I = D\, v_1 + D\, v_2 + D\, v_m$. Then, we find $a_1, a_2 \in D$ such that:

$$I = D\,(v_1 + a_1\, v_m) + D\,(v_2 + a_2\, v_m).$$

Using the fact that v is unimodular, i.e., $\sum_{i=1}^{m} D\, v_i = D$, we obtain

$$D\,(v_1 + a_1\, v_m) + D\,(v_2 + a_2\, v_m) + \sum_{i=3}^{m-1} D\, v_i = D,$$

showing that the vector $(v_1 + a_1\, v_m, v_2 + a_2\, v_m, v_3, \ldots, v_{m-1})^T$ is unimodular. Hence, using Stafford, we then have a constructive way to perform step 1 of Algorithm 2, and thus, the complete Algorithm 2, as step 3 can be performed using the command LeftInverse of OreModules.

We note that a simple constructive algorithm for the computation of two generators of left ideals over $D = A_1(\mathbb{Q})$ is developed in [8, 19]. However, we do not know yet whether or not we can use it in order to compute the $a_i \in D$ satisfying the relation (27) for a stable vector $v = (v_1, \ldots, v_m)^T \in D$.

Combining Theorem 3 with 3 and 4 of Example 11, we then obtain the following result.

Corollary 4. *[34] If k is a field containing $\mathbb{Q}$ and $D = A_n(k)$ or $B_n(k)$, then any stably free left D-module M satisfying $\mathrm{rank}_D(M) \geq 2$ is free.*

With the aid of the functions Involution, Mult and LeftInverse of OreModules, Algorithms 2 and 3 become constructive. Moreover, using the command MinimalFreeResolution (see Remark 3), we have a way to compute a finite free resolution of M of the form (39) and to check whether or not M is a stably free left D-module with $\mathrm{rank}_D(M) \geq 2$ (see [3] for another algorithm checking stably freeness using the computation of certain extension modules $\mathrm{ext}^i_D(\widetilde{N}, D)$, where $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\theta(R))$ and θ is the involution defined in 2 of Example 2). We conclude that Algorithm 4 can be performed constructively.

Another algorithm for computing bases of free modules over $A_n(k)$ has also been developed in [7] following the proof given by J. T. Stafford [34]. However, despite the interest of [7], Algorithm 4 seems to be easier to understand and to implement. Indeed, it is conceptually nothing else than Gaussian elimination as soon as a constructive version of Theorem 4 is available.

Example 16. Let us consider $D = A_3(\mathbb{Q})$, $R = (-\partial_1 + x_3 \quad -\partial_2 \quad -\partial_3)$ and the left D-module $M = D^{1\times 3}/(D\,R)$. We easily check that $S = (\partial_3 \quad 0 \quad \partial_1 - x_3)^T$ is a right-inverse of R, a fact showing that M is a stably free left D-module of rank 2. Hence, by Corollary 4, we obtain that M is a free left D-module. Let us compute a basis of M following Algorithm 3. We first compute the formal adjoint $\widetilde{R} = (\partial_1 + x_3 \quad \partial_2 \quad \partial_1)^T$ of R. We then need to compute an elementary matrix G such that $G\,\widetilde{R} = (1 \quad 0 \quad 0)^T$. However, such an elementary matrix G has already been computed in Example 13 and was denoted by E. Therefore, if we form the matrix Q by selecting the last two columns of the formal adjoint of $\widetilde{E}$, then we obtain that the system $(\partial_1 - x_3)\,y_1(x) + \partial_2\,y_2(x) + \partial_3\,y_3(x) = 0$ admits the following injective parametrization

$$\left\{\begin{array}{l} y_1(x) = ((1 - \theta(v_1''))\,(\partial_2 + \partial_3))\,z_1(x) + ((1 - \theta(v_1''))\,(\partial_1 - x_3) + 1)\,z_2(x), \\ y_2(x) = (-\theta(v_2'')\,(\partial_2 + \partial_3) + 1)\,z_1(x) - \theta(v_2'')\,(\partial_1 - x_3)\,z_2(x), \\ y_3(x) = (-(1 + \theta(v_2''))\,(\partial_2 + \partial_3) + 1)\,z_1(x) - (1 + \theta(v_2''))\,(\partial_1 - x_3)\,z_2(x), \end{array}\right. \tag{40}$$

where θ denotes the standard involution of $A_3(\mathbb{Q})$ and:

$$\left\{\begin{array}{l} v_1'' = (\partial_1 - \partial_3 + x_3 - 1)\,(\partial_2 + \partial_3), \\ v_2'' = -(\partial_1 - \partial_3 + x_3 - 1)\,(\partial_1 + x_3). \end{array}\right.$$

