Time-delay Example 2

Example from the paper by C. Califano, S. Li, C.H. Moog.

Let us consider the following differential time-delay nonlinear system:

```
eqs = {x1'[t] → 2 x3[t] (x1[t - 1] - x2[t - 1]^3) u1[t],
   (x2'[t] → x2[t - 1] u2[t]),
   x3'[t] → (x1[t - 1] - x2[t - 1]^3) u1[t]};
vars = {x1[t], x2[t], x3[t], u1[t], u2[t]};
TableForm[eqs]
```

$$x_1'[t] \to 2\, u_1[t]\, \left(x_1[-1+t] - x_2[-1+t]^3\right) x_3[t]$$
$$x_2'[t] \to u_2[t]\, x_2[-1+t]$$
$$x_3'[t] \to u_1[t]\, \left(x_1[-1+t] - x_2[-1+t]^3\right)$$

Let us introduce the following Ore algebra A of differential time-delay operators

```
replA = ModelToReplacementRules[eqs, t];
A = OreAlgebraWithRelations[Der[t], S[-1][t], replA]
```

$$\mathbb{K}(t)\left[D_t;\ 1,\ D_t\right]\left[\left(S_t^{-1}\right);\ \#1\ /.\ t \to t-1\ \&,\ 0\ \&\right]$$

and the matrix R of ordinary differential time-delay operators which defines the generic linearization of the above nonlinear system

```
R = ToOrePolynomialD[eqs, vars, A]
```

$$\{\{D_t - 2\, u_1[t]\, x_3[t]\, \left(S_t^{-1}\right),\ 6\, u_1[t]\, x_2[-1+t]^2\, x_3[t]\, \left(S_t^{-1}\right),$$
$$-2\, u_1[t]\, x_1[-1+t] + 2\, u_1[t]\, x_2[-1+t]^3,\ -2\, x_1[-1+t]\, x_3[t] + 2\, x_2[-1+t]^3\, x_3[t],\ 0\},$$
$$\{0,\ D_t - u_2[t]\, \left(S_t^{-1}\right),\ 0,\ 0,\ -x_2[-1+t]\},$$
$$\{-u_1[t]\, \left(S_t^{-1}\right),\ 3\, u_1[t]\, x_2[-1+t]^2\, \left(S_t^{-1}\right),\ D_t,\ -x_1[-1+t] + x_2[-1+t]^3,\ 0\}\}$$

and the left A-module, finitely presented by R, defined by M = $A^{1\times 5} / A^{1\times 3} R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} D_t - 2\, u_1[1-t]\, x_3[1-t]\, \left(S_t^{-1}\right) & 0 & -u_1[1-t]\, \left(S_t^{-1}\right) \\ 6\, u_1[1-t]\, x_2[-t]^2\, x_3[1-t]\, \left(S_t^{-1}\right) & D_t - u_2[1-t]\, \left(S_t^{-1}\right) & 3\, u_1[1-t]\, x_2[-t]^2\, \left(S_t^{-1}\right) \\ -2\, u_1[-t]\, x_1[-1-t] + 2\, u_1[-t]\, x_2[-1-t]^3 & 0 & D_t \\ -2\, x_1[-1-t]\, x_3[-t] + 2\, x_2[-1-t]^3\, x_3[-t] & 0 & -x_1[-1-t] + x_2[-1-t]^3 \\ 0 & -x_2[-1-t] & 0 \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
{Ann, Rp, Q} = Exti[Radj, A, 1];
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```
Ann
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$$\{\{D_t,\ 0,\ 0,\ 0\},\ \{0,\ -u_1[t]\, D_t + u_1'[t],\ 0,\ 0\},$$
$$\{0,\ 0,\ -u_1[t]\, u_2[t]\, D_t + (u_2[t]\, u_1'[t] + u_1[t]\, u_2'[t]),\ 0\},\ \{0,\ 0,\ 0,$$
$$\left(-u_1[-2+t]\, u_1[-1+t]\, u_1[t]^3\, u_2[-2+t]\, u_2[-1+t]^3\, x_2[-2+t]^3\, x_2[-1+t]^2 + u_1[-2+t]\right.$$
$$\left. u_1[-1+t]\, u_1[t]^3\, u_2[-2+t]^2\, u_2[-1+t]^2\, x_2[-3+t]\, x_2[-2+t]\, x_2[-1+t]^3\right) D_t\, \left(S_t^{-1}\right) +$$
$$\left(-2\, u_1[-2+t]^2\, u_1[-1+t]^2\, u_1[t]\, u_2[-2+t]^3\, u_2[-1+t]\, x_2[-3+t]^4\, x_2[-2+t] +\right.$$

