Time-delay Example 4

Example from the paper by C. Califano, S. Li, C.H. Moog.

Let us consider the following differential time-delay nonlinear system:

```
eqs = {x1'[t] → 0,
    x2'[t] → x1[t - 1] u[t],
    x3'[t] → (x5[t - 1] - x2[t - 1]) u[t],
    x4'[t] → 0,
    x5'[t] → x4[t - 1] u[t],
    x6'[t] → x5[t - 1] u[t]};
vars = {x1[t], x2[t], x3[t], x4[t], x5[t], x6[t], u[t]};
TableForm[eqs]
```

$x_1'[t] \to 0$
$x_2'[t] \to u[t]\, x_1[-1+t]$
$x_3'[t] \to u[t]\, (-x_2[-1+t] + x_5[-1+t])$
$x_4'[t] \to 0$
$x_5'[t] \to u[t]\, x_4[-1+t]$
$x_6'[t] \to u[t]\, x_5[-1+t]$

Let us introduce the following Ore algebra A of differential time-delay operators

```
replA = ModelToReplacementRules[eqs, t];
A = OreAlgebraWithRelations[Der[t], S[-1][t], replA]
```

$\mathbb{K}(t)[D_t;\, 1,\, D_t]\left[\left(S_t^{-1}\right);\, \#1 \,/.\, t \to t-1\, \&,\, 0\, \&\right]$

and the matrix R of ordinary differential time-delay operators which defines the generic linearization of the above nonlinear system

```
R = ToOrePolynomialD[eqs, vars, A]
```

$\{\{D_t, 0, 0, 0, 0, 0, 0\}, \{-u[t]\,(S_t^{-1}), D_t, 0, 0, 0, 0, -x_1[-1+t]\},$
$\{0, u[t]\,(S_t^{-1}), D_t, 0, -u[t]\,(S_t^{-1}), 0, x_2[-1+t] - x_5[-1+t]\}, \{0, 0, 0, D_t, 0, 0, 0\},$
$\{0, 0, 0, -u[t]\,(S_t^{-1}), D_t, 0, -x_4[-1+t]\}, \{0, 0, 0, 0, -u[t]\,(S_t^{-1}), D_t, -x_5[-1+t]\}\}$

and the left A-module, finitely presented by R, defined by M = $A^{1\times 7} / A^{1\times 6} R$.

Let us compute the adjoint of R:

```
Radj = Involution[R, A]
```

$\{\{D_t, -u[1-t]\,(S_t^{-1}), 0, 0, 0, 0\}, \{0, D_t, u[1-t]\,(S_t^{-1}), 0, 0, 0\},$
$\{0, 0, D_t, 0, 0, 0\}, \{0, 0, 0, D_t, -u[1-t]\,(S_t^{-1}), 0\},$
$\{0, 0, -u[1-t]\,(S_t^{-1}), 0, D_t, -u[1-t]\,(S_t^{-1})\}, \{0, 0, 0, 0, 0, D_t\},$
$\{0, -x_1[-1-t], x_2[-1-t] - x_5[-1-t], 0, -x_4[-1-t], -x_5[-1-t]\}\}$

Let us check whether or not M is torsion-free:

```
{Ann, Rp, Q} = Exti[Radj, A, 1];
Ann
```

$$
\begin{aligned}
&\{\{D_t, 0, 0, 0, 0, 0, 0\},\\
&\{0, u[-1+t]\, u[t]^2\, D_t^3 + \left(-u[t]^2\, u'[-1+t] - 2\, u[-1+t]\, u[t]\, u'[t]\right) D_t^2 +\\
&\quad \left(u[t]\, u'[-1+t]\, u'[t] + 2\, u[-1+t]\, u'[t]^2 - u[-1+t]\, u[t]\, u''[t]\right) D_t, 0, 0, 0, 0, 0\},\\
&\{0, 0, u[t]\, D_t^2 - u'[t]\, D_t, 0, 0, 0, 0\}, \{0, 0, 0, D_t, 0, 0, 0\},\\
&\{0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, -u[t]\, D_t + u'[t], 0\},\\
&\{0, 0, 0, 0, 0, 0, u[-1+t]\, u[t]\, D_t + (-u[t]\, u'[-1+t] - 2\, u[-1+t]\, u'[t])\}\}
\end{aligned}
$$

```
Rp
```

$$
\begin{aligned}
&\{\{0, 0, 0, 1, 0, 0, 0\},\\
&\{0, 0, x_4[-2+t]\, x_4[-1+t], 0, x_2[-1+t]\, x_4[-2+t] - x_1[-2+t]\, x_5[-1+t],\\
&\quad x_1[-2+t]\, x_4[-1+t] - x_4[-2+t]\, x_4[-1+t], 0\},\\
&\{0, -x_4[-1+t], 0, 0, x_1[-1+t], 0, 0\}, \{1, 0, 0, 0, 0, 0, 0\},\\
&\{0, 0, 0, 0, -u[t]\, \left(S_t^{-1}\right), D_t, -x_5[-1+t]\}, \{0, 0, 0, 0, -D_t, 0, x_4[-1+t]\},\\
&\{0, 0, 0, 0, 0, u[t]\, D_t^2 - u'[t]\, D_t, -u[t]\, x_5[-1+t]\, D_t -\\
&\quad u[t]^2\, x_4[-2+t]\, \left(S_t^{-1}\right) + (-u[-1+t]\, u[t]\, x_4[-2+t] + x_5[-1+t]\, u'[t])\}\}
\end{aligned}
$$

```
Q
```