If we develop the expressions in (40) (see [2] for more details), we can check that we have

$$\left\{\begin{array}{lcl} z_1(x) & = & (-\partial_1^2 + \partial_1\,\partial_3 - x_3\,\partial_3 + (2\,x_3 - 1)\,\partial_1 + x_3 - x_3^2 + 1)\,y_2(x) \\ & & +(\partial_1^2 - \partial_1\,\partial_3 + x_3\,\partial_3 - (2\,x_3 - 1)\,\partial_1 + x_3^2 - x_3)\,y_3(x), \\ z_2(x) & = & y_1(x) + (-\partial_3^2 + \partial_1\,\partial_2 - \partial_2\,\partial_3 + \partial_1\,\partial_3 + \partial_2 - (x_3 - 1)\,\partial_3 - x_3 - 2)\,y_2(x) \\ & & +(\partial_3^2 - \partial_1\,\partial_2 + \partial_2\,\partial_3 - \partial_1\,\partial_3 + (x_3 - 1)\,\partial_3 + (x_3 - 1)\,\partial_2 + 2)\,y_3(x). \end{array}\right.$$

showing that $\{z_1,\, z_2\}$ is a basis of the free left D-module M.

We refer the reader to [31] for more detailed examples and applications.

Finally, Corollary 4 shows that we should investigate when a stably free module of rank 1 over the algebras $D = A_n(k)$ or $B_n(k)$ is free. Using the fact that D is a domain, this problem is equivalent to recognize when a stably free left ideal of D is principal. We shall study this problem in the future. However, let us illustrate this last idea on an example which is relevant in control theory [35].

Example 17. Let us consider the system $\dot{x}(t) = t^k\,u(t)$, where $k \in \mathbb{Z}_+$, and define $D = A_1(\mathbb{Q})$, $R_k = \left(\frac{d}{dt} \quad -t^k\right)$ and the left D-module $M_k = D^{1\times 2}/(D\,R_k)$. As R_k has full row rank, we know that M_k is stably free iff the left D-module $\widetilde{N} = D^{1\times q}/(D^{1\times p}\,\widetilde{R_k})$, where $\widetilde{R_k} = \left(-\frac{d}{dt} \quad -t^k\right)^T$ is the formal adjoint of R_k, is the zero module [3, 25]. Using the definition of $\widetilde{N}$, we then obtain:

$$\left\{\begin{array}{l} -\dot{\lambda} = 0, \\ -t^k\,\lambda = 0, \end{array}\right. \Rightarrow t^k\,\dot{\lambda} + k\,t^{k-1}\,\lambda = 0 \Rightarrow t^{k-1}\,\lambda = 0 \Rightarrow \ldots \Rightarrow \lambda = 0 \Rightarrow \widetilde{N} = 0.$$

Hence, the left D-module M_k is stably free for all $k \in \mathbb{Z}_+$. Now, we can prove that we have the following exact sequence:

$$0 \longrightarrow D \xrightarrow{.R_k} D^{1\times 2} \xrightarrow{.Q_k} D \longrightarrow D/(D^{1\times 2}\,Q_k) \longrightarrow 0, \quad Q_k = \left(\begin{array}{c} t^{k+1} \\ t\,\frac{d}{dt} + k + 1 \end{array}\right).$$

Since $P_k = D/(D^{1\times 2}\, Q_k)$ is a non-trivial torsion left D-module, the matrix Q_k is called a *minimal parametrization* of M_k. See [3, 26] for more details. Hence, we obtain that

$$M_k = D^{1\times 2}/(D\, R_k) \cong D^{1\times 2}\, Q_k = D\, t^{k+1} + D\left(t\,\frac{d}{dt} + k + 1\right),$$

showing that M_k is isomorphic to the left ideal I_k of D generated by t^{k+1} and $t\,\frac{d}{dt} + k + 1$. Using the fact that D is a domain, we obtain that M_k is a free left D-module iff I_k is a principal left ideal of D.

