Stirred tank

H. Kwakernaak, R. Sivan, Linear optimal control systems, Wiley-Interscience, 1972, p. 449.
Let us introduce the Ore algebra A of shift operators defined by

```
A = OreAlgebra[S[t], θ, κ, c0, c1, c2, v0]
```

$\mathbb{K}(t)[\theta, \kappa, c_0, c_1, c_2, v_0][S_t; S_t, 0]$

and the matrix R of shift operators, which defines the system

```
R0 =
  {{S[t] - κ, 0, -2 θ (1 - κ), -2 θ (1 - κ)},
   {0, v0 (S[t] - κ^2), -(θ (c1 - c0) (1 - κ^2)), -(θ (c2 - c0) (1 - κ^2))}};
R = ToOrePolynomial[R0, A]
```

$$\{\{S_t - \kappa,\ 0,\ 2\theta\kappa - 2\theta,\ 2\theta\kappa - 2\theta\},\ \{0,\ -\kappa^2 v_0 + S_t v_0,\ -\theta\kappa^2 c_0 + \theta\kappa^2 c_1 + \theta c_0 - \theta c_1,\ -\theta\kappa^2 c_0 + \theta\kappa^2 c_2 + \theta c_0 - \theta c_2\}\}$$

and the left A-module, finitely presented by R, defined by M = $A^{1\times4}/A^{1\times2}R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} S_t + \kappa & 0 \\ 0 & \kappa^2 v_0 - S_t v_0 \\ 2\theta\kappa + 2\theta & -\theta\kappa^2 c_0 + \theta\kappa^2 c_1 + \theta c_0 - \theta c_1 \\ 2\theta\kappa + 2\theta & -\theta\kappa^2 c_0 + \theta\kappa^2 c_2 + \theta c_0 - \theta c_2 \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
{Ann, Rp, Q} = Exti[Radj, A, 1];
```

```
MatrixForm[Ann]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Rp
```

$$\{\{S_t - \kappa,\ 0,\ 2\theta\kappa - 2\theta,\ 2\theta\kappa - 2\theta\},\ \{S_t\kappa c_0 - S_t\kappa c_1 - \kappa^2 c_0 + \kappa^2 c_1 + S_t c_0 - S_t c_1 - \kappa c_0 + \kappa c_1,\ -2\kappa^2 v_0 + 2 S_t v_0,\ 0,\ -2\theta\kappa^2 c_1 + 2\theta\kappa^2 c_2 + 2\theta c_1 - 2\theta c_2\}\}$$

```
Q
```

$$\{\{-2\theta\kappa c_1 + 2\theta\kappa c_2 + 2\theta c_1 - 2\theta c_2,\ 0,\ -2\theta\kappa^3 v_0 + 2 S_t\theta\kappa v_0 + 2\theta\kappa^2 v_0 - 2 S_t\theta v_0\},\ \{0,\ -\theta\kappa^2 c_1 + \theta\kappa^2 c_2 + \theta c_1 - \theta c_2,\ -S_t\theta\kappa^2 c_0 + S_t\theta\kappa^2 c_2 + \theta\kappa^3 c_0 - \theta\kappa^3 c_2 + S_t\theta c_0 - S_t\theta c_2 - \theta\kappa c_0 + \theta\kappa c_2\},\ \{S_t c_0 - S_t c_2 - \kappa c_0 + \kappa c_2,\ -\kappa^2 v_0 + S_t v_0,\ 0\},\ \{-S_t c_0 + S_t c_1 + \kappa c_0 - \kappa c_1,\ \kappa^2 v_0 - S_t v_0,\ S_t\kappa^2 v_0 - \kappa^3 v_0 - S_t^2 v_0 + S_t\kappa v_0\}\}$$

Since the first matrix is identity, M is torsion-free. Let us now compute the obstructions for M to be projective:

```
P = ObstructionToProjectiveness[R, A]
```

$$\left\{\left\{\left\{\theta\,\kappa^2-\theta\right\},\ \left\{-S_t\,\theta\,v_0+\theta\,v_0\right\},\ \left\{-S_t^2\,v_0+S_t\,\kappa^2\,v_0+S_t\,\kappa\,v_0-\kappa^3\,v_0\right\}\right\},\ \left\{\left\{-c_1+c_2\right\},\ \left\{S_t\,c_0-S_t\,c_2-\kappa\,c_0+\kappa\,c_2\right\},\ \left\{-S_t\,v_0+\kappa^2\,v_0\right\}\right\}\right\}$$

Let us deduce the obstructions in the parameters θ, κ, c_0, c_1, c_2, v_0 for M to be a projective left A-module.

```
int = OreIntersection[Flatten@P, {θ, κ, c₀, c₁, c₂, v₀}, A]
```

$$\left\{-c_1+c_2,\ \theta\,\kappa^2-\theta,\ \kappa^2\,c_0\,v_0-\kappa^2\,c_2\,v_0-\kappa\,c_0\,v_0+\kappa\,c_2\,v_0,\ \theta\,\kappa\,c_0\,v_0-\theta\,\kappa\,c_2\,v_0-\theta\,c_0\,v_0+\theta\,c_2\,v_0\right\}$$

```
obst = Factor /@ iOrePolynomialToNormal[int]
```

$$\left\{-c_1+c_2,\ \theta\,(-1+\kappa)\,(1+\kappa),\ (-1+\kappa)\,\kappa\,(c_0-c_2)\,v_0,\ \theta\,(-1+\kappa)\,(c_0-c_2)\,v_0\right\}$$

We obtain that the system is controllable i.e. flat if one of the entries obst is not equal to zero.
In what follows, we assume that the system is flat (one of the entries of obst is not zero). Let us compute flat output of the system.

```
B = OreAlgebra[S[t]]
```

$\mathbb{K}(t)[S_t;\ S_t,\ 0]$

```
R = ToOrePolynomial[
   {{S[t] - κ, 0, -2 θ (1 - κ), -2 θ (1 - κ)},
    {0, v₀ (S[t] - κ²), -(θ (c₁ - c₀) (1 - κ²)), -(θ (c₂ - c₀) (1 - κ²))}}, B]
```

$$\left\{\left\{S_t-\kappa,\ 0,\ -2\,\theta+2\,\theta\,\kappa,\ -2\,\theta+2\,\theta\,\kappa\right\},\ \left\{0,\ v_0\,S_t-\kappa^2\,v_0,\ \theta\,c_0-\theta\,\kappa^2\,c_0-\theta\,c_1+\theta\,\kappa^2\,c_1,\ \theta\,c_0-\theta\,\kappa^2\,c_0-\theta\,c_2+\theta\,\kappa^2\,c_2\right\}\right\}$$

```
MatrixForm[Radj = Involution[R, B]]
```

$$\begin{pmatrix} S_t-\kappa & 0 \\ 0 & v_0\,S_t-\kappa^2\,v_0 \\ -2\,\theta+2\,\theta\,\kappa & \theta\,c_0-\theta\,\kappa^2\,c_0-\theta\,c_1+\theta\,\kappa^2\,c_1 \\ -2\,\theta+2\,\theta\,\kappa & \theta\,c_0-\theta\,\kappa^2\,c_0-\theta\,c_2+\theta\,\kappa^2\,c_2 \end{pmatrix}$$