$$
\begin{aligned}
&\{\{0, 0\},\\
&\{\left(u[t]\, u[1+t]\, x_1[-1+t]\, u'[-2+t] - u[-2+t]\, u[1+t]\, x_1[-1+t]\, u'[t]\right) D_t \left(S_t^{-1}\right) +\\
&\quad (-u[-1+t]\, u[2+t]\, x_1[-1+t]\, u'[t] + u[-1+t]\, u[t]\, x_1[-1+t]\, u'[2+t])\, D_t +\\
&\quad \left(2\, u[t]\, x_1[-1+t]\, u'[-2+t]\, u'[1+t] - 2\, u[-2+t]\, x_1[-1+t]\, u'[t]\, u'[1+t] +\right.\\
&\qquad \left. u[t]\, u[1+t]\, x_1[-1+t]\, u''[-2+t] - u[-2+t]\, u[1+t]\, x_1[-1+t]\, u''[t]\right) \left(S_t^{-1}\right) +\\
&\quad (-2\, u[2+t]\, x_1[-1+t]\, u'[-1+t]\, u'[t] + 2\, u[t]\, x_1[-1+t]\, u'[-1+t]\, u'[2+t] -\\
&\qquad u[-1+t]\, u[2+t]\, x_1[-1+t]\, u''[t] + u[-1+t]\, u[t]\, x_1[-1+t]\, u''[2+t]),\\
&\quad \left(u[1+t]\, x_1[-1+t]\, u'[-1+t] - u[-1+t]\, x_1[-1+t]\, u'[1+t]\right) D_t^2 +\\
&\quad (3\, u[1+t]\, x_1[-1+t]\, u''[-1+t] - 3\, u[-1+t]\, x_1[-1+t]\, u''[1+t])\, D_t +\\
&\quad \left(3\, x_1[-1+t]\, u'[1+t]\, u''[-1+t] - 3\, x_1[-1+t]\, u'[-1+t]\, u''[1+t] +\right.\\
&\qquad \left. 2\, u[1+t]\, x_1[-1+t]\, u^{(3)}[-1+t] - 2\, u[-1+t]\, x_1[-1+t]\, u^{(3)}[1+t]\right)\},\\
&\{\left(-u[t]\, u[1+t]\, x_2[-1+t]\, u'[-2+t] + u[t]\, u[1+t]\, x_5[-1+t]\, u'[-2+t] +\right.\\
&\qquad \left. u[-2+t]\, u[1+t]\, x_2[-1+t]\, u'[t] - u[-2+t]\, u[1+t]\, x_5[-1+t]\, u'[t]\right) D_t \left(S_t^{-1}\right) +\\
&\quad \left(-u[-1+t]\, u[t]^2\, x_1[-2+t]\, u'[-3+t] + u[-1+t]\, u[t]^2\, x_4[-2+t]\, u'[-3+t] +\right.\\
&\qquad \left. u[-3+t]\, u[t]^2\, x_1[-2+t]\, u'[-1+t] - u[-3+t]\, u[t]^2\, x_4[-2+t]\, u'[-1+t]\right) \left(S_t^{-1}\right)^2 +\\
&\quad (u[-1+t]\, u[2+t]\, x_2[-1+t]\, u'[t] - u[-1+t]\, u[2+t]\, x_5[-1+t]\, u'[t] -\\
&\qquad u[-1+t]\, u[t]\, x_2[-1+t]\, u'[2+t] + u[-1+t]\, u[t]\, x_5[-1+t]\, u'[2+t])\, D_t +\\
&\quad \left(u[-1+t]\, u[t]\, u[1+t]\, x_1[-2+t]\, u'[-2+t] - u[-1+t]\, u[t]\, u[1+t]\, x_4[-2+t]\right.\\
&\qquad u'[-2+t] + u[-2+t]\, u[t]\, u[1+t]\, x_1[-2+t]\, u'[-1+t] - u[-2+t]\, u[t]\\
&\qquad u[1+t]\, x_4[-2+t]\, u'[-1+t] - u[-2+t]\, u[-1+t]\, u[1+t]\, x_1[-2+t]\, u'[t] +\\
&\qquad u[-2+t]\, u[-1+t]\, u[1+t]\, x_4[-2+t]\, u'[t] - u[-2+t]\, u[-1+t]\, u[t]\\
&\qquad x_1[-2+t]\, u'[1+t] + u[-2+t]\, u[-1+t]\, u[t]\, x_4[-2+t]\, u'[1+t] -\\
&\qquad 2\, u[t]\, x_2[-1+t]\, u'[-2+t]\, u'[1+t] + 2\, u[t]\, x_5[-1+t]\, u'[-2+t]\, u'[1+t] +\\
&\qquad 2\, u[-2+t]\, x_2[-1+t]\, u'[t]\, u'[1+t] - 2\, u[-2+t]\, x_5[-1+t]\, u'[t]\, u'[1+t] -\\
&\qquad u[t]\, u[1+t]\, x_2[-1+t]\, u''[-2+t] + u[t]\, u[1+t]\, x_5[-1+t]\, u''[-2+t] +\\
&\qquad \left. u[-2+t]\, u[1+t]\, x_2[-1+t]\, u''[t] - u[-2+t]\, u[1+t]\, x_5[-1+t]\, u''[t]\right) \left(S_t^{-1}\right) +\\
&\quad \left(-u[-1+t]^2\, u[2+t]\, x_1[-2+t]\, u'[t] + u[-1+t]^2\, u[2+t]\, x_4[-2+t]\, u'[t] +\right.\\
&\qquad 2\, u[2+t]\, x_2[-1+t]\, u'[-1+t]\, u'[t] - 2\, u[2+t]\, x_5[-1+t]\, u'[-1+t]\, u'[t] +
\end{aligned}
$$

