Discrete example

Let us consider the following nonlinear difference system:

```
eqs = {
    x₁[t+1] → (1 - a₁) x₁[t] + a₁ x₂[t] + b₁ x₁[t] u[t],
    x₂[t+1] → a₂ x₁[t] + (1 - a₂ x₂[t])};
vars = {x₁[t], x₂[t], u[t]};
TableForm[eqs]
```

$x_1[1+t] \to (1-a_1)\, x_1[t] + b_1\, u[t]\, x_1[t] + a_1\, x_2[t]$

$x_2[1+t] \to 1 + a_2\, x_1[t] - a_2\, x_2[t]$

Let us introduce the Ore algebra A defined by

```
replA = ModelToReplacementRules[eqs, t];
A = OreAlgebraWithRelations[S[t], replA]
```

$\mathbb{K}(t)[S_t;\, S_t,\, 0]$

and the matrix R of ordinary differential operators, which defines the generic linearization of the above nonlinear system

```
MatrixForm[R = ToOrePolynomialD[eqs, vars, A]]
```

$$\begin{pmatrix} S_t + (-1 + a_1 - b_1\, u[t]) & -a_1 & -b_1\, x_1[t] \\ -a_2 & S_t + a_2 & 0 \end{pmatrix}$$

and the left A-module, finitely presented by R, defined by M = $A^{1\times 3} / A^{1\times 2} R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} S_t + (-1 + a_1 - b_1\, u[-t]) & -a_2 \\ -a_1 & S_t + a_2 \\ -b_1\, x_1[-t] & 0 \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
{Ann, Rp, Q} = Exti[Radj, A, 1];
MatrixForm[Ann]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
MatrixForm[Rp]
```

$$\begin{pmatrix} -a_2 & S_t + a_2 & 0 \\ -S_t + (1 - a_1 + b_1\, u[t]) & a_1 & b_1\, x_1[t] \end{pmatrix}$$

```
Q
```

$$\{\{(b_1 x_1[-1+t] x_1[t]^2 - a_1 b_1 x_1[-1+t] x_1[t]^2 + b_1^2 u[t] x_1[-1+t] x_1[t]^2 + a_1 b_1 x_1[-1+t] x_1[t] x_2[t]) S_t + a_2 b_1 x_1[-2+t] x_1[-1+t] x_1[t]\}, \{a_2 b_1 x_1[-2+t] x_1[-1+t] x_1[t]\},$$

$$\{(a_1 x_1[t] - a_1^2 x_1[t] + a_1 b_1 u[t] x_1[t] + x_1[t]^2 - 3 a_1 x_1[t]^2 + 3 a_1^2 x_1[t]^2 - a_1^3 x_1[t]^2 + a_1 a_2 x_1[t]^2 - a_1^2 a_2 x_1[t]^2 + 2 b_1 u[t] x_1[t]^2 - 4 a_1 b_1 u[t] x_1[t]^2 + 2 a_1^2 b_1 u[t] x_1[t]^2 + a_1 a_2 b_1 u[t] x_1[t]^2 + b_1^2 u[t]^2 x_1[t]^2 - a_1 b_1^2 u[t]^2 x_1[t]^2 + b_1 u[1+t] x_1[t]^2 - 2 a_1 b_1 u[1+t] x_1[t]^2 + a_1^2 b_1 u[1+t] x_1[t]^2 + 2 b_1^2 u[t] u[1+t] x_1[t]^2 - 2 a_1 b_1^2 u[t] u[1+t] x_1[t]^2 + b_1^3 u[t]^2 u[1+t] x_1[t]^2 + a_1^2 x_2[t] + 2 a_1 x_1[t] x_2[t] - 4 a_1^2 x_1[t] x_2[t] + 2 a_1^3 x_1[t] x_2[t] - a_1 a_2 x_1[t] x_2[t] + 2 a_1^2 a_2 x_1[t] x_2[t] + 2 a_1 b_1 u[t] x_1[t] x_2[t] - 2 a_1^2 b_1 u[t] x_1[t] x_2[t] - a_1 a_2 b_1 u[t] x_1[t] x_2[t] + 2 a_1 b_1 u[1+t] x_1[t] x_2[t] - 2 a_1^2 b_1 u[1+t] x_1[t] x_2[t] + 2 a_1 b_1^2 u[t] u[1+t] x_1[t] x_2[t] + a_1^2 x_2[t]^2 - a_1^3 x_2[t]^2 - a_1^2 a_2 x_2[t]^2 + a_1^2 b_1 u[1+t] x_2[t]^2) S_t^2 +$$

$$(-x_1[-1+t] x_1[t] + 2 a_1 x_1[-1+t] x_1[t] - a_1^2 x_1[-1+t] x_1[t] + a_2 x_1[-1+t] x_1[t] - a_1 a_2 x_1[-1+t] x_1[t] - 2 b_1 u[t] x_1[-1+t] x_1[t] + 2 a_1 b_1 u[t] x_1[-1+t] x_1[t] + a_2 b_1 u[t] x_1[-1+t] x_1[t] - b_1^2 u[t]^2 x_1[-1+t] x_1[t] - a_1 x_1[-1+t] x_2[t] + a_1^2 x_1[-1+t] x_2[t] + a_1 a_2 x_1[-1+t] x_2[t] - a_1 b_1 u[t] x_1[-1+t] x_2[t]) S_t + (-a_2 x_1[-2+t] x_1[-1+t] - a_2 b_1 u[t] x_1[-2+t] x_1[-1+t])\}\}$$

Since the first matrix is identity, M is torsion-free, which proves that the generic linearization is controllable. Let us now check whether or not M is free, i.e. that generic linearization is flat.