$2\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_1[t]\,u_2[-3+t]\,u_2[-2+t]^2\,u_2[-1+t]$
$x_2[-4+t]\,x_2[-3+t]^2\,x_2[-2+t]^2\big)\,D_t+\big(3\,u_1[-2+t]\,u_1[-1+t]$
$u_1[t]^3\,u_2[-2+t]^2\,u_2[-1+t]^3\,x_2[-3+t]\,x_2[-2+t]^2\,x_2[-1+t]^2-$
$3\,u_1[-2+t]\,u_1[-1+t]\,u_1[t]^3\,u_2[-2+t]^3\,u_2[-1+t]^2\,x_2[-3+t]^2\,x_2[-1+t]^3+$
$u_1[-1+t]\,u_1[t]^3\,u_2[-2+t]\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]^2\,u_1'[-2+t]-$
$u_1[-1+t]\,u_1[t]^3\,u_2[-2+t]^2\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^3\,u_1'[-2+t]+$
$2\,u_1[-2+t]\,u_1[t]^3\,u_2[-2+t]\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]^2\,u_1'[-1+t]-$
$2\,u_1[-2+t]\,u_1[t]^3\,u_2[-2+t]^2\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]$
$x_2[-1+t]^3\,u_1'[-1+t]+u_1[-2+t]\,u_1[-1+t]\,u_1[t]^3\,u_2[-1+t]^3$
$x_2[-2+t]^3\,x_2[-1+t]^2\,u_2'[-2+t]-u_1[-2+t]\,u_1[-1+t]\,u_1[t]^3$
$u_2[-2+t]\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^3\,u_2'[-2+t]\big)\,\big(S_t^{-1}\big)+$
$\big(6\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_1[t]\,u_2[-2+t]^4\,u_2[-1+t]\,x_2[-3+t]^5-$
$6\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_1[t]\,u_2[-3+t]\,u_2[-2+t]^3\,u_2[-1+t]\,x_2[-4+t]$
$x_2[-3+t]^3\,x_2[-2+t]+2\,u_1[-2+t]^2\,u_1[-1+t]\,u_1[t]\,u_2[-2+t]^3\,u_2[-1+t]$
$x_2[-3+t]^4\,x_2[-2+t]\,u_1'[-1+t]-2\,u_1[-2+t]^2\,u_1[-1+t]\,u_1[t]\,u_2[-3+t]$
$u_2[-2+t]^2\,u_2[-1+t]\,x_2[-4+t]\,x_2[-3+t]^2\,x_2[-2+t]^2\,u_1'[-1+t]+$
$4\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_2[-2+t]^3\,u_2[-1+t]\,x_2[-3+t]^4\,x_2[-2+t]\,u_1'[t]-$
$4\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_2[-3+t]\,u_2[-2+t]^2\,u_2[-1+t]\,x_2[-4+t]$
$x_2[-3+t]^2\,x_2[-2+t]^2\,u_1'[t]+2\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_1[t]\,u_2[-2+t]^3$
$x_2[-3+t]^4\,x_2[-2+t]\,u_2'[-1+t]-2\,u_1[-2+t]^2\,u_1[-1+t]^2\,u_1[t]$
$u_2[-3+t]\,u_2[-2+t]^2\,x_2[-4+t]\,x_2[-3+t]^2\,x_2[-2+t]^2\,u_2'[-1+t]\big)\big\},$
$\big\{0,\ 0,\ 0,\ \big(-u_1[-1+t]^2\,u_1[t]^2\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]+$
$u_1[-1+t]^2\,u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2\big)\,D_t^2+$
$\big(2\,u_1[-1+t]^2\,u_1[t]^2\,u_2[-1+t]^4\,x_2[-2+t]^4+$
$u_1[-1+t]^2\,u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]^3\,x_2[-3+t]\,x_2[-2+t]^2\,x_2[-1+t]-$
$2\,u_1[-1+t]^2\,u_1[t]^2\,u_2[-2+t]^2\,u_2[-1+t]^2\,x_2[-3+t]^2\,x_2[-1+t]^2-$
$u_1[-1+t]^2\,u_1[t]^2\,u_2[-3+t]\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-4+t]\,x_2[-2+t]\,x_2[-1+t]^2+$
$2\,u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]\,u_1'[-1+t]-$
$2\,u_1[-1+t]\,u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2$
$u_1'[-1+t]+4\,u_1[-1+t]^2\,u_1[t]\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]\,u_1'[t]-$
$4\,u_1[-1+t]^2\,u_1[t]\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2\,u_1'[t]-$
$u_1[-1+t]^2\,u_1[t]^2\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2\,u_2'[-2+t]+$
$3\,u_1[-1+t]^2\,u_1[t]^2\,u_2[-1+t]^2\,x_2[-2+t]^3\,x_2[-1+t]\,u_2'[-1+t]-2\,u_1[-1+t]^2$
$u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2\,u_2'[-1+t]\big)\,D_t+$
$\big(-6\,u_1[-1+t]^2\,u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]^4\,x_2[-3+t]\,x_2[-2+t]^3+$
$3\,u_1[-1+t]^2\,u_1[t]^2\,u_2[-2+t]^2\,u_2[-1+t]^3\,x_2[-3+t]^2\,x_2[-2+t]\,x_2[-1+t]+$
$3\,u_1[-1+t]^2\,u_1[t]^2\,u_2[-3+t]\,u_2[-2+t]\,u_2[-1+t]^3\,x_2[-4+t]\,x_2[-2+t]^2$
$x_2[-1+t]-2\,u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]^4\,x_2[-2+t]^4\,u_1'[-1+t]-$
$u_1[-1+t]\,u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]^3\,x_2[-3+t]\,x_2[-2+t]^2\,x_2[-1+t]\,u_1'[-1+t]+$
$2\,u_1[-1+t]\,u_1[t]^2\,u_2[-2+t]^2\,u_2[-1+t]^2\,x_2[-3+t]^2\,x_2[-1+t]^2\,u_1'[-1+t]+$
$u_1[-1+t]\,u_1[t]^2\,u_2[-3+t]\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-4+t]\,x_2[-2+t]$
$x_2[-1+t]^2\,u_1'[-1+t]-2\,u_1[t]^2\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]\,u_1'[-1+t]^2+$
$2\,u_1[t]^2\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2\,u_1'[-1+t]^2-$
$4\,u_1[-1+t]^2\,u_1[t]\,u_2[-1+t]^4\,x_2[-2+t]^4\,u_1'[t]-$
$2\,u_1[-1+t]^2\,u_1[t]\,u_2[-2+t]\,u_2[-1+t]^3\,x_2[-3+t]\,x_2[-2+t]^2\,x_2[-1+t]\,u_1'[t]+$
$4\,u_1[-1+t]^2\,u_1[t]\,u_2[-2+t]^2\,u_2[-1+t]^2\,x_2[-3+t]^2\,x_2[-1+t]^2\,u_1'[t]+$
$2\,u_1[-1+t]^2\,u_1[t]\,u_2[-3+t]\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-4+t]\,x_2[-2+t]\,x_2[-1+t]^2$
$u_1'[t]-4\,u_1[-1+t]\,u_1[t]\,u_2[-1+t]^3\,x_2[-2+t]^3\,x_2[-1+t]\,u_1'[-1+t]\,u_1'[t]+$
$4\,u_1[-1+t]\,u_1[t]\,u_2[-2+t]\,u_2[-1+t]^2\,x_2[-3+t]\,x_2[-2+t]\,x_2[-1+t]^2$

$u_1'[-1+t]\ u_1'[t] - 6\ u_1[-1+t]^2\ u_2[-1+t]^3\ x_2[-2+t]^3\ x_2[-1+t]\ u_1'[t]^2 +$
$6\ u_1[-1+t]^2\ u_2[-2+t]\ u_2[-1+t]^2\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1'[t]^2 +$
$3\ u_1[-1+t]^2\ u_1[t]^2\ u_2[-1+t]^3\ x_2[-3+t]\ x_2[-2+t]^2\ x_2[-1+t]\ u_2'[-2+t] +$
$u_1[-1+t]\ u_1[t]^2\ u_2[-1+t]^2\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1'[-1+t]\ u_2'[-2+t] +$
$2\ u_1[-1+t]^2\ u_1[t]\ u_2[-1+t]^2\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1'[t]\ u_2'[-2+t] -$
$2\ u_1[-1+t]^2\ u_1[t]^2\ u_2[-1+t]^3\ x_2[-2+t]^4\ u_2'[-1+t] - 4\ u_1[-1+t]^2$
$u_1[t]^2\ u_2[-2+t]\ u_2[-1+t]^2\ x_2[-3+t]\ x_2[-2+t]^2\ x_2[-1+t]\ u_2'[-1+t] +$
$2\ u_1[-1+t]^2\ u_1[t]^2\ u_2[-2+t]^2\ u_2[-1+t]\ x_2[-3+t]^2\ x_2[-1+t]^2\ u_2'[-1+t] +$
$u_1[-1+t]^2\ u_1[t]^2\ u_2[-3+t]\ u_2[-2+t]\ u_2[-1+t]\ x_2[-4+t]\ x_2[-2+t]$
$x_2[-1+t]^2\ u_2'[-1+t] - 3\ u_1[-1+t]\ u_1[t]^2\ u_2[-1+t]^2\ x_2[-2+t]^3$
$x_2[-1+t]\ u_1'[-1+t]\ u_2'[-1+t] + 2\ u_1[-1+t]\ u_1[t]^2\ u_2[-2+t]$
$u_2[-1+t]\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1'[-1+t]\ u_2'[-1+t] -$
$6\ u_1[-1+t]^2\ u_1[t]\ u_2[-1+t]^2\ x_2[-2+t]^3\ x_2[-1+t]\ u_1'[t]\ u_2'[-1+t] + 4\ u_1[-1+t]^2$
$u_1[t]\ u_2[-2+t]\ u_2[-1+t]\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1'[t]\ u_2'[-1+t] +$
$u_1[-1+t]^2\ u_1[t]^2\ u_2[-1+t]\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_2'[-2+t]\ u_2'[-1+t] -$
$3\ u_1[-1+t]^2\ u_1[t]^2\ u_2[-1+t]\ x_2[-2+t]^3\ x_2[-1+t]\ u_2'[-1+t]^2 +$
$2\ u_1[-1+t]^2\ u_1[t]^2\ u_2[-2+t]\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_2'[-1+t]^2 +$
$u_1[-1+t]\ u_1[t]^2\ u_2[-1+t]^3\ x_2[-2+t]^3\ x_2[-1+t]\ u_1''[-1+t] -$
$u_1[-1+t]\ u_1[t]^2\ u_2[-2+t]\ u_2[-1+t]^2\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1''[-1+t] +$
$2\ u_1[-1+t]^2\ u_1[t]\ u_2[-1+t]^3\ x_2[-2+t]^3\ x_2[-1+t]\ u_1''[t] -$
$2\ u_1[-1+t]^2\ u_1[t]\ u_2[-2+t]\ u_2[-1+t]^2\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_1''[t] +$
$u_1[-1+t]^2\ u_1[t]^2\ u_2[-1+t]^2\ x_2[-2+t]^3\ x_2[-1+t]\ u_2''[-1+t] - u_1[-1+t]^2$
$u_1[t]^2\ u_2[-2+t]\ u_2[-1+t]\ x_2[-3+t]\ x_2[-2+t]\ x_2[-1+t]^2\ u_2''[-1+t])\}\}$