$$u[-1+t]^2 u[t]\, x_1[-2+t]\, u'[2+t] - u[-1+t]^2 u[t]\, x_4[-2+t]\, u'[2+t] -$$
$$2\, u[t]\, x_2[-1+t]\, u'[-1+t]\, u'[2+t] + 2\, u[t]\, x_5[-1+t]\, u'[-1+t]\, u'[2+t] +$$
$$u[-1+t]\, u[2+t]\, x_2[-1+t]\, u''[t] - u[-1+t]\, u[2+t]\, x_5[-1+t]\, u''[t] -$$
$$u[-1+t]\, u[t]\, x_2[-1+t]\, u''[2+t] + u[-1+t]\, u[t]\, x_5[-1+t]\, u''[2+t]\Big),$$
$$\Big(-u[1+t]\, x_2[-1+t]\, u'[-1+t] + u[1+t]\, x_5[-1+t]\, u'[-1+t] +$$
$$u[-1+t]\, x_2[-1+t]\, u'[1+t] - u[-1+t]\, x_5[-1+t]\, u'[1+t]\Big)\, D_t^2 +$$
$$\Big(-u[t]^2 x_1[-2+t]\, u'[-2+t] + u[t]^2 x_4[-2+t]\, u'[-2+t] +$$
$$u[-2+t]\, u[t]\, x_1[-2+t]\, u'[t] - u[-2+t]\, u[t]\, x_4[-2+t]\, u'[t]\Big)\, D_t\, \left(S_t^{-1}\right) +$$
$$\Big(u[-1+t]\, u[1+t]\, x_1[-2+t]\, u'[-1+t] - u[-1+t]\, u[1+t]\, x_4[-2+t]\, u'[-1+t] -$$
$$u[-1+t]^2 x_1[-2+t]\, u'[1+t] + u[-1+t]^2 x_4[-2+t]\, u'[1+t] -$$
$$3\, u[1+t]\, x_2[-1+t]\, u''[-1+t] + 3\, u[1+t]\, x_5[-1+t]\, u''[-1+t] +$$
$$3\, u[-1+t]\, x_2[-1+t]\, u''[1+t] - 3\, u[-1+t]\, x_5[-1+t]\, u''[1+t]\Big)\, D_t +$$
$$\Big(u[t]\, x_1[-2+t]\, u'[-2+t]\, u'[t] - u[t]\, x_4[-2+t]\, u'[-2+t]\, u'[t] -$$
$$u[-2+t]\, x_1[-2+t]\, u'[t]^2 + u[-2+t]\, x_4[-2+t]\, u'[t]^2 -$$
$$2\, u[t]^2 x_1[-2+t]\, u''[-2+t] + 2\, u[t]^2 x_4[-2+t]\, u''[-2+t] +$$
$$2\, u[-2+t]\, u[t]\, x_1[-2+t]\, u''[t] - 2\, u[-2+t]\, u[t]\, x_4[-2+t]\, u''[t]\Big)\, \left(S_t^{-1}\right) +$$
$$\Big(-u[1+t]\, x_1[-2+t]\, u'[-1+t]^2 + u[1+t]\, x_4[-2+t]\, u'[-1+t]^2 +$$
$$u[-1+t]\, x_1[-2+t]\, u'[-1+t]\, u'[1+t] - u[-1+t]\, x_4[-2+t]\, u'[-1+t]\, u'[1+t] +$$
$$2\, u[-1+t]\, u[1+t]\, x_1[-2+t]\, u''[-1+t] - 2\, u[-1+t]\, u[1+t]\, x_4[-2+t]\, u''[-1+t] -$$
$$3\, x_2[-1+t]\, u'[1+t]\, u''[-1+t] + 3\, x_5[-1+t]\, u'[1+t]\, u''[-1+t] -$$
$$2\, u[-1+t]^2 x_1[-2+t]\, u''[1+t] + 2\, u[-1+t]^2 x_4[-2+t]\, u''[1+t] +$$
$$3\, x_2[-1+t]\, u'[-1+t]\, u''[1+t] - 3\, x_5[-1+t]\, u'[-1+t]\, u''[1+t] -$$
$$2\, u[1+t]\, x_2[-1+t]\, u^{(3)}[-1+t] + 2\, u[1+t]\, x_5[-1+t]\, u^{(3)}[-1+t] +$$
$$2\, u[-1+t]\, x_2[-1+t]\, u^{(3)}[1+t] - 2\, u[-1+t]\, x_5[-1+t]\, u^{(3)}[1+t]\Big)\Big\},\ \{0,\ 0\},$$
$$\Big\{\Big(u[t]\, u[1+t]\, x_4[-1+t]\, u'[-2+t] - u[-2+t]\, u[1+t]\, x_4[-1+t]\, u'[t]\Big)\, D_t\, \left(S_t^{-1}\right) +$$
$$(-u[-1+t]\, u[2+t]\, x_4[-1+t]\, u'[t] + u[-1+t]\, u[t]\, x_4[-1+t]\, u'[2+t])\, D_t +$$
$$\Big(2\, u[t]\, x_4[-1+t]\, u'[-2+t]\, u'[1+t] - 2\, u[-2+t]\, x_4[-1+t]\, u'[t]\, u'[1+t] +$$
$$u[t]\, u[1+t]\, x_4[-1+t]\, u''[-2+t] - u[-2+t]\, u[1+t]\, x_4[-1+t]\, u''[t]\Big)\, \left(S_t^{-1}\right) +$$
$$(-2\, u[2+t]\, x_4[-1+t]\, u'[-1+t]\, u'[t] + 2\, u[t]\, x_4[-1+t]\, u'[-1+t]\, u'[2+t] -$$
$$u[-1+t]\, u[2+t]\, x_4[-1+t]\, u''[t] + u[-1+t]\, u[t]\, x_4[-1+t]\, u''[2+t]),$$
$$\Big(u[1+t]\, x_4[-1+t]\, u'[-1+t] - u[-1+t]\, x_4[-1+t]\, u'[1+t]\Big)\, D_t^2 +$$
$$(3\, u[1+t]\, x_4[-1+t]\, u''[-1+t] - 3\, u[-1+t]\, x_4[-1+t]\, u''[1+t])\, D_t +$$
$$\Big(3\, x_4[-1+t]\, u'[1+t]\, u''[-1+t] - 3\, x_4[-1+t]\, u'[-1+t]\, u''[1+t] +$$
$$2\, u[1+t]\, x_4[-1+t]\, u^{(3)}[-1+t] - 2\, u[-1+t]\, x_4[-1+t]\, u^{(3)}[1+t]\Big)\Big\},$$
$$\Big\{\Big(u[t]\, u[1+t]\, x_5[-1+t]\, u'[-2+t] - u[-2+t]\, u[1+t]\, x_5[-1+t]\, u'[t]\Big)\, D_t\, \left(S_t^{-1}\right) +$$
$$\Big(u[-1+t]\, u[t]^2 x_4[-2+t]\, u'[-3+t] - u[-3+t]\, u[t]^2 x_4[-2+t]\, u'[-1+t]\Big)\, \left(S_t^{-1}\right)^2 +$$
$$(-u[-1+t]\, u[2+t]\, x_5[-1+t]\, u'[t] + u[-1+t]\, u[t]\, x_5[-1+t]\, u'[2+t])\, D_t +$$
$$\Big(-u[-1+t]\, u[t]\, u[1+t]\, x_4[-2+t]\, u'[-2+t] -$$
