Nonholonomic car

Let us consider the following nonlinear ordinary differential system:

```
eqs = {x'[t] → v[t] Cos[θ[t]], y'[t] → v[t] Sin[θ[t]], θ'[t] → v[t]/l Tan[φ[t]]};
vars = {x[t], y[t], θ[t], v[t], φ[t]};
TableForm[eqs]
```

$x'[t] \to \mathrm{Cos}[\theta[t]]\, v[t]$
$y'[t] \to \mathrm{Sin}[\theta[t]]\, v[t]$
$\theta'[t] \to \frac{\mathrm{Tan}[\varphi[t]]\, v[t]}{l}$

Let us introduce the Ore algebra A defined by

```
replA = ModelToReplacementRules[eqs, t];
A = OreAlgebraWithRelations[Der[t], replA]
```

$\mathbb{K}(t)[D_t;\, 1,\, D_t]$

and the matrix R of ordinary differential operators, which defines the generic linearization of the above nonlinear system

```
MatrixForm[R = ToOrePolynomialD[eqs, vars, A]]
```

$$\begin{pmatrix} D_t & 0 & \mathrm{Sin}[\theta[t]]\, v[t] & -\mathrm{Cos}[\theta[t]] & 0 \\ 0 & D_t & -\mathrm{Cos}[\theta[t]]\, v[t] & -\mathrm{Sin}[\theta[t]] & 0 \\ 0 & 0 & D_t & -\frac{\mathrm{Tan}[\varphi[t]]}{l} & -\frac{\mathrm{Sec}[\varphi[t]]^2\, v[t]}{l} \end{pmatrix}$$

and the left A-module, finitely presented by R, defined by M = $A^{1\times 5} / A^{1\times 3} R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} -D_t & 0 & 0 \\ 0 & -D_t & 0 \\ \mathrm{Sin}[\theta[t]]\, v[t] & -\mathrm{Cos}[\theta[t]]\, v[t] & -D_t \\ -\mathrm{Cos}[\theta[t]] & -\mathrm{Sin}[\theta[t]] & -\frac{\mathrm{Tan}[\varphi[t]]}{l} \\ 0 & 0 & -\frac{\mathrm{Sec}[\varphi[t]]^2\, v[t]}{l} \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
{Ann, Rp, Q} = Exti[Radj, A, 1];
MatrixForm[Ann]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm[Rp]
```

$$\begin{pmatrix} 0 & 0 & -l\, v[t]\, D_t & \mathrm{Tan}[\varphi[t]]\, v[t] & \mathrm{Sec}[\varphi[t]]^2\, v[t]^2 \\ 0 & -D_t & \mathrm{Cos}[\theta[t]]\, v[t] & \mathrm{Sin}[\theta[t]] & 0 \\ D_t & 0 & \mathrm{Sin}[\theta[t]]\, v[t] & -\mathrm{Cos}[\theta[t]] & 0 \end{pmatrix}$$

```
Q
```

$$\{\{-l\,\text{Cos}[\theta[t]]\,v[t],\ -l\,\text{Sec}[\varphi[t]]^2\,\text{Sin}[\theta[t]]\,v[t]^4\},\ \{-l\,\text{Sin}[\theta[t]]\,v[t],\ l\,\text{Cos}[\theta[t]]\,\text{Sec}[\varphi[t]]^2\,v[t]^4\},\ \{-\text{Tan}[\varphi[t]]\,v[t],\ l\,\text{Sec}[\varphi[t]]^2\,v[t]^3\,D_t + (4\,l\,\text{Sec}[\varphi[t]]^2\,v[t]^2\,v'[t] + 2\,l\,\text{Sec}[\varphi[t]]^2\,\text{Tan}[\varphi[t]]\,v[t]^3\,\varphi'[t])\},\ \{-l\,v[t]\,D_t - l\,v'[t],\ -\text{Sec}[\varphi[t]]^2\,\text{Tan}[\varphi[t]]\,v[t]^5\},\ \{-l\,\varphi'[t],\ l^2\,v[t]^2\,D_t^2 + (7\,l^2\,v[t]\,v'[t] + 4\,l^2\,\text{Tan}[\varphi[t]]\,v[t]^2\,\varphi'[t])\,D_t + (\text{Tan}[\varphi[t]]^2\,v[t]^4 + 8\,l^2\,v'[t]^2 + 14\,l^2\,\text{Tan}[\varphi[t]]\,v[t]\,v'[t]\,\varphi'[t] + 4\,l^2\,\text{Sec}[\varphi[t]]^2\,v[t]^2\,\varphi'[t]^2 - 2\,l^2\,\text{Cos}[2\,\varphi[t]]\,\text{Sec}[\varphi[t]]^2\,v[t]^2\,\varphi'[t]^2 + 4\,l^2\,v[t]\,v''[t] - l^2\,\text{Sec}[\varphi[t]]^2\,\text{Sin}[2\,\varphi[t]]\,v[t]^2\,\varphi''[t] + 4\,l^2\,\text{Tan}[\varphi[t]]\,v[t]^2\,\varphi''[t])\}\}$$

Since the first matrix is identity, M is torsion-free, which proves that the generic linearization is controllable. Let us now check whether or not M is free, i.e. that generic linearization is flat.

```
T = LeftInverse[Q, A]
```

$$\left\{\left\{-\frac{\text{Cos}[\theta[t]]}{l\,v[t]},\ -\frac{\text{Sin}[\theta[t]]}{l\,v[t]},\ 0,\ 0,\ 0\right\},\ \left\{-\frac{\text{Cos}[\varphi[t]]^2\,\text{Sin}[\theta[t]]}{l\,v[t]^4},\ \frac{\text{Cos}[\theta[t]]\,\text{Cos}[\varphi[t]]^2}{l\,v[t]^4},\ 0,\ 0,\ 0\right\}\right\}$$

Since Q admits a left inverse T, we conclude that M is a free left A-module of rank 2. In particular, a basis of M, i.e. a flat output
$\xi = (\xi_1, \xi_2)$ of the generic linearization is defined by

```
ApplyMatrix[T, vars]
```

$$\left\{-\frac{1}{l\,v[t]}\,(\text{Cos}[\theta[t]]\,x[t] + \text{Sin}[\theta[t]]\,y[t]),\ \frac{1}{l\,v[t]^4}\,\text{Cos}[\varphi[t]]^2\,(-\text{Sin}[\theta[t]]\,x[t] + \text{Cos}[\theta[t]]\,y[t])\right\}$$

Moreover, a parametrization of the flat system is defined by

```
Thread[vars -> ApplyMatrix[Q, {ξ1[t], ξ2[t]}]]
```

$$\{x[t] \to -l\,v[t]\,(\text{Cos}[\theta[t]]\,\xi_1[t] + \text{Sec}[\varphi[t]]^2\,\text{Sin}[\theta[t]]\,v[t]^3\,\xi_2[t]),\ y[t] \to -l\,\text{Sin}[\theta[t]]\,v[t]\,\xi_1[t] + l\,\text{Cos}[\theta[t]]\,\text{Sec}[\varphi[t]]^2\,v[t]^4\,\xi_2[t],\ \theta[t] \to v[t]\,(-\text{Tan}[\varphi[t]]\,\xi_1[t] + l\,\text{Sec}[\varphi[t]]^2\,v[t]\,(2\,\xi_2[t]\,(2\,v'[t] + \text{Tan}[\varphi[t]]\,v[t]\,\varphi'[t]) + v[t]\,\xi_2'[t])),\ v[t] \to -\text{Sec}[\varphi[t]]^2\,\text{Tan}[\varphi[t]]\,v[t]^5\,\xi_2[t] - l\,\xi_1[t]\,v'[t] - l\,v[t]\,\xi_1'[t],\ \varphi[t] \to -l\,\xi_1[t]\,\varphi'[t] + l^2\,v[t]\,(7\,v'[t] + 4\,\text{Tan}[\varphi[t]]\,v[t]\,\varphi'[t])\,\xi_2'[t] + \xi_2[t]\,(\text{Tan}[\varphi[t]]^2\,v[t]^4 + 8\,l^2\,v'[t]^2 + 2\,l^2\,v[t]\,(7\,\text{Tan}[\varphi[t]]\,v'[t]\,\varphi'[t] + 2\,v''[t]) - l^2\,\text{Sec}[\varphi[t]]^2\,v[t]^2\,(2\,(-2 + \text{Cos}[2\,\varphi[t]])\,\varphi'[t]^2 - \text{Sin}[2\,\varphi[t]]\,\varphi''[t])) + l^2\,v[t]^2\,\xi_2''[t]\}$$

From this point we will use some procedures which are not freely available (see NLControl website http://www.nlcontrol.ioc.ee).
Let us check if the flat output of the generic linearization can be simply lifted to flat outputs of the nonlinear system. To do that we consider the one-forms corresponding to the flat outputs of the generic linearization. We try to integrate them.

```
BookForm[difs = ApplyMatrixD[T, {x[t], y[t], θ[t], u1[t], u2[t]}]]
```

$$\left\{-\frac{\text{Cos}[\theta[t]]}{l\,v[t]}\,dx[t]-\frac{\text{Sin}[\theta[t]]}{l\,v[t]}\,dy[t],\ \frac{\text{Cos}[\theta[t]]\,\text{Cos}[\varphi[t]]^2}{l\,v[t]^4}\,dy[t]-\frac{\text{Cos}[\varphi[t]]^2\,\text{Sin}[\theta[t]]}{l\,v[t]^4}\,dx[t]\right\}$$

```
BookForm[sp = SimplifyBasis[SpanK[difs, t]]]
```

SpanK[dx[t], dy[t]]

```
IntegrateOneForms[sp]
```

{x[t], y[t]}

Finally we obtain that x and y define a flat output of the nonlinear system.