Rp

$\{\{1,\ 0,\ -2\ x_3[t],\ 0,\ 0\},\ \{0,\ -3\ u_1[t]\ x_2[-1+t]^2\ (S_t^{-1}),$
$-D_t + 2\ u_1[t]\ x_3[-1+t]\ (S_t^{-1}),\ x_1[-1+t] - x_2[-1+t]^3,\ 0\},$
$\{0,\ -3\ u_1[t]\ x_2[-1+t]^2\ D_t,\ -u_2[t]\ D_t + 2\ u_1[t]\ u_2[t]\ x_3[-1+t]\ (S_t^{-1}),$
$u_2[t]\ x_1[-1+t] - u_2[t]\ x_2[-1+t]^3,\ 3\ u_1[t]\ x_2[-1+t]^3\},$
$\{0,\ 0,\ u_1[-1+t]\ u_1[t]\ x_2[-2+t]^2\ x_2[-1+t]\ D_t^2 +$
$(-u_1[t]^2\ u_2[-1+t]\ x_2[-1+t]^3 - 2\ u_1[-1+t]\ u_1[t]^2\ x_2[-2+t]^2\ x_2[-1+t]\ x_3[-1+t])$
$D_t\ (S_t^{-1}) + 2\ u_1[-1+t]\ u_1[t]^2\ u_2[-1+t]\ x_2[-1+t]^3\ x_3[-2+t]\ (S_t^{-1})^2 +$
$(-2\ u_1[-1+t]\ u_1[t]\ u_2[-1+t]\ x_2[-2+t]^3 - u_1[-1+t]\ x_2[-2+t]^2\ x_2[-1+t]\ u_1'[t])$
$D_t + (-2\ u_1[-1+t]^2\ u_1[t]^2\ x_1[-2+t]\ x_2[-2+t]^2\ x_2[-1+t] + 2\ u_1[-1+t]^2\ u_1[t]^2$
$x_2[-2+t]^5\ x_2[-1+t] + 4\ u_1[-1+t]\ u_1[t]^2\ u_2[-1+t]\ x_2[-2+t]^3\ x_3[-1+t])\ (S_t^{-1}),$
$(-u_1[-1+t]\ u_1[t]\ x_1[-1+t]\ x_2[-2+t]^2\ x_2[-1+t] +$
$u_1[-1+t]\ u_1[t]\ x_2[-2+t]^2\ x_2[-1+t]^4)\ D_t +$
$(u_1[t]^2\ u_2[-1+t]\ x_1[-2+t]\ x_2[-1+t]^3 - u_1[t]^2\ u_2[-1+t]\ x_2[-2+t]^3\ x_2[-1+t]^3)$
$(S_t^{-1}) + (2\ u_1[-1+t]\ u_1[t]\ u_2[-1+t]\ x_1[-1+t]\ x_2[-2+t]^3 +$
$u_1[-1+t]\ u_1[t]\ u_2[-1+t]\ x_2[-2+t]^3\ x_2[-1+t]^3 -$
$2\ u_1[-1+t]^2\ u_1[t]\ x_1[-2+t]\ x_2[-2+t]^2\ x_2[-1+t]\ x_3[-1+t] +$
$2\ u_1[-1+t]^2\ u_1[t]\ x_2[-2+t]^5\ x_2[-1+t]\ x_3[-1+t] + u_1[-1+t]\ x_1[-1+t]$
$x_2[-2+t]^2\ x_2[-1+t]\ u_1'[t] - u_1[-1+t]\ x_2[-2+t]^2\ x_2[-1+t]^4\ u_1'[t]),$
$3\ u_1[-1+t]\ u_1[t]^2\ x_2[-2+t]^3\ x_2[-1+t]^3\ (S_t^{-1})\}\}$

Q

$\{\{0, -6 x_1[-1+t]^2 x_2[-1+t]^2 x_2[t]^3 x_3[t] + 12 x_1[-1+t] x_2[-1+t]^5 x_2[t]^3 x_3[t] -$

$6 x_2[-1+t]^8 x_2[t]^3 x_3[t], -6 u_1[t]^2 x_2[-2+t]^3 x_2[-1+t]^4 x_2[t]^3 x_3[t] (S_t^{-1})\},$

$\{x_1[t] x_2[-1+t]^2 x_2[t] - x_2[-1+t]^2 x_2[t]^4, -2 x_1[-1+t]^2 x_2[-1+t]^2 x_2[t] x_3[t] +$

$4 x_1[-1+t] x_2[-1+t]^5 x_2[t] x_3[t] - 2 x_2[-1+t]^8 x_2[t] x_3[t],$

$u_1[1+t] x_2[-1+t]^3 x_2[t]^2 x_2[1+t]^3 D_t - 2 u_1[t]^2 x_2[-2+t]^3 x_2[-1+t]^4 x_2[t] x_3[t]$

$(S_t^{-1}) + (3 u_1[1+t] u_2[1+t] x_2[-1+t]^3 x_2[t]^3 x_2[1+t]^2 +$

$4 u_1[1+t] u_2[t] x_2[-1+t]^4 x_2[t] x_2[1+t]^3 + 3 u_1[1+t] u_2[-1+t] x_2[-2+t]$

$x_2[-1+t]^2 x_2[t]^2 x_2[1+t]^3 + 2 x_2[-1+t]^3 x_2[t]^2 x_2[1+t]^3 u_1'[1+t])\},$

$\{0, -3 x_1[-1+t]^2 x_2[-1+t]^2 x_2[t]^3 + 6 x_1[-1+t] x_2[-1+t]^5 x_2[t]^3 -$

$3 x_2[-1+t]^8 x_2[t]^3, -3 u_1[t]^2 x_2[-2+t]^3 x_2[-1+t]^4 x_2[t]^3 (S_t^{-1})\},$

$\{3 u_1[t] x_2[-2+t]^2 x_2[-1+t]^3 (S_t^{-1}),$

$(-3 x_1[-1+t] x_2[-1+t]^2 x_2[t]^3 + 3 x_2[-1+t]^5 x_2[t]^3) D_t +$

$(-9 u_2[t] x_1[-1+t] x_2[-1+t]^3 x_2[t]^2 + 9 u_2[t] x_2[-1+t]^6 x_2[t]^2 -$

$6 u_2[-1+t] x_1[-1+t] x_2[-2+t] x_2[-1+t] x_2[t]^3 + 24 u_2[-1+t] x_2[-2+t]$

$x_2[-1+t]^4 x_2[t]^3 - 12 u_1[-1+t] x_1[-2+t] x_2[-1+t]^2 x_2[t]^3 x_3[-1+t] +$

$12 u_1[-1+t] x_2[-2+t]^3 x_2[-1+t]^2 x_2[t]^3 x_3[-1+t]), 0\},$

$\{(x_1[t] x_2[-1+t] x_2[t] - x_2[-1+t] x_2[t]^4) D_t +$

$(-u_2[t] x_1[-1+t] x_2[-2+t]^2 + u_2[t] x_2[-2+t]^2 x_2[-1+t]^3) (S_t^{-1}) +$

$(u_2[t] x_1[t] x_2[-1+t]^2 + 2 u_2[-1+t] x_1[t] x_2[-2+t] x_2[t] -$

$4 u_2[t] x_2[-1+t]^2 x_2[t]^3 - 2 u_2[-1+t] x_2[-2+t] x_2[t]^4 +$

$2 u_1[t] x_1[-1+t] x_2[-1+t] x_2[t] x_3[t] - 2 u_1[t] x_2[-1+t]^4 x_2[t] x_3[t]),$