$$u[-2+t]\, u[t]\, u[1+t]\, x_4[-2+t]\, u'[-1+t] + u[-2+t]\, u[-1+t]$$
$$u[1+t]\, x_4[-2+t]\, u'[t] + u[-2+t]\, u[-1+t]\, u[t]\, x_4[-2+t]\, u'[1+t] +$$
$$2\, u[t]\, x_5[-1+t]\, u'[-2+t]\, u'[1+t] - 2\, u[-2+t]\, x_5[-1+t]\, u'[t]\, u'[1+t] +$$
$$u[t]\, u[1+t]\, x_5[-1+t]\, u''[-2+t] - u[-2+t]\, u[1+t]\, x_5[-1+t]\, u''[t]\Big)\, \left(S_t^{-1}\right) +$$
$$\Big(u[-1+t]^2 u[2+t]\, x_4[-2+t]\, u'[t] - 2\, u[2+t]\, x_5[-1+t]\, u'[-1+t]\, u'[t] -$$
$$u[-1+t]^2 u[t]\, x_4[-2+t]\, u'[2+t] + 2\, u[t]\, x_5[-1+t]\, u'[-1+t]\, u'[2+t] -$$
$$u[-1+t]\, u[2+t]\, x_5[-1+t]\, u''[t] + u[-1+t]\, u[t]\, x_5[-1+t]\, u''[2+t]\Big),$$
$$\Big(u[1+t]\, x_5[-1+t]\, u'[-1+t] - u[-1+t]\, x_5[-1+t]\, u'[1+t]\Big)\, D_t^2 +$$
$$\Big(u[t]^2 x_4[-2+t]\, u'[-2+t] - u[-2+t]\, u[t]\, x_4[-2+t]\, u'[t]\Big)\, D_t\, \left(S_t^{-1}\right) +$$

```
  (-u[-1 + t] u[1 + t] x4[-2 + t] u'[-1 + t] + u[-1 + t]^2 x4[-2 + t] u'[1 + t] +
     3 u[1 + t] x5[-1 + t] u''[-1 + t] - 3 u[-1 + t] x5[-1 + t] u''[1 + t]) Dt +
  (-u[t] x4[-2 + t] u'[-2 + t] u'[t] + u[-2 + t] x4[-2 + t] u'[t]^2 +
     2 u[t]^2 x4[-2 + t] u''[-2 + t] - 2 u[-2 + t] u[t] x4[-2 + t] u''[t]) (St^-1) +
  (u[1 + t] x4[-2 + t] u'[-1 + t]^2 - u[-1 + t] x4[-2 + t] u'[-1 + t] u'[1 + t] -
     2 u[-1 + t] u[1 + t] x4[-2 + t] u''[-1 + t] + 3 x5[-1 + t] u'[1 + t] u''[-1 + t] +
     2 u[-1 + t]^2 x4[-2 + t] u''[1 + t] - 3 x5[-1 + t] u'[-1 + t] u''[1 + t] +
     2 u[1 + t] x5[-1 + t] u^(3)[-1 + t] - 2 u[-1 + t] x5[-1 + t] u^(3)[1 + t])},
{(u[t] u[1 + t] u'[-2 + t] - u[-2 + t] u[1 + t] u'[t]) Dt^2 (St^-1) +
  (-u[-1 + t] u[2 + t] u'[t] + u[-1 + t] u[t] u'[2 + t]) Dt^2 +
  (3 u[t] u'[-2 + t] u'[1 + t] - 3 u[-2 + t] u'[t] u'[1 + t] +
     2 u[t] u[1 + t] u''[-2 + t] - 2 u[-2 + t] u[1 + t] u''[t]) Dt (St^-1) +
  (-3 u[2 + t] u'[-1 + t] u'[t] + 3 u[t] u'[-1 + t] u'[2 + t] -
     2 u[-1 + t] u[2 + t] u''[t] + 2 u[-1 + t] u[t] u''[2 + t]) Dt +
  (u[1 + t] u'[t] u''[-2 + t] + 3 u[t] u'[1 + t] u''[-2 + t] - u[1 + t] u'[-2 + t] u''[t] -
     3 u[-2 + t] u'[1 + t] u''[t] + 2 u[t] u'[-2 + t] u''[1 + t] - 2 u[-2 + t] u'[t] u''[1 + t] +
     u[t] u[1 + t] u^(3)[-2 + t] - u[-2 + t] u[1 + t] u^(3)[t]) (St^-1) +
  (-2 u[2 + t] u'[t] u''[-1 + t] + 2 u[t] u'[2 + t] u''[-1 + t] - 3 u[2 + t] u'[-1 + t] u''[t] -
     u[-1 + t] u'[2 + t] u''[t] + 3 u[t] u'[-1 + t] u''[2 + t] + u[-1 + t] u'[t] u''[2 + t] -
     u[-1 + t] u[2 + t] u^(3)[t] + u[-1 + t] u[t] u^(3)[2 + t]),
 (u[1 + t] u'[-1 + t] - u[-1 + t] u'[1 + t]) Dt^3 +
  (4 u[1 + t] u''[-1 + t] - 4 u[-1 + t] u''[1 + t]) Dt^2 +
  (6 u'[1 + t] u''[-1 + t] - 6 u'[-1 + t] u''[1 + t] + 5 u[1 + t] u^(3)[-1 + t] -
     5 u[-1 + t] u^(3)[1 + t]) Dt + (5 u'[1 + t] u^(3)[-1 + t] -
     5 u'[-1 + t] u^(3)[1 + t] + 2 u[1 + t] u^(4)[-1 + t] - 2 u[-1 + t] u^(4)[1 + t])}}
```

```
aut = AutonomousElements[R,
   {dx1[t], dx2[t], dx3[t], dx4[t], dx5[t], dx6[t], du[t]}, τ, A, Relations → True];
```

```
aut[[1]]
```

```
{τ[1][t] → dx4[t],
 τ[2][t] → (dx6[t] (x1[-2 + t] - x4[-2 + t]) + dx3[t] x4[-2 + t]) x4[-1 + t] +
   dx5[t] (x2[-1 + t] x4[-2 + t] - x1[-2 + t] x5[-1 + t]),
 τ[3][t] → dx5[t] x1[-1 + t] - dx2[t] x4[-1 + t], τ[4][t] → dx1[t],
 τ[5][t] → -u[t] dx5[-1 + t] - du[t] x5[-1 + t] + dx6'[t],
 τ[6][t] → du[t] x4[-1 + t] - dx5'[t],
 τ[7][t] → -du[-1 + t] u[t]^2 x4[-2 + t] - u[t] x5[-1 + t] du'[t] +
   du[t] (-u[-1 + t] u[t] x4[-2 + t] + x5[-1 + t] u'[t]) - u'[t] dx6'[t] + u[t] dx6''[t]}
```