Given a left ideal J of $A_1(\mathbb{Q})$, we define by $L(a) = a_r(t) \neq 0$ (resp., $\mathrm{ord}(a) = r$) the *leading term* (resp., *order*) of an element $a = \sum_{i=0}^{r} a_i(t)\, \frac{d^i}{dt^i} \in D$ and we denote by J_m the family of ideals of $\mathbb{Q}[t]$ given by:

$$J_m = \{L(a) \in \mathbb{Q}[t] \mid a \in J,\ \mathrm{ord}(a) = m\} \cup \{0\}.$$

We easily check that $J_m \subseteq J_{m+1}$. Now, if J is a principal left ideal of D, then we easily check that we have $J_m = J_{m+1}$ for all $m \geq 0$ as:

$$\forall\, a \in D, \quad \begin{cases} L\left(\frac{d}{dt}\, a\right) = L(a), \\ L(t\, a) = t\, L(a). \end{cases}$$

For the left ideal $I_k = D\, t^{k+1} + D\left(t\,\frac{d}{dt} + k + 1\right)$, we get $(I_k)_0 = (t^{k+1})$ and $(I_k)_1 = (t)$, which proves that $(I_k)_0 \subsetneq (I_k)_1$ as soon as we have $k \geq 1$, and thus, the left D-module M_k is not free when $k \geq 1$. When $k = 0$, we check that $I_0 = D\, t + D\left(t\,\frac{d}{dt} + 1\right) = D\, t$ as we have $\frac{d}{dt}\, t = t\,\frac{d}{dt} + 1$. Hence, I_0 is a principal left ideal of D, and thus, M_0 is a free left D-module.

To finish, we give a few more results concerning different Ore algebras.

Theorem 5. *(Theorem 11.1.14 and 11.1.17 in [20]) If D is a left noetherian ring, then any stably free left D-module M with* $\mathrm{rank}_D(M) \geq \mathrm{Kdim}(D) + 1$ *is free, where* Kdim *denotes the* Krull dimension.

The next proposition gives some bounds on the Krull dimension of Ore algebras.

Proposition 10. *(Proposition 6.5.4 in [20]) Let A be a left noetherian ring, σ an automorphism of A and δ a σ-derivation. Then, we have:*

1. $\mathrm{Kdim}(A) \leq \mathrm{Kdim}(A[\partial; \sigma, \delta]) \leq \mathrm{Kdim}(A) + 1$.
2. *If $\delta = 0$, then we have* $\mathrm{Kdim}(A[\partial; \sigma, 0]) = \mathrm{Kdim}(A) + 1$.
3. *If A is a left artinian ring, then* $\mathrm{Kdim}(A[\partial; \sigma, \delta]) = 1$.

In particular, we obtain the following examples of Krull dimensions.

Example 18.

1. $\mathrm{Kdim}(B_1(k)) = 1$.
2. (Theorem 6.6.15 in [20]) If k is a field containing $\mathbb{Q}$, then $\mathrm{Kdim}(A_n(k)) = n$.
3. If $D = A_1(k)[\partial_2; \sigma_2, 0]$, where $\sigma_2(a(t)) = a(t - h)$, $h \in \mathbb{R}_+$, denotes the ring of differential time-delay operators with polynomial coefficients in t, then $\mathrm{Kdim}(D) = 2$. More generally, if $D = A_1(k)[\partial_2; \sigma_2, 0] \ldots [\partial_n; \sigma_n, 0]$, where $\sigma_i(a(t)) = a(t - h_i)$, $h_i \in \mathbb{R}_+$, and the $\mathbb{Q}$-vector space formed by the h_i is n-dimensional, then $\mathrm{Kdim}(D) = n + 1$.