$(-2 x_1[-1+t]^2 x_2[-1+t] x_2[t] x_3[t] + 4 x_1[-1+t] x_2[-1+t]^4 x_2[t] x_3[t] -$

$2 x_2[-1+t]^7 x_2[t] x_3[t]) D_t + (2 u_2[t] x_1[-2+t]^2 x_2[-2+t]^2 x_3[-1+t] -$

$4 u_2[t] x_1[-2+t] x_2[-2+t]^5 x_3[-1+t] + 2 u_2[t] x_2[-2+t]^8 x_3[-1+t]) (S_t^{-1}) +$

$(-2 u_1[t] x_1[-1+t]^3 x_2[-1+t] x_2[t] + 6 u_1[t] x_1[-1+t]^2 x_2[-1+t]^4 x_2[t] -$

$6 u_1[t] x_1[-1+t] x_2[-1+t]^7 x_2[t] + 2 u_1[t] x_2[-1+t]^{10} x_2[t] -$

$2 u_2[t] x_1[-1+t]^2 x_2[-1+t]^2 x_3[t] + 4 u_2[t] x_1[-1+t] x_2[-1+t]^5 x_3[t] -$

$2 u_2[t] x_2[-1+t]^8 x_3[t] - 4 u_2[-1+t] x_1[-1+t]^2 x_2[-2+t] x_2[t] x_3[t] +$

$20 u_2[-1+t] x_1[-1+t] x_2[-2+t] x_2[-1+t]^3 x_2[t] x_3[t] -$

$16 u_2[-1+t] x_2[-2+t] x_2[-1+t]^6 x_2[t] x_3[t] -$

$8 u_1[-1+t] x_1[-2+t] x_1[-1+t] x_2[-1+t] x_2[t] x_3[-1+t] x_3[t] +$

$8 u_1[-1+t] x_1[-1+t] x_2[-2+t]^3 x_2[-1+t] x_2[t] x_3[-1+t] x_3[t] +$

$8 u_1[-1+t] x_1[-2+t] x_2[-1+t]^4 x_2[t] x_3[-1+t] x_3[t] -$

$8 u_1[-1+t] x_2[-2+t]^3 x_2[-1+t]^4 x_2[t] x_3[-1+t] x_3[t]),$

$u_1[1+t] x_2[-1+t]^2 x_2[t]^2 x_2[1+t]^3 D_t^2 + (-u_1[t] u_2[t] x_2[-2+t]^3 x_2[-1+t] x_2[t]^3 -$

$2 u_1[t]^2 x_2[-2+t]^3 x_2[-1+t]^3 x_2[t] x_3[t]) D_t (S_t^{-1}) +$

$2 u_1[-1+t]^2 u_2[t] x_2[-3+t]^3 x_2[-2+t]^4 x_3[-1+t] (S_t^{-1})^2 +$

$(6 u_1[1+t] u_2[1+t] x_2[-1+t]^2 x_2[t]^3 x_2[1+t]^2 +$

$6 u_1[1+t] u_2[t] x_2[-1+t]^3 x_2[t] x_2[1+t]^3 + 6 u_1[1+t] u_2[-1+t] x_2[-2+t]$

$x_2[-1+t] x_2[t]^2 x_2[1+t]^3 + 3 x_2[-1+t]^2 x_2[t]^2 x_2[1+t]^3 u_1'[1+t]) D_t +$

$(-2 u_1[t]^3 x_1[-1+t] x_2[-2+t]^3 x_2[-1+t]^3 x_2[t] +$

$2 u_1[t]^3 x_2[-2+t]^3 x_2[-1+t]^6 x_2[t] - 3 u_1[t] u_2[t]^2 x_2[-2+t]^3 x_2[-1+t]^2 x_2[t]^2 -$

$4 u_1[t] u_2[-1+t] u_2[t] x_2[-2+t]^4 x_2[t]^3 - 3 u_1[t] u_2[-2+t] u_2[t] x_2[-3+t]$

$x_2[-2+t]^2 x_2[-1+t] x_2[t]^3 - 2 u_1[t]^2 u_2[t] x_2[-2+t]^3 x_2[-1+t]^4 x_3[t] -$

$8 u_1[t]^2 u_2[-1+t] x_2[-2+t]^4 x_2[-1+t]^2 x_2[t] x_3[t] -$

$6 u_1[t]^2 u_2[-2+t] x_2[-3+t] x_2[-2+t]^2 x_2[-1+t]^3 x_2[t] x_3[t] - 2 u_2[t] x_2[-2+t]^3$

$$x_2[-1+t]\, x_2[t]^3\, u_1'[t] - 4\, u_1[t]\, x_2[-2+t]^3\, x_2[-1+t]^3\, x_2[t]\, x_3[t]\, u_1'[t]\big)\, \left(S_t^{-1}\right) +$$
$$\big(6\, u_1[1+t]\, u_2[1+t]^2\, x_2[-1+t]^2\, x_2[t]^4\, x_2[1+t] +$$
$$21\, u_1[1+t]\, u_2[t]\, u_2[1+t]\, x_2[-1+t]^3\, x_2[t]^2\, x_2[1+t]^2 +$$
$$18\, u_1[1+t]\, u_2[-1+t]\, u_2[1+t]\, x_2[-2+t]\, x_2[-1+t]\, x_2[t]^3\, x_2[1+t]^2 +$$
$$4\, u_1[1+t]\, u_2[t]^2\, x_2[-1+t]^4\, x_2[1+t]^3 + 22\, u_1[1+t]\, u_2[-1+t]\, u_2[t]\, x_2[-2+t]$$
$$x_2[-1+t]^2\, x_2[t]\, x_2[1+t]^3 + 6\, u_1[1+t]\, u_2[-1+t]^2\, x_2[-2+t]^2\, x_2[t]^2\, x_2[1+t]^3 +$$
$$3\, u_1[1+t]\, u_2[-2+t]\, u_2[-1+t]\, x_2[-3+t]\, x_2[-1+t]\, x_2[t]^2\, x_2[1+t]^3 +$$
$$9\, u_2[1+t]\, x_2[-1+t]^2\, x_2[t]^3\, x_2[1+t]^2\, u_1'[1+t] + 8\, u_2[t]\, x_2[-1+t]^3\, x_2[t]$$
$$x_2[1+t]^3\, u_1'[1+t] + 9\, u_2[-1+t]\, x_2[-2+t]\, x_2[-1+t]\, x_2[t]^2\, x_2[1+t]^3\, u_1'[1+t] +$$
$$3\, u_1[1+t]\, x_2[-2+t]\, x_2[-1+t]\, x_2[t]^2\, x_2[1+t]^3\, u_2'[-1+t] +$$
$$4\, u_1[1+t]\, x_2[-1+t]^3\, x_2[t]\, x_2[1+t]^3\, u_2'[t] + 3\, u_1[1+t]\, x_2[-1+t]^2$$
$$x_2[t]^3\, x_2[1+t]^2\, u_2'[1+t] + 2\, x_2[-1+t]^2\, x_2[t]^2\, x_2[1+t]^3\, u_1''[1+t]\big)\big\}\big\}$$