The autonomous elements τ[1], ..., τ[7] satisfy the following equations:

```
aut[[2]]
```

```
{τ[1]'[t] == 0, τ[2]'[t] (2 u[-1 + t] u'[t]^2 + u[t] (u'[-1 + t] u'[t] - u[-1 + t] u''[t])) -
   u[t] (u[t] u'[-1 + t] + 2 u[-1 + t] u'[t]) τ[2]''[t] + u[-1 + t] u[t]^2 τ[2]^(3)[t] == 0,
 -u'[t] τ[3]'[t] + u[t] τ[3]''[t] == 0, τ[4]'[t] == 0, τ[5][t] == 0,
 τ[6][t] u'[t] - u[t] τ[6]'[t] == 0,
 -τ[7][t] (u[t] u'[-1 + t] + 2 u[-1 + t] u'[t]) + u[-1 + t] u[t] τ[7]'[t] == 0}
```

The A-linear relations among the autonomous elements are given by:

aut[[3]]

$$\Big\{\tau[4]'[t],\ -\frac{u[t]\ x_4[-1+t]\ \tau[4][-1+t]+x_1[-1+t]\ \tau[6][t]+\tau[3]'[t]}{x_4[-1+t]},$$
$$\frac{1}{x_4[-2+t]\ x_4[-1+t]}\big(-u[t]\ x_4[-1+t]\ \tau[3][-1+t]+x_4[-2+t]\ x_4[-1+t]\ \tau[5][t]+x_2[-1+t]\ x_4[-2+t]\ \tau[6][t]-x_1[-2+t]\ (x_4[-1+t]\ \tau[5][t]+x_5[-1+t]\ \tau[6][t])+\tau[2]'[t]\big),$$
$$\tau[1]'[t],\ -u[t]\ \tau[1][-1+t]-\tau[6][t],\ \tau[5][t],$$
$$u[t]^2\ \tau[6][-1+t]+\tau[7][t]+\tau[5][t]\ u'[t]-u[t]\ \tau[5]'[t]\Big\}$$