   A similar result holds if we use $B_1(k)$ instead of $A_1(k)$.
4. If $D = k(x_1, \ldots, x_n)[\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ denotes the algebra of forward shifts with rational coefficients, where σ_i and δ_i are defined by

   $$\sigma_i(a)(x_1, \ldots, x_n) = a(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n), \quad \delta_i = 0, \quad 1 \leq i \leq n,$$

   then $\mathrm{Kdim}(D) = n$.

5. If $D = k[x_1, \ldots, x_n][\partial_1; \sigma_1, \delta_1] \ldots [\partial_n; \sigma_n, \delta_n]$ denotes the algebra of forward shifts with polynomial coefficients, where σ_i and δ_i are defined as in 4, then $\mathrm{Kdim}(D) = 2\, n$.

Using Theorem 5 and, for instance, 3 of Example 18, we deduce that any stably free left module M over the ring of differential time-delay operators $D = A_1(k)[\partial_2; \sigma_2, 0]$ with $\mathrm{rank}_D(M) \geq 3$ is free. Unfortunately, we cannot use this result in order to check whether or not the stably free differential time-delay system defined in Example 10 is free as its rank equals 1. However, we still can use the system theoretic interpretation of stably freeness developed in Section 4 in order to do some motion planning as shown in [21]. Finally, we note that the lower bound on the rank of the module given in Theorem 5 can generally be improved as it is the case, for instance, for the Weyl algebra $A_n(k)$ (see 1 of Example 18 and Corollary 4).

6 Conclusion

In this paper, we have shown how to use the concept of stable range of a ring D in order to reduce the computation of bases of free left D-modules to Gaussian elimination. In the case of the Weyl algebras $D = A_n(k)$ or $B_n(k)$ over a field k of characteristic 0, by using the recent constructive versions of the result of J. T. Stafford on the number of generators of left D-ideals [14, 11, 34], Algorithm 4 gives an effective way for the computation of bases of stably free left D-modules of rank greater or equal to 2. This algorithm has been implemented in the package [31] developed under OreModules [2]. Finally, it seems to us that Algorithm 4 is simpler and more tractable than the algorithm developed in [7].

As noticed in [33], different injective parametrizations of (36) can be obtained. This result is easily explained by the fact that there are different ways to obtain the elements $a_i \in D$ satisfying (27). In the Weyl algebra case, we have chosen to apply Stafford's result, i.e., Theorem 4, to the vector formed by the first two and the last component of the vector V_i defined in Algorithm 3. See Section 5.2 for more details. This is indeed a particular choice and Algorithm 3 can be optimized by firstly inspecting the components of V_i in order to get simpler $a_1, a_2 \in D$ satisfying (27). In particular, this means that some heuristics must be added in the implementation of Algorithm 3 in order to simplify and speed-up the computation of the bases. Some of them have been implemented in [31], but much work in this direction still needs to be done in the future.

Another aspect which can be used in order to optimize Algorithm 4 is to allow us to use more general transformations than only the elementary ones. Indeed, an inspection of the proof of Theorem 3 shows that we only need that G is an invertible matrix over D, i.e., $G \in \mathrm{GL}(p, D)$. Algorithm 2 gives a general way to compute $E \in E(m, D)$ satisfying $E\, v = (1, 0, \ldots, 0)^T$ for any stable vector $v \in U_c(m, D)$. But, in some particular cases, it is possible to find a simpler $E \in \mathrm{GL}(m, D)$ satisfying $E\, v = (1, 0, \ldots, 0)^T$ which can avoid the multiplication by the factor $v_1' - 1 - v_m$ in (29), and thus, lower the order of the final basis. Finally, much work must to be done in order to optimize the time-consuming algorithms of J. T. Stafford's result developed in [14, 11].

All these questions will be studied in the future as well as their extension to different classes of Ore algebras (e.g., the algebra of differential time-delay operators). Applications of the different algorithms developed in this paper to control theory and, in particular, to the effective computation of flat outputs of flat linear systems over Ore algebras will be developed in forthcoming publications.

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