Since the first matrix is not identity, we deduce that M admits nontrivial torsion elements and thus the corresponding system admits autonomous elements τ[1], ..., τ[4], defined by:

```
aut = AutonomousElements[R,
   {dx1[t], dx2[t], dx3[t], du1[t], du2[t]}, τ, A, Relations → True];
```

```
aut[[1]]
```

$$\big\{\tau[1][t] \to dx_1[t] - 2\, dx_3[t]\, x_3[t],\ \tau[2][t] \to -3\, dx_2[-1+t]\, u_1[t]\, x_2[-1+t]^2 +$$
$$du_1[t]\, \left(x_1[-1+t] - x_2[-1+t]^3\right) + 2\, dx_3[-1+t]\, u_1[t]\, x_3[-1+t] - dx_3'[t],$$
$$\tau[3][t] \to 3\, du_2[t]\, u_1[t]\, x_2[-1+t]^3 + du_1[t]\, u_2[t]\, \left(x_1[-1+t] - x_2[-1+t]^3\right) +$$
$$2\, dx_3[-1+t]\, u_1[t]\, u_2[t]\, x_3[-1+t] - 3\, u_1[t]\, x_2[-1+t]^2\, dx_2'[t] - u_2[t]\, dx_3'[t],$$
$$\tau[4][t] \to 3\, du_2[-1+t]\, u_1[-1+t]\, u_1[t]^2\, x_2[-2+t]^3\, x_2[-1+t]^3 +$$
$$du_1[-1+t]\, u_1[t]^2\, u_2[-1+t]\, \left(x_1[-2+t] - x_2[-2+t]^3\right)\, x_2[-1+t]^3 +$$
$$2\, dx_3[-2+t]\, u_1[-1+t]\, u_1[t]^2\, u_2[-1+t]\, x_2[-1+t]^3\, x_3[-2+t] +$$
$$2\, dx_3[-1+t]\, u_1[-1+t]\, u_1[t]^2\, x_2[-2+t]^2$$
$$\left(u_1[-1+t]\, \left(-x_1[-2+t] + x_2[-2+t]^3\right)\, x_2[-1+t] + 2\, u_2[-1+t]\, x_2[-2+t]\, x_3[-1+t]\right) +$$
$$u_1[-1+t]\, u_1[t]\, x_2[-2+t]^2\, x_2[-1+t]\, \left(-x_1[-1+t] + x_2[-1+t]^3\right)\, du_1'[t] - u_1[t]^2$$
$$x_2[-1+t]\, \left(u_2[-1+t]\, x_2[-1+t]^2 + 2\, u_1[-1+t]\, x_2[-2+t]^2\, x_3[-1+t]\right)\, dx_3'[-1+t] -$$
$$u_1[-1+t]\, x_2[-2+t]^2\, dx_3'[t]\, (2\, u_1[t]\, u_2[-1+t]\, x_2[-2+t] + x_2[-1+t]\, u_1'[t]) +$$
$$du_1[t]\, u_1[-1+t]\, x_2[-2+t]^2\, \Big(u_1[t]\, \Big(u_2[-1+t]\, x_2[-2+t]\, \left(2\, x_1[-1+t] + x_2[-1+t]^3\right) +$$
$$2\, u_1[-1+t]\, \left(-x_1[-2+t] + x_2[-2+t]^3\right)\, x_2[-1+t]\, x_3[-1+t]\Big) + x_2[-1+t]$$
$$\left(x_1[-1+t] - x_2[-1+t]^3\right)\, u_1'[t]\Big) + u_1[-1+t]\, u_1[t]\, x_2[-2+t]^2\, x_2[-1+t]\, dx_3''[t]\big\}$$

The autonomous elements τ[1],...,τ[4] satisfy the following equations:

```
aut[[2]]
```

$$\big\{\tau[1]'[t] == 0,\ \tau[2][t]\, u_1'[t] - u_1[t]\, \tau[2]'[t] == 0,$$
$$u_1[t]\, \tau[3][t]\, u_2'[t] + u_2[t]\, (\tau[3][t]\, u_1'[t] - u_1[t]\, \tau[3]'[t]) == 0,$$
$$u_1[t]^3\, u_2[-1+t]^2\, x_2[-1+t]^2$$
$$\left(u_2[-1+t]\, x_2[-2+t]^2 - u_2[-2+t]\, x_2[-3+t]\, x_2[-1+t]\right)\, \tau[4][-1+t]$$
$$\Big(u_1[-1+t]\, u_2[-2+t]\, x_2[-2+t]\, u_1'[-2+t] + u_1[-2+t]\, \Big(2\, u_2[-2+t]\, x_2[-2+t]$$
$$u_1'[-1+t] + u_1[-1+t]\, \left(3\, u_2[-2+t]^2\, x_2[-3+t] + x_2[-2+t]\, u_2'[-2+t]\right)\Big)\Big) +$$
$$2\, u_1[-2+t]^2\, u_1[-1+t]\, u_2[-2+t]^2\, x_2[-3+t]^2\, \Big(u_2[-2+t]\, x_2[-3+t]^2 -$$
$$u_2[-3+t]\, x_2[-4+t]\, x_2[-2+t]\Big)\, \tau[4][t]$$
$$(u_1[t]\, u_2[-1+t]\, x_2[-2+t]\, u_1'[-1+t] + u_1[-1+t]\, (2\, u_2[-1+t]\, x_2[-2+t]\, u_1'[t] +$$
$$u_1[t]\, (3\, u_2[-2+t]\, u_2[-1+t]\, x_2[-3+t] + x_2[-2+t]\, u_2'[-1+t]))) +$$
$$u_1[-2+t]\, u_1[-1+t]\, u_1[t]^3\, u_2[-2+t]\, u_2[-1+t]^2\, x_2[-2+t]\, x_2[-1+t]^2$$