Let us now prove that the set of autonomous elements can be generated by T[1], ...,T[4] (our guess or inspection of aut[[3]]). Let us introduce the matrix L, defining aut[[3]]:

L = ToOrePolynomialD[aut[[3]],
{τ[1][t], τ[2][t], τ[3][t], τ[4][t], τ[5][t], τ[6][t], τ[7][t]}, A]

$$\Big\{\{0,0,0,D_t,0,0,0\},\ \Big\{0,0,-\frac{1}{x_4[-1+t]}D_t,\ -u[t]\,(S_t^{-1}),\ 0,\ -\frac{x_1[-1+t]}{x_4[-1+t]},\ 0\Big\},$$
$$\Big\{0,\ \frac{1}{x_4[-2+t]\ x_4[-1+t]}D_t,\ -\frac{u[t]}{x_4[-2+t]}(S_t^{-1}),\ 0,\ 1-\frac{x_1[-2+t]}{x_4[-2+t]},\ \frac{x_2[-1+t]}{x_4[-1+t]}-\frac{x_1[-2+t]\ x_5[-1+t]}{x_4[-2+t]\ x_4[-1+t]},\ 0\Big\},$$
$$\{D_t,0,0,0,0,0,0\},\ \Big\{-u[t]\,(S_t^{-1}),0,0,0,0,-1,0\Big\},$$
$$\{0,0,0,0,1,0,0\},\ \Big\{0,0,0,0,-u[t]\,D_t+u'[t],\ u[t]^2\,(S_t^{-1}),\ 1\Big\}\Big\}$$

Let us consider the following matrix

MatrixForm[γ = PadRight[#, 7] & /@ IdentityMatrix[4]]

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\end{pmatrix}$$

which corresponds to the position of *τ*[1], ...,*τ*[4]. To express *τ*[5], *τ*[6] and *τ*[7] in terms of *τ*[1], ..., *τ*[4] we first check whether or not the matrix Q formed by stacking L with *γ* admits a left inverse.

S1 = LeftInverse[Q = Join[L, γ], A]

$$\Big\{\{0,0,0,0,0,0,0,1,0,0,0\},$$
$$\{0,0,0,0,0,0,0,0,1,0,0\},\ \{0,0,0,0,0,0,0,0,0,1,0\},$$
$$\{0,0,0,0,0,0,0,0,0,0,1\},\ \{0,0,0,0,0,1,0,0,0,0,0\},$$
$$\Big\{0,\ -\frac{x_4[-1+t]}{x_1[-1+t]},\ 0,0,0,0,0,0,0,\ -\frac{1}{x_1[-1+t]}D_t,\ -\frac{u[t]\ x_4[-1+t]}{x_1[-1+t]}(S_t^{-1})\Big\},$$
$$\Big\{0,\ \frac{u[t]^2\ x_4[-2+t]}{x_1[-2+t]}(S_t^{-1}),\ 0,0,0,\ u[t]\,D_t-u'[t],\ 1,\ 0,0,\ \frac{u[t]^2}{x_1[-3+t]}D_t\,(S_t^{-1}),\ \frac{u[-1+t]\ u[t]^2\ x_4[-2+t]}{x_1[-2+t]}(S_t^{-1})^2\Big\}\Big\}$$

If we consider the last four columns of the left inverse S1 of Q, i.e.

```
S2 = Take[#, -4] & /@ S1
```

$$\left\{\{1, 0, 0, 0\}, \{0, 1, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 0\}, \left\{0, 0, -\frac{1}{x_1[-1+t]} D_t, -\frac{u[t]\, x_4[-1+t]}{x_1[-1+t]} \left(S_t^{-1}\right)\right\}, \left\{0, 0, \frac{u[t]^2}{x_1[-3+t]} D_t \left(S_t^{-1}\right), \frac{u[-1+t]\, u[t]^2\, x_4[-2+t]}{x_1[-2+t]} \left(S_t^{-1}\right)^2\right\}\right\}$$

then we obtain:

```
Thread[Table[τ[i][t], {i, 7}] ->
  ApplyMatrix[S2, {τ[1][t], τ[2][t], τ[3][t], τ[4][t]}]]
```

$$\left\{\tau[1][t] \to \tau[1][t], \tau[2][t] \to \tau[2][t], \tau[3][t] \to \tau[3][t], \tau[4][t] \to \tau[4][t], \tau[5][t] \to 0, \tau[6][t] \to -\frac{u[t]\, x_4[-1+t]\, \tau[4][-1+t] + \tau[3]'[t]}{x_1[-1+t]}, \tau[7][t] \to u[t]^2 \left(\frac{u[-1+t]\, x_4[-2+t]\, \tau[4][-2+t]}{x_1[-2+t]} + \frac{\tau[3]'[-1+t]}{x_1[-3+t]}\right)\right\}$$

From this point we will use some procedures which are not freely available (see NLControl website http://www.nlcontrol.ioc.ee).
Let us integrate the one-form defined by *τ*[1], namely

```
BookForm[difs = {ApplyMatrixD[Rp[[1]], Join[Xt, Ut]]}]
```

$\{dx_4[t]\}$

```
BookForm[sp = SpanK[difs, t]]
```

SpanK[$dx_4[t]$]

```
IntegrateOneForms[sp]
```

$\{x_4[t]\}$

Let us integrate the one-form defined by *τ*[4], namely

```
BookForm[difs = {ApplyMatrixD[Rp[[4]], vars]}]
```

