Tank model 2

Let us introduce the following Ore algebra A of differential time-delay operators

```
A = OreAlgebra[Der[t], S[-1][t]]
```

$\mathbb{K}(t)[D_t; 1, D_t][(S_t^{-1}); \#1 /. t \to t-1 \&, 0 \&]$

and the matrix R of ordinary differential time-delay operators, which defines a tank model described in Petit, N. and Rouchon, P.,
Dynamics and solutions to some control problems for water-tank systems, IEEE Trans. Automatic Control, 47 (2002), 595-609.

```
MatrixForm[R = ToOrePolynomial[{
    {Der[t], -Der[t] S[-1][t]^2, α Der[t]^2 S[-1][t]},
    {Der[t] S[-1][t]^2, -Der[t], α Der[t]^2 S[-1][t]}
  }, A]]
```

$$\begin{pmatrix} D_t & -D_t\,(S_t^{-1})^2 & \alpha\,D_t^2\,(S_t^{-1}) \\ D_t\,(S_t^{-1})^2 & -D_t & \alpha\,D_t^2\,(S_t^{-1}) \end{pmatrix}$$

and the left A-module, finitely presented by R, defined by M = $A^{1\times3}/A^{1\times2}\,R$.
Let us check whether or not M can be decomposed. We first compute the endomorphisms of M defined a constant matrix P:

```
MatrixForm[morph = Morphisms[R, R, {0, 0}, p, A]]
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

$$\begin{pmatrix} p[1][1] & p[4][1] & 0 \\ p[4][1] & p[1][1] & 0 \\ 0 & 0 & p[1][1]-p[4][1] \end{pmatrix}$$

From this P, let us try to compute idempotent endomorphisms of M:

```
MatrixForm /@ (PAll = IdempotentMorphisms[morph, R, p, A])
```

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\}$$

Considering the nontrivial idempotent endomorphism PAll[[2]], we obtain the following decomposition:

```
MatrixForm[Decomposition[R, PAll[[2]], A]]
```

$$\begin{pmatrix} D_t\,(S_t^{-1})^2 - D_t & 0 & 0 \\ 0 & 2\,\alpha\,D_t^2\,(S_t^{-1}) & -D_t\,(S_t^{-1})^2 - D_t \end{pmatrix}$$

Considering the nontrivial idempotent endomorphism PAll[[3]], we obtain the following decomposition:

```
MatrixForm[Decomposition[R, PAll[[3]], A]]
```

$$\begin{pmatrix} 2\,\alpha\,D_t^2\,\left(S_t^{-1}\right) & -D_t\,\left(S_t^{-1}\right)^2 - D_t & 0 \\ 0 & 0 & D_t\,\left(S_t^{-1}\right)^2 - D_t \end{pmatrix}$$