$$\left(-u_2[-1+t]\, x_2[-2+t]^2 + u_2[-2+t]\, x_2[-3+t]\, x_2[-1+t]\right) \tau[4]'[-1+t] +$$
$$2\, u_1[-2+t]^2\, u_1[-1+t]^2\, u_1[t]\, u_2[-2+t]^2\, u_2[-1+t]\, x_2[-3+t]^2\, x_2[-2+t]$$
$$\left(-u_2[-2+t]\, x_2[-3+t]^2 + u_2[-3+t]\, x_2[-4+t]\, x_2[-2+t]\right) \tau[4]'[t] == 0,$$
$$2\, u_1[t]^2\, u_2[-1+t]^2\, x_2[-2+t]\, x_2[-1+t]$$
$$\left(-u_2[-1+t]\, x_2[-2+t]^2 + u_2[-2+t]\, x_2[-3+t]\, x_2[-1+t]\right) \tau[4][t]\, u_1'[-1+t]^2 +$$
$$u_1[-1+t]\, u_1[t]\, u_2[-1+t]\, \left(4\, u_2[-1+t]\, x_2[-2+t]\, x_2[-1+t]\right.$$
$$\left(-u_2[-1+t]\, x_2[-2+t]^2 + u_2[-2+t]\, x_2[-3+t]\, x_2[-1+t]\right) \tau[4][t]\, u_1'[-1+t]$$
$$u_1'[t] + u_1[t]\, \left(-2\, u_2[-1+t]^3\, x_2[-2+t]^4\, \tau[4][t]\, u_1'[-1+t] + 2\, u_2[-2+t]\right.$$
$$x_2[-3+t]\, x_2[-2+t]\, x_2[-1+t]^2\, \tau[4][t]\, u_1'[-1+t]\, u_2'[-1+t] + u_2[-1+t]^2$$
$$x_2[-2+t]^2\, x_2[-1+t]\, (-u_2[-2+t]\, x_2[-3+t]\, \tau[4][t]\, u_1'[-1+t] + x_2[-2+t]$$
$$(2\, u_1'[-1+t]\, \tau[4]'[t] + \tau[4][t]\, u_1''[-1+t])) + u_2[-1+t]\, x_2[-1+t]$$
$$\left(2\, u_2[-2+t]^2\, x_2[-3+t]^2\, x_2[-1+t]\, \tau[4][t]\, u_1'[-1+t] + x_2[-2+t]\, \tau[4][t]\right.$$
$$u_1'[-1+t]\, \left(x_2[-3+t]\, x_2[-1+t]\, u_2'[-2+t] - 3\, x_2[-2+t]^2\, u_2'[-1+t]\right) +$$
$$u_2[-2+t]\, x_2[-2+t]\, x_2[-1+t]\, (u_2[-3+t]\, x_2[-4+t]\, \tau[4][t]\, u_1'[-1+t] -$$
$$\left.\left.\left. x_2[-3+t]\, (2\, u_1'[-1+t]\, \tau[4]'[t] + \tau[4][t]\, u_1''[-1+t]))\right)\right)\right) +$$
$$u_1[-1+t]^2\, \left(6\, u_2[-1+t]^2\, x_2[-2+t]\, x_2[-1+t]\, \left(-u_2[-1+t]\, x_2[-2+t]^2 +\right.\right.$$
$$\left. u_2[-2+t]\, x_2[-3+t]\, x_2[-1+t]\right) \tau[4][t]\, u_1'[t]^2 +$$
$$2\, u_1[t]\, u_2[-1+t]\, \left(-2\, u_2[-1+t]^3\, x_2[-2+t]^4\, \tau[4][t]\, u_1'[t] +\right.$$
$$2\, u_2[-2+t]\, x_2[-3+t]\, x_2[-2+t]\, x_2[-1+t]^2\, \tau[4][t]\, u_1'[t]\, u_2'[-1+t] +$$
$$u_2[-1+t]^2\, x_2[-2+t]^2\, x_2[-1+t]\, (-u_2[-2+t]\, x_2[-3+t]\, \tau[4][t]\, u_1'[t] +$$
$$x_2[-2+t]\, (2\, u_1'[t]\, \tau[4]'[t] + \tau[4][t]\, u_1''[t])) + u_2[-1+t]\, x_2[-1+t]$$
$$\left(2\, u_2[-2+t]^2\, x_2[-3+t]^2\, x_2[-1+t]\, \tau[4][t]\, u_1'[t] + x_2[-2+t]\, \tau[4][t]\right.$$
$$u_1'[t]\, \left(x_2[-3+t]\, x_2[-1+t]\, u_2'[-2+t] - 3\, x_2[-2+t]^2\, u_2'[-1+t]\right) +$$
$$u_2[-2+t]\, x_2[-2+t]\, x_2[-1+t]\, (u_2[-3+t]\, x_2[-4+t]\, \tau[4][t]$$
$$\left.\left. u_1'[t] - x_2[-3+t]\, (2\, u_1'[t]\, \tau[4]'[t] + \tau[4][t]\, u_1''[t]))\right)\right) + u_1[t]^2$$
$$\left(u_2[-2+t]^2\, u_2[-1+t]\, x_2[-3+t]^2\, x_2[-1+t]\, \left(3\, u_2[-1+t]^2\, x_2[-2+t]\, \tau[4][t] +\right.\right.$$
$$\left. 2\, x_2[-1+t]\, \tau[4][t]\, u_2'[-1+t] - 2\, u_2[-1+t]\, x_2[-1+t]\, \tau[4]'[t]\right) +$$
$$u_2[-2+t]\, x_2[-2+t]\, \left(-6\, u_2[-1+t]^4\, x_2[-3+t]\, x_2[-2+t]^2\, \tau[4][t] +\right.$$
$$2\, x_2[-3+t]\, x_2[-1+t]^2\, \tau[4][t]\, u_2'[-1+t]^2 + u_2[-1+t]^3\, x_2[-2+t]$$
$$x_2[-1+t]\, (3\, u_2[-3+t]\, x_2[-4+t]\, \tau[4][t] + x_2[-3+t]\, \tau[4]'[t]) +$$
$$u_2[-1+t]\, x_2[-1+t]^2\, (u_2[-3+t]\, x_2[-4+t]\, \tau[4][t]\, u_2'[-1+t] -$$
$$x_2[-3+t]\, (2\, u_2'[-1+t]\, \tau[4]'[t] + \tau[4][t]\, u_2''[-1+t])) +$$
$$u_2[-1+t]^2\, x_2[-1+t]\, (-u_2[-3+t]\, x_2[-4+t]\, x_2[-1+t]\, \tau[4]'[t] +$$
$$\left. x_2[-3+t]\, (-4\, x_2[-2+t]\, \tau[4][t]\, u_2'[-1+t] + x_2[-1+t]\, \tau[4]''[t]))\right) +$$
$$u_2[-1+t]\, x_2[-2+t]\, \left(x_2[-1+t]\, \tau[4][t]\, u_2'[-1+t]\, \left(x_2[-3+t]\, x_2[-1+t]\right.\right.$$
$$\left. u_2'[-2+t] - 3\, x_2[-2+t]^2\, u_2'[-1+t]\right) + 2\, u_2[-1+t]^3\, x_2[-2+t]^3$$
$$\tau[4]'[t] + u_2[-1+t]\, x_2[-1+t]\, \left(-x_2[-3+t]\, x_2[-1+t]\, u_2'[-2+t]\right.$$
$$\left. \tau[4]'[t] + x_2[-2+t]^2\, (3\, u_2'[-1+t]\, \tau[4]'[t] + \tau[4][t]\, u_2''[-1+t])\right) -$$
$$u_2[-1+t]^2\, x_2[-2+t]\, (-3\, x_2[-3+t]\, x_2[-1+t]\, \tau[4][t]\, u_2'[-2+t] + x_2[-2+$$
$$\left.\left.\left.\left. t]\, (2\, x_2[-2+t]\, \tau[4][t]\, u_2'[-1+t] + x_2[-1+t]\, \tau[4]''[t]))\right)\right)\right)\right) == 0\}$$

The A-linear relations among the autonomous elements are given by:

```
aut[[3]]
```

$$\left\{-2\,u_1[t]\,x_3[t]\,\tau[1][-1+t]-2\,x_3[t]\,\tau[2][t]+\tau[1]'[t],\ \frac{u_2[t]\,\tau[2][t]-\tau[3][t]}{3\,u_1[t]\,x_2[-1+t]^2},\right.$$
$$-u_1[t]\,\tau[1][-1+t]-\tau[2][t],\ u_1[t]^2\,x_2[-1+t]^3\,\tau[3][-1+t]-$$
$$\tau[4][t]+u_1[-1+t]\,x_2[-2+t]^2\,(x_2[-1+t]\,\tau[2][t]\,u_1'[t]+$$
$$\left.u_1[t]\,(2\,u_2[-1+t]\,x_2[-2+t]\,\tau[2][t]-x_2[-1+t]\,\tau[2]'[t]))\right\}$$