$\{dx_1[t]\}$

```
BookForm[sp = SpanK[difs, t]]
```

SpanK[$dx_1[t]$]

```
IntegrateOneForms[sp]
```

$\{x_1[t]\}$

We are not able to integrate the one-forms defined by *τ*[2] and *τ*[3], since they both contain delays.

```
BookForm[difs = {ApplyMatrixD[Rp[[2]], Join[Xt, Ut]]}]
```

$\{(x_1[-2+t] - x_4[-2+t])\, x_4[-1+t]\, dx_6[t] + x_4[-2+t]\, x_4[-1+t]\, dx_3[t] + (x_2[-1+t]\, x_4[-2+t] - x_1[-2+t]\, x_5[-1+t])\, dx_5[t]\}$

```
BookForm[difs = {ApplyMatrixD[Rp[[3]], Join[Xt, Ut]]}]
```

$\{x_1[-1+t]\, dx_5[t] - x_4[-1+t]\, dx_2[t]\}$

We have obtained two autonomous elements x_1 and x_4 for the nonlinear system.

Serre' s reduction techniques

Let us now prove that L is equivalent to a diagonal matrix. Let us consider the column vector Λ

```
MatrixForm[Λ = Transpose[-PadRight[#, 7] & /@ IdentityMatrix[4]]]
```

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the matrix P obtained by concatenation of L and Λ:

```
P = Join[L, Λ, 2]
```

$$\Big\{\{0, 0, 0, D_t, 0, 0, 0, -1, 0, 0, 0\},$$
$$\Big\{0, 0, -\frac{1}{x_4[-1+t]} D_t, -u[t]\, (S_t^{-1}), 0, -\frac{x_1[-1+t]}{x_4[-1+t]}, 0, 0, -1, 0, 0\Big\},$$
$$\Big\{0, \frac{1}{x_4[-2+t]\, x_4[-1+t]} D_t, -\frac{u[t]}{x_4[-2+t]} (S_t^{-1}), 0, 1 - \frac{x_1[-2+t]}{x_4[-2+t]}, \frac{x_2[-1+t]}{x_4[-1+t]} - \frac{x_1[-2+t]\, x_5[-1+t]}{x_4[-2+t]\, x_4[-1+t]}, 0, 0, 0, -1, 0\Big\},$$
$$\{D_t, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1\}, \Big\{-u[t]\, (S_t^{-1}), 0, 0, 0, 0, -1, 0, 0, 0, 0, 0\Big\},$$
$$\{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0\},$$
$$\Big\{0, 0, 0, 0, -u[t]\, D_t + u'[t], u[t]^2\, (S_t^{-1}), 1, 0, 0, 0, 0\Big\}\Big\}$$

Let us check again that the matrix P admits a right inverse:

```
MatrixForm[Pinv = RightInverse[P, A]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & u[t]^2\,(S_t^{-1}) & u[t]\,D_t - u'[t] & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & \frac{x_1[-1+t]}{x_4[-1+t]} & 0 & 0 \\ 0 & 0 & -1 & 0 & \frac{-x_2[-1+t]\,x_4[-2+t]+x_1[-2+t]\,x_5[-1+t]}{x_4[-2+t]\,x_4[-1+t]} & 1-\frac{x_1[-2+t]}{x_4[-2+t]} & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

We deduce that the module E, finitely presented by P, is stably free. Let us now check whether or not E is free.

```
Q = MinimalParametrization[P, A]
```

$$\{\{0, 0, 0, -x_4[-1+t]\,x_4[t]\}, \{-x_4[-2+t]\,x_4[-1+t], 0, 0, 0\}, \{0, -x_4[-1+t], 0, 0\},$$
$$\{0, 0, -1, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, u[t]\,x_4[-2+t]\,x_4[-1+t]\,(S_t^{-1})\},$$
$$\{0, 0, 0, -u[-1+t]\,u[t]^2\,x_4[-3+t]\,x_4[-2+t]\,(S_t^{-1})^2\},$$
$$\{0, 0, -D_t, 0\}, \{0, D_t, u[t]\,(S_t^{-1}), -u[t]\,x_1[-1+t]\,x_4[-2+t]\,(S_t^{-1})\},$$
$$\{-D_t, u[t]\,(S_t^{-1}), 0, (u[t]\,x_2[-1+t]\,x_4[-2+t]-u[t]\,x_1[-2+t]\,x_5[-1+t])\,(S_t^{-1})\},$$
$$\{0, 0, 0, -x_4[-1+t]\,x_4[t]\,D_t\}\}$$

```
LeftInverse[Q, A]
```

$$\left\{\left\{0, -\frac{1}{x_4[-2+t]\,x_4[-1+t]}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\},\right.$$
$$\left\{0, 0, -\frac{1}{x_4[-1+t]}, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \{0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0\},$$
$$\left.\left\{-\frac{1}{x_4[-1+t]\,x_4[t]}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right\}\right\}$$

We obtain that E is a free module. Let us extract blocs from the matrix Q as follows:

```
MatrixForm[Q1 = Take[Q, 7]]
```

$$\begin{pmatrix} 0 & 0 & 0 & -x_4[-1+t]\,x_4[t] \\ -x_4[-2+t]\,x_4[-1+t] & 0 & 0 & 0 \\ 0 & -x_4[-1+t] & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u[t]\,x_4[-2+t]\,x_4[-1+t]\,(S_t^{-1}) \\ 0 & 0 & 0 & -u[-1+t]\,u[t]^2\,x_4[-3+t]\,x_4[-2+t]\,(S_t^{-1})^2 \end{pmatrix}$$