```
{Ann, Rp, Q} = Exti[Radj, B, 1];
```

```
MatrixForm[Ann]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
Rp
```

$$\left\{\left\{0,\ v_0\,S_t-\kappa^2\,v_0,\ \theta\,c_0-\theta\,\kappa^2\,c_0-\theta\,c_1+\theta\,\kappa^2\,c_1,\ \theta\,c_0-\theta\,\kappa^2\,c_0-\theta\,c_2+\theta\,\kappa^2\,c_2\right\},\ \left\{-S_t+\kappa,\ 0,\ 2\,\theta-2\,\theta\,\kappa,\ 2\,\theta-2\,\theta\,\kappa\right\}\right\}$$

Q

$$\{\{-2\,\theta\,c_1\,v_0+2\,\theta\,\kappa\,c_1\,v_0+2\,\theta\,c_2\,v_0-2\,\theta\,\kappa\,c_2\,v_0,\ 2\,\theta\,c_1\,v_0-2\,\theta\,\kappa\,c_1\,v_0-2\,\theta\,c_2\,v_0+2\,\theta\,\kappa\,c_2\,v_0\},$$
$$\{\theta\,c_0\,c_1-\theta\,\kappa^2\,c_0\,c_1-\theta\,c_0\,c_2+\theta\,\kappa^2\,c_0\,c_2-\theta\,c_1\,c_2+\theta\,\kappa^2\,c_1\,c_2+\theta\,c_2^2-\theta\,\kappa^2\,c_2^2,$$
$$-\theta\,c_0\,c_1+\theta\,\kappa^2\,c_0\,c_1+\theta\,c_1^2-\theta\,\kappa^2\,c_1^2+\theta\,c_0\,c_2-\theta\,\kappa^2\,c_0\,c_2-\theta\,c_1\,c_2+\theta\,\kappa^2\,c_1\,c_2\},$$
$$\{\kappa\,c_0\,v_0-\kappa^2\,c_0\,v_0-\kappa\,c_2\,v_0+\kappa^2\,c_2\,v_0,$$
$$(c_1\,v_0-c_2\,v_0)\,S_t+\left(-\kappa\,c_0\,v_0+\kappa^2\,c_0\,v_0-\kappa^2\,c_1\,v_0+\kappa\,c_2\,v_0\right)\},$$
$$\{(-c_1\,v_0+c_2\,v_0)\,S_t+\left(-\kappa\,c_0\,v_0+\kappa^2\,c_0\,v_0+\kappa\,c_1\,v_0-\kappa^2\,c_2\,v_0\right),$$
$$\kappa\,c_0\,v_0-\kappa^2\,c_0\,v_0-\kappa\,c_1\,v_0+\kappa^2\,c_1\,v_0\}\}$$

Finally the flat output $\xi = (\xi_1, \xi_2)$ is defined by:

T = LeftInverse[Q, B]

$$\left\{\left\{\frac{-c_0+c_1}{2\,\theta\,(-1+\kappa)\,(c_1-c_2)^2\,v_0},\ -\frac{1}{\theta\,\left(-1+\kappa^2\right)\,(c_1-c_2)^2},\ 0,\ 0\right\},\right.$$
$$\left.\left\{\frac{-c_0+c_2}{2\,\theta\,(-1+\kappa)\,(c_1-c_2)^2\,v_0},\ -\frac{1}{\theta\,\left(-1+\kappa^2\right)\,(c_1-c_2)^2},\ 0,\ 0\right\}\right\}$$

Thread[{ξ₁[t], ξ₂[t]} → ApplyMatrix[T, {x₁[t], x₂[t], u₁[t], u₂[t]}]]

$$\{\xi_1[t]\to(-(1+\kappa)\,c_0\,x_1[t]+(1+\kappa)\,c_1\,x_1[t]-2\,v_0\,x_2[t])\,/$$
$$\left(2\,\theta\,(-1+\kappa)\,(1+\kappa)\,(c_1-c_2)^2\,v_0\right),\ \xi_2[t]\to$$
$$(-(1+\kappa)\,c_0\,x_1[t]+(1+\kappa)\,c_2\,x_1[t]-2\,v_0\,x_2[t])\,/\,\left(2\,\theta\,(-1+\kappa)\,(1+\kappa)\,(c_1-c_2)^2\,v_0\right)\}$$