Let us now prove that the set of autonomous elements can be generated by $\tau[1]$ (our guess or inspection of aut[[3]]). Let us introduce the matrix L, defining aut[[3]]:

```
L = ToOrePolynomialD[aut[[3]], {τ[1][t], τ[2][t], τ[3][t], τ[4][t]}, A]
```

$$\left\{\left\{D_t-2\,u_1[t]\,x_3[t]\,\left(S_t^{-1}\right),\ -2\,x_3[t],\ 0,\ 0\right\},\right.$$
$$\left\{0,\ \frac{u_2[t]}{3\,u_1[t]\,x_2[-1+t]^2},\ -\frac{1}{3\,u_1[t]\,x_2[-1+t]^2},\ 0\right\},$$
$$\left\{-u_1[t]\,\left(S_t^{-1}\right),\ -1,\ 0,\ 0\right\},\ \left\{0,\ -u_1[-1+t]\,u_1[t]\,x_2[-2+t]^2\,x_2[-1+t]\,D_t+\right.$$
$$\left(2\,u_1[-1+t]\,u_1[t]\,u_2[-1+t]\,x_2[-2+t]^3+u_1[-1+t]\,x_2[-2+t]^2\,x_2[-1+t]\,u_1'[t]\right),$$
$$\left.\left.u_1[t]^2\,x_2[-1+t]^3\,\left(S_t^{-1}\right),\ -1\right\}\right\}$$

Let us consider the following row vector

```
γ = {{1, 0, 0, 0}};
```

which corresponds to the position of $\tau[1]$. To express $\tau[2]$, $\tau[3]$ and $\tau[4]$ in terms of $\tau[1]$ we first check whether or not the matrix Q formed by stacking L with γ admits a left inverse.

```
S1 = LeftInverse[Q = Join[L, γ], A]
```

$$\left\{\{0,\ 0,\ 0,\ 0,\ 1\},\ \left\{0,\ 0,\ -1,\ 0,\ -u_1[t]\,\left(S_t^{-1}\right)\right\},\right.$$
$$\left\{0,\ -3\,u_1[t]\,x_2[-1+t]^2,\ -u_2[t],\ 0,\ -u_1[t]\,u_2[t]\,\left(S_t^{-1}\right)\right\},$$
$$\left\{u_1[-1+t]\,u_1[t]^2\,x_2[-2+t]^2\,x_2[-1+t]\,\left(S_t^{-1}\right),\right.$$
$$-3\,u_1[-1+t]\,u_1[t]^2\,x_2[-2+t]^2\,x_2[-1+t]^3\,\left(S_t^{-1}\right),$$
$$u_1[-1+t]\,u_1[t]\,x_2[-2+t]^2\,x_2[-1+t]\,D_t-$$
$$u_1[t]^2\,x_2[-1+t]\,\left(u_2[-1+t]\,x_2[-1+t]^2+2\,u_1[-1+t]\,x_2[-2+t]^2\,x_3[-1+t]\right)\,\left(S_t^{-1}\right)-$$
$$u_1[-1+t]\,x_2[-2+t]^2\,(2\,u_1[t]\,u_2[-1+t]\,x_2[-2+t]+x_2[-1+t]\,u_1'[t]),$$
$$-1,\ -u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]\,x_2[-1+t]^3\,\left(S_t^{-1}\right)^2-$$
$$\left.\left.2\,u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]\,x_2[-2+t]^3\,\left(S_t^{-1}\right)\right\}\right\}$$

If we consider the last column of the left inverse S1 of Q, i.e.

```
S2 = List /@ Transpose[S1][[-1]]
```

$$\left\{\{1\},\ \left\{-u_1[t]\,\left(S_t^{-1}\right)\right\},\ \left\{-u_1[t]\,u_2[t]\,\left(S_t^{-1}\right)\right\},\right.$$
$$\left\{-u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]\,x_2[-1+t]^3\,\left(S_t^{-1}\right)^2-\right.$$
$$\left.\left.2\,u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]\,x_2[-2+t]^3\,\left(S_t^{-1}\right)\right\}\right\}$$

then we obtain:

```
Thread[{τ[1][t], τ[2][t], τ[3][t], τ[4][t]} -> ApplyMatrix[S2, {τ[1][t]}]]
```

$\{\tau[1][t] \to \tau[1][t],\ \tau[2][t] \to -u_1[t]\ \tau[1][-1+t],\ \tau[3][t] \to -u_1[t]\ u_2[t]\ \tau[1][-1+t],\ \tau[4][t] \to -u_1[-1+t]\ u_1[t]^2\ u_2[-1+t]\ (x_2[-1+t]^3\ \tau[1][-2+t] + 2\ x_2[-2+t]^3\ \tau[1][-1+t])\}$

From this point we will use some procedures which are not freely available (see NLControl website http://www.nlcontrol.ioc.ee).
Let us integrate the one-form defined by T[1], namely

```
difs = {ApplyMatrixD[Rp[[1]], vars]}
```

$\{De[1, \{x_1[t]\}] + De[-2\ x_3[t], \{x_3[t]\}]\}$

```
BookForm[sp = SpanK[difs, t]]
```

$SpanK[dx_1[t] - 2\ x_3[t]\ dx_3[t]]$

```
Integrability[sp]
```

True

```
IntegrateOneForms[sp]
```

$\{x_1[t] - x_3[t]^2\}$

We finally obtain the above autonomous element of the nonlinear system.

Serre' s reduction techniques

Let us now prove that L is equivalent to a diagonal matrix. Let us consider the column vector Λ

```
Λ = {{-1}, {0}, {0}, {0}}
```

{{-1}, {0}, {0}, {0}}

and the matrix P obtained by concatenation of L and Λ:

```
P = Join[L, Λ, 2]
```

$$\Big\{\{D_t - 2\ u_1[t]\ x_3[t]\ (S_t^{-1}),\ -2\ x_3[t],\ 0,\ 0,\ -1\},\ \Big\{0,\ \frac{u_2[t]}{3\ u_1[t]\ x_2[-1+t]^2},\ -\frac{1}{3\ u_1[t]\ x_2[-1+t]^2},\ 0,\ 0\Big\},\ \{-u_1[t]\ (S_t^{-1}),\ -1,\ 0,\ 0,\ 0\},\ \{0,\ -u_1[-1+t]\ u_1[t]\ x_2[-2+t]^2\ x_2[-1+t]\ D_t + (2\ u_1[-1+t]\ u_1[t]\ u_2[-1+t]\ x_2[-2+t]^3 + u_1[-1+t]\ x_2[-2+t]^2\ x_2[-1+t]\ u_1'[t]),\ u_1[t]^2\ x_2[-1+t]^3\ (S_t^{-1}),\ -1,\ 0\}\Big\}$$

Let us check again that the matrix P admits a right inverse:

Pinv = RightInverse[P, A]

$$\{\{0, 0, 0, 0\}, \{0, 0, -1, 0\}, \{0, -3\, u_1[t]\, x_2[-1+t]^2, -u_2[t], 0\},$$
$$\{0, -3\, u_1[-1+t]\, u_1[t]^2\, x_2[-2+t]^2\, x_2[-1+t]^3\, (S_t^{-1}),$$
$$u_1[-1+t]\, u_1[t]\, x_2[-2+t]^2\, x_2[-1+t]\, D_t - u_1[t]^2\, u_2[-1+t]\, x_2[-1+t]^3\, (S_t^{-1}) -$$
$$u_1[-1+t]\, x_2[-2+t]^2\, (2\, u_1[t]\, u_2[-1+t]\, x_2[-2+t] + x_2[-1+t]\, u_1'[t]),$$
$$-1\}, \{-1, 0, 2\, x_3[t], 0\}\}$$

We deduce that the module E, finitely presented by P, is stably free. Let us now check whether or not E is free.

