Lie-Poisson structure E_2

Example from Bender, C. M., Dunne, G.V. and Mead, L. R., Underdetermined systems of partial differential equations. Journal of Mathematical Physics, vol 41 (2000), pp. 6388-6398.

Let us introduce the following Ore algebra A of partial differential operators (A is first Weyl algebra in three variables)

```
A = OreAlgebra[x1, x2, x3, Der[x1], Der[x2], Der[x3]]
```

$\mathbb{K}[x_1, x_2, x_3][D_{x_1}; 1, D_{x_1}][D_{x_2}; 1, D_{x_2}][D_{x_3}; 1, D_{x_3}]$

and the matrix R of partial differential operators, which defines the system

```
R0 = {{x1 Der[x3], x2 Der[x3], 0},
   {x2 Der[x1] - x1 Der[x2], -1, x2 Der[x3]},
   {-1, x1 Der[x2] - x2 Der[x1], x1 Der[x3]}};
MatrixForm[R = ToOrePolynomial[R0, A]]
```

$$\begin{pmatrix} x_1 D_{x_3} & x_2 D_{x_3} & 0 \\ -x_1 D_{x_2} + x_2 D_{x_1} & -1 & x_2 D_{x_3} \\ -1 & x_1 D_{x_2} - x_2 D_{x_1} & x_1 D_{x_3} \end{pmatrix}$$

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} -x_1 D_{x_3} & x_1 D_{x_2} - x_2 D_{x_1} & -1 \\ -x_2 D_{x_3} & -1 & -x_1 D_{x_2} + x_2 D_{x_1} \\ 0 & -x_2 D_{x_3} & -x_1 D_{x_3} \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
MatrixForm /@ ({Ann, Rp, Q} = Exti[Radj, A, 1])
```

$$\left\{ \begin{pmatrix} x_2 D_{x_3} & 0 & 0 \\ x_1 D_{x_3} & 0 & 0 \\ x_1 D_{x_2} - x_2 D_{x_1} & 0 & 0 \\ 0 & D_{x_3} & 0 \\ 0 & x_1 D_{x_2} - x_2 D_{x_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -D_{x_1} & -D_{x_2} & -D_{x_3} \\ x_1 & x_2 & 0 \\ -1 & x_1 D_{x_2} - x_2 D_{x_1} & x_1 D_{x_3} \end{pmatrix}, \begin{pmatrix} x_2 D_{x_3} \\ -x_1 D_{x_3} \\ x_1 D_{x_2} - x_2 D_{x_1} \end{pmatrix} \right\}$$

Since the first matrix is not identity, we deduce that M admits nontrivial torsion elements and thus the corresponding system admits autonomous elements $\tau[1]$, $\tau[2]$, $\tau[3]$, defined by:

```
aut = AutonomousElements[R,
  {u[x1, x2, x3], v[x1, x2, x3], w[x1, x2, x3]}, τ, A, Relations → True]
```

$$\{\{\tau[1][x_1, x_2, x_3] \to -\{0, 0, 0\}[x_1, x_2, x_3] - w^{(0,0,1)}[x_1, x_2, x_3] - v^{(0,1,0)}[x_1, x_2, x_3] - \{x, y, z\}^{(1,0,0)}[x_1, x_2, x_3],$$
$$\tau[2][x_1, x_2, x_3] \to x_2\, v[x_1, x_2, x_3] + x_1 \{x, y, z\}[x_1, x_2, x_3],$$
$$\tau[3][x_1, x_2, x_3] \to -\{x, y, z\}[x_1, x_2, x_3] + x_1 \left(w^{(0,0,1)}[x_1, x_2, x_3] + v^{(0,1,0)}[x_1, x_2, x_3]\right) - x_2\, v^{(1,0,0)}[x_1, x_2, x_3]\},$$
$$\{x_2\, \tau[1]^{(0,0,1)}[x_1, x_2, x_3] == 0,\ x_1\, \tau[1]^{(0,0,1)}[x_1, x_2, x_3] == 0,$$
$$x_1\, \tau[1]^{(0,1,0)}[x_1, x_2, x_3] - x_2\, \tau[1]^{(1,0,0)}[x_1, x_2, x_3] == 0,\ \tau[2]^{(0,0,1)}[x_1, x_2, x_3] == 0,$$
$$x_1\, \tau[2]^{(0,1,0)}[x_1, x_2, x_3] - x_2\, \tau[2]^{(1,0,0)}[x_1, x_2, x_3] == 0,\ \tau[3][x_1, x_2, x_3] == 0\},$$
$$\{\tau[2]^{(0,0,1)}[x_1, x_2, x_3],\ -x_2\, \tau[1][x_1, x_2, x_3] - \tau[2]^{(0,1,0)}[x_1, x_2, x_3],$$
$$\tau[3][x_1, x_2, x_3],\ x_1\, \tau[1][x_1, x_2, x_3] + \tau[3][x_1, x_2, x_3] + \tau[2]^{(1,0,0)}[x_1, x_2, x_3]\}\}$$

The first list gives the definition of the autonomous elements T[1], τ[2], τ[3], the second list gives the equations they satisfy and the last list gives the relations between τ[1], τ[2], τ[3]. Note that τ[3] is zero.

Let us now introduce the second Weyl algebra B in three variables x_1, x_2, x_3

```
B = OreAlgebra[Der[x1], Der[x2], Der[x3]]
```

$\mathbb{K}(x_1, x_2, x_3)[D_{x_1}; 1, D_{x_1}][D_{x_2}; 1, D_{x_2}][D_{x_3}; 1, D_{x_3}]$

and the matrix R of partial differential operators in B, which defines the system

```
MatrixForm[R = ToOrePolynomial[R0, B]]
```

$$\begin{pmatrix} x_1 D_{x_3} & x_2 D_{x_3} & 0 \\ x_2 D_{x_1} - x_1 D_{x_2} & -1 & x_2 D_{x_3} \\ -1 & -x_2 D_{x_1} + x_1 D_{x_2} & x_1 D_{x_3} \end{pmatrix}$$

and the left B-module, finitely presented by R, defined by N = $B^{1\times 3} / B^{1\times 3} R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, B]]
```

$$\begin{pmatrix} -x_1 D_{x_3} & -x_2 D_{x_1} + x_1 D_{x_2} & -1 \\ -x_2 D_{x_3} & -1 & x_2 D_{x_1} - x_1 D_{x_2} \\ 0 & -x_2 D_{x_3} & -x_1 D_{x_3} \end{pmatrix}$$

Let us check whether or not N is torsion-free:

```
MatrixForm /@ ({Ann, Rp, Q} = Exti[Radj, B, 1])
```

$$\left\{\begin{pmatrix} D_{x_3} & 0 \\ x_2 D_{x_1} - x_1 D_{x_2} & 0 \\ 0 & D_{x_3} \\ 0 & -x_2 D_{x_1} + x_1 D_{x_2} \end{pmatrix}, \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 x_2 D_{x_1} - x_1^2 D_{x_2} - x_2 & -x_1^2 D_{x_3} \end{pmatrix}, \begin{pmatrix} x_2 D_{x_3} \\ -x_1 D_{x_3} \\ -x_2 D_{x_1} + x_1 D_{x_2} \end{pmatrix}\right\}$$

Since the first matrix is not identity, we deduce that N admits nontrivial torsion elements and thus the corresponding system admits autonomous elements τ[1], τ[2], defined by:

```
aut = AutonomousElements[R,
  {u[x1, x2, x3], v[x1, x2, x3], w[x1, x2, x3]}, τ, B, Relations → True]
```

$$\{\{\tau[1][x_1, x_2, x_3] \to x_2\, v[x_1, x_2, x_3] + x_1 \{x, y, z\}[x_1, x_2, x_3],$$
$$\tau[2][x_1, x_2, x_3] \to -x_1^2 \left(w^{(0,0,1)}[x_1, x_2, x_3] + v^{(0,1,0)}[x_1, x_2, x_3]\right) + x_2 \left(-v[x_1, x_2, x_3] + x_1 v^{(1,0,0)}[x_1, x_2, x_3]\right)\},$$
$$\{\tau[1]^{(0,0,1)}[x_1, x_2, x_3] == 0,\ -x_1 \tau[1]^{(0,1,0)}[x_1, x_2, x_3] + x_2 \tau[1]^{(1,0,0)}[x_1, x_2, x_3] == 0,$$
$$\tau[2]^{(0,0,1)}[x_1, x_2, x_3] == 0,\ x_1 \tau[2]^{(0,1,0)}[x_1, x_2, x_3] - x_2 \tau[2]^{(1,0,0)}[x_1, x_2, x_3] == 0\},$$
$$\{\tau[1]^{(0,0,1)}[x_1, x_2, x_3],\ -\frac{1}{x_1^2}\left(x_1^2 \tau[1]^{(0,1,0)}[x_1, x_2, x_3] + x_2 \left(\tau[1][x_1, x_2, x_3] + \tau[2][x_1, x_2, x_3] - x_1 \tau[1]^{(1,0,0)}[x_1, x_2, x_3]\right)\right),$$
$$-\frac{\tau[1][x_1, x_2, x_3] + \tau[2][x_1, x_2, x_3]}{x_1}\}\}$$

The first list gives the definition of the autonomous elements τ[1], τ[2], the second list gives the equations they satisfy and the last list gives the relations between τ[1], τ[2]. Note that as we are now computing over the second Weyl algebra B, some denominators appear in the relations.