```
T = LeftInverse[Q, A]
```

$$\left\{\left\{0, \frac{1}{a_2 b_1 x_1[-2+t] x_1[-1+t] x_1[t]}, 0\right\}\right\}$$

Since Q admits a left inverse T, we conclude that M is a free left A-module of rank 1. In particular, a basis of M, i.e. a flat output ξ of the generic linearization is defined by

```
ApplyMatrix[T, vars]
```

$$\left\{\frac{x_2[t]}{a_2 b_1 x_1[-2+t] x_1[-1+t] x_1[t]}\right\}$$

Moreover, a parametrization of the flat system is defined by

```
Thread[vars → ApplyMatrix[Q, {ξ[t]}]]
```

$$\{x_1[t] \to b_1 x_1[-1+t] x_1[t] (a_2 \xi[t] x_1[-2+t] + \xi[1+t] ((1 - a_1 + b_1 u[t]) x_1[t] + a_1 x_2[t])),$$

$$x_2[t] \to a_2 b_1 \xi[t] x_1[-2+t] x_1[-1+t] x_1[t],$$

$$u[t] \to -a_2 (1 + b_1 u[t]) \xi[t] x_1[-2+t] x_1[-1+t] - (-1 + a_1 + a_2 - b_1 u[t]) \xi[1+t] x_1[-1+t] ((-1 + a_1 - b_1 u[t]) x_1[t] - a_1 x_2[t]) - \xi[2+t] ((-1 + a_1 - b_1 u[t]) x_1[t] - a_1 x_2[t]) ((1 + b_1 u[t]) (1 + b_1 u[1+t]) x_1[t] + a_1^2 (x_1[t] - x_2[t]) + a_1 (1 + (-2 + a_2 - b_1 (u[t] + u[1+t]))) x_1[t] + (1 - a_2 + b_1 u[1+t]) x_2[t]))\}$$

From this point we will use some procedures which are not freely available (see NLControl website http://www.nlcontrol.ioc.ee).
Let us check if the flat output of the generic linearization can be simply lifted to flat outputs of the nonlinear system. To do that we consider the one-forms corresponding to the flat outputs of the generic linearization. We try to integrate them.

```
BookForm[difs = ApplyMatrixD[T, {x1[t], x2[t], u[t]}]]
```

$$\left\{\frac{1}{a_2\, b_1\, x_1[-2