Q = MinimalParametrization[P, A]

$$\{\{-1\}, \{u_1[t]\, (S_t^{-1})\}, \{u_1[t]\, u_2[t]\, (S_t^{-1})\},$$
$$\{-u_1[-1+t]\, u_1[t]^2\, x_2[-2+t]^2\, x_2[-1+t]\, D_t\, (S_t^{-1}) +$$
$$u_1[-1+t]\, u_1[t]^2\, u_2[-1+t]\, x_2[-1+t]^3\, (S_t^{-1})^2 +$$
$$2\, u_1[-1+t]\, u_1[t]^2\, u_2[-1+t]\, x_2[-2+t]^3\, (S_t^{-1})\}, \{-D_t\}\}$$

LeftInverse[Q, A]

$$\{\{-1, 0, 0, 0, 0\}\}$$

We obtain that E is a free module. Let us extract blocs from the matrix Q as follows:

Q1 = Take[Q, 4]

$$\{\{-1\}, \{u_1[t]\, (S_t^{-1})\}, \{u_1[t]\, u_2[t]\, (S_t^{-1})\},$$
$$\{-u_1[-1+t]\, u_1[t]^2\, x_2[-2+t]^2\, x_2[-1+t]\, D_t\, (S_t^{-1}) +$$
$$u_1[-1+t]\, u_1[t]^2\, u_2[-1+t]\, x_2[-1+t]^3\, (S_t^{-1})^2 +$$
$$2\, u_1[-1+t]\, u_1[t]^2\, u_2[-1+t]\, x_2[-2+t]^3\, (S_t^{-1})\}\}$$

Q2 = Take[Q, -1]

$$\{\{-D_t\}\}$$

By theory we know that the left A-module N finitely presented by L is isomorphic to the module finitely presented by Q2.
Let now check that the matrix L is equivalent to the diagonal matrix diag(1,1,1,Q2). To do that let us compute a basis of the left kernel of Q1:

U1 = LeftKernel[Q1, A]

$$\{\{0, -u_2[t], 1, 0\},$$
$$\{0, 0, -u_1[-1+t]\, u_1[t]\, u_2[t]\, x_2[-2+t]^2\, x_2[-1+t]\, D_t + u_1[t]^2\, u_2[t]^2\, x_2[-1+t]^3\, (S_t^{-1}) +$$
$$(2\, u_1[-1+t]\, u_1[t]\, u_2[-1+t]\, u_2[t]\, x_2[-2+t]^3 + u_1[-1+t]\, u_2[t]\, x_2[-2+t]^2$$
$$x_2[-1+t]\, u_1'[t] + u_1[-1+t]\, u_1[t]\, x_2[-2+t]^2\, x_2[-1+t]\, u_2'[t]),$$
$$-u_2[t]^2\}, \{u_1[t]\, u_2[t]\, (S_t^{-1}), 0, 1, 0\}\}$$

LeftKernel[U1, A]

Inj[3]

Let W1 be a right inverse of U1

W1 = RightInverse[U1, A]

$$\left\{\{0,0,0\},\left\{-\frac{1}{u_2[t]},0,\frac{1}{u_2[t]}\right\},\{0,0,1\},\left\{0,-\frac{1}{u_2[t]^2},-\frac{1}{u_2[t]}u_1[-1+t]\,u_1[t]\,x_2[-2+t]^2\,x_2[-1+t]\,D_t+u_1[t]^2\,x_2[-1+t]^3\left(S_t^{-1}\right)+\frac{1}{u_2[t]^2}u_1[-1+t]\,x_2[-2+t]^2\left(u_2[t]\,x_2[-1+t]\,u_1'[t]+u_1[t]\left(2\,u_2[-1+t]\,u_2[t]\,x_2[-2+t]+x_2[-1+t]\,u_2'[t]\right)\right)\right\}\right\}$$

and W be the unimodular matrix formed by W1 and Q1:

W = Join[W1, Q1, 2]

$$\left\{\{0,0,0,-1\},\left\{-\frac{1}{u_2[t]},0,\frac{1}{u_2[t]},u_1[t]\left(S_t^{-1}\right)\right\},\left\{0,0,1,u_1[t]\,u_2[t]\left(S_t^{-1}\right)\right\},\left\{0,-\frac{1}{u_2[t]^2},-\frac{1}{u_2[t]}u_1[-1+t]\,u_1[t]\,x_2[-2+t]^2\,x_2[-1+t]\,D_t+u_1[t]^2\,x_2[-1+t]^3\left(S_t^{-1}\right)+\frac{1}{u_2[t]^2}u_1[-1+t]\,x_2[-2+t]^2\left(u_2[t]\,x_2[-1+t]\,u_1'[t]+u_1[t]\left(2\,u_2[-1+t]\,u_2[t]\,x_2[-2+t]+x_2[-1+t]\,u_2'[t]\right)\right),-u_1[-1+t]\,u_1[t]^2\,x_2[-2+t]^2\,x_2[-1+t]\,D_t\left(S_t^{-1}\right)+u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]\,x_2[-1+t]^3\left(S_t^{-1}\right)^2+2\,u_1[-1+t]\,u_1[t]^2\,u_2[-1+t]\,x_2[-2+t]^3\left(S_t^{-1}\right)\right\}\right\}$$

Let us check again that W is unimodular:

LeftInverse[W, A]

$$\left\{\{0,-u_2[t],1,0\},\left\{0,0,-u_1[-1+t]\,u_1[t]\,u_2[t]\,x_2[-2+t]^2\,x_2[-1+t]\,D_t+u_1[t]^2\,u_2[t]^2\,x_2[-1+t]^3\,S[-1][t]+u_1[-1+t]\,x_2[-2+t]^2\left(u_2[t]\,x_2[-1+t]\,u_1'[t]+u_1[t]\left(2\,u_2[-1+t]\,u_2[t]\,x_2[-2+t]+x_2[-1+t]\,u_2'[t]\right)\right),-u_2[t]^2\right\},\{u_1[t]\,u_2[t]\,S[-1][t],0,1,0\},\{-1,0,0,0\}\right\}$$

Let us form the following matrix

X = Join[OreDot[L, W1], Λ, 2]

$$\left\{\left\{\frac{2\,x_3[t]}{u_2[t]},0,-\frac{2\,x_3[t]}{u_2[t]},-1\right\},\left\{-\frac{1}{3\,u_1[t]\,x_2[-1+t]^2},0,0,0\right\},\left\{\frac{1}{u_2[t]},0,-\frac{1}{u_2[t]},0\right\},\left\{\frac{1}{u_2[t]}u_1[-1+t]\,u_1[t]\,x_2[-2+t]^2\,x_2[-1+t]\,D_t+\left(-\frac{1}{u_2[t]}2\,u_1[-1+t]\,u_1[t]\,u_2[-1+t]\,x_2[-2+t]^3-\frac{1}{u_2[t]}u_1[-1+t]\,x_2[-2+t]^2\,x_2[-1+t]\,u_1'[t]-\frac{1}{u_2[t]^2}u_1[-1+t]\,u_1[t]\,x_2[-2+t]^2\,x_2[-1+t]\,u_2'[t]\right),\frac{1}{u_2[t]^2},0,0\right\}\right\}$$

Let us check that X is unimodular:

```
V = LeftInverse[X, A]
```

$$\{\{0, -3\,u_1[t]\,x_2[-1+t]^2, 0, 0\}, \{0, 3\,u_1[-1+t]\,u_1[t]^2\,u_2[t]\,x_2[-2+t]^2\,x_2[-1+t]^3\,D_t - 3\,u_1[-1+t]\,u_1[t]^2\,x_2[-2+t]^2\,x_2[-1+t]^3\,u_2'[t], 0, u_2[t]^2\}, \{0, -3\,u_1[t]\,x_2[-1+t]^2, -u_2[t], 0\}, \{-1, 0, 2\,x_3[t], 0\}\}$$

Finally we obtain that L is equivalent to the following diagonal matrix:

```
MatrixForm[OreDot[V, L, W]]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & D_t \end{pmatrix}$$