```
MatrixForm[Q2 = Take[Q, -4]]
```

$$\begin{pmatrix} 0 & 0 & -D_t & 0 \\ 0 & D_t & u[t]\,(S_t^{-1}) & -u[t]\,x_1[-1+t]\,x_4[-2+t]\,(S_t^{-1}) \\ -D_t & u[t]\,(S_t^{-1}) & 0 & (u[t]\,x_2[-1+t]\,x_4[-2+t]-u[t]\,x_1[-2+t]\,x_5[-1+t])\,(S_t^{-1}) \\ 0 & 0 & 0 & -x_4[-1+t]\,x_4[t]\,D_t \end{pmatrix}$$

By theory we know that the left A-module N finitely presented by L is isomorphic to the module finitely presented by Q2.
Let now check that the matrix L is equivalent to the diagonal matrix diag(I(3), Q2). To do that let us compute a basis of the left kernel of Q1:

```
U1 = LeftKernel[Q1, A]
```

$\{\{0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, u[t]^2 (S_t^{-1}), 1\}, \{-u[t] (S_t^{-1}), 0, 0, 0, 0, -1, 0\}\}$

```
LeftKernel[U1, A]
```

Inj[3]

Let W1 be a right inverse of U1

```
W1 = RightInverse[U1, A]
```

$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}, \{1, 0, 0\}, \{0, 0, -1\}, \{0, 1, u[t]^2 (S_t^{-1})\}\}$

and W be the unimodular matrix formed by W1 and Q1:

```
W = Join[W1, Q1, 2]
```

$$\{\{0, 0, 0, 0, 0, 0, -x_4[-1+t]\, x_4[t]\}, \{0, 0, 0, -x_4[-2+t]\, x_4[-1+t], 0, 0, 0\}, \{0, 0, 0, 0, -x_4[-1+t], 0, 0\}, \{0, 0, 0, 0, 0, -1, 0\}, \{1, 0, 0, 0, 0, 0, 0\}, \{0, 0, -1, 0, 0, 0, u[t]\, x_4[-2+t]\, x_4[-1+t]\, (S_t^{-1})\}, \{0, 1, u[t]^2 (S_t^{-1}), 0, 0, 0, -u[-1+t]\, u[t]^2\, x_4[-3+t]\, x_4[-2+t]\, (S_t^{-1})^2\}\}$$

Let us check again that W is unimodular:

```
LeftInverse[W, A]
```

$$\{\{0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, u[t]^2 (S_t^{-1}), 1\}, \{-u[t] (S_t^{-1}), 0, 0, 0, 0, -1, 0\}, \{0, -\frac{1}{x_4[-2+t]\, x_4[-1+t]}, 0, 0, 0, 0, 0\}, \{0, 0, -\frac{1}{x_4[-1+t]}, 0, 0, 0, 0\}, \{0, 0, 0, -1, 0, 0, 0\}, \{-\frac{1}{x_4[-1+t]\, x_4[t]}, 0, 0, 0, 0, 0, 0\}\}$$

Let us form the following matrix

```
X = Join[OreDot[L, W1], Λ, 2]
```

$$\{\{0, 0, 0, -1, 0, 0, 0\}, \{0, 0, \frac{x_1[-1+t]}{x_4[-1+t]}, 0, -1, 0, 0\}, \{1 - \frac{x_1[-2+t]}{x_4[-2+t]}, 0, -\frac{x_2[-1+t]}{x_4[-1+t]} + \frac{x_1[-2+t]\, x_5[-1+t]}{x_4[-2+t]\, x_4[-1+t]}, 0, 0, -1, 0\}, \{0, 0, 0, 0, 0, 0, -1\}, \{0, 0, 1, 0, 0, 0, 0\}, \{1, 0, 0, 0, 0, 0, 0\}, \{-u[t]\, D_t + u'[t], 1, 0, 0, 0, 0, 0\}\}$$

Let us check that X is unimodular:

```
V = LeftInverse[X, A]
```

$\{\{0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, u[t]\, D_t - u'[t], 1\}, \{0, 0, 0, 0, 1, 0, 0\}, \{-1, 0, 0, 0, 0, 0, 0\}, \{0, -1, 0, 0, \frac{x_1[-1+t]}{x_4[-1+t]}, 0, 0\}, \{0, 0, -1, 0, (-x_2[-1+t]\, x_4[-2+t] + x_1[-2+t]\, x_5[-1+t]) / (x_4[-2+t]\, x_4[-1+t]), 1 - \frac{x_1[-2+t]}{x_4[-2+t]}, 0\}, \{0, 0, 0, -1, 0, 0, 0\}\}$

Finally we obtain that L is equivalent to the following diagonal matrix:

```
OreDot[V, L, W]
```

$\{\{1, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, D_t, 0\}, \{0, 0, 0, 0, -D_t, -u[t]\, (S_t^{-1}), u[t]\, x_1[-1+t]\, x_4[-2+t]\, (S_t^{-1})\}, \{0, 0, 0, D_t, -u[t]\, (S_t^{-1}), 0, (-u[t]\, x_2[-1+t]\, x_4[-2+t] + u[t]\, x_1[-2+t]\, x_5[-1+t])\, (S_t^{-1})\}, \{0, 0, 0, 0, 0, 0, x_4[-1+t]\, x_4[t]\, D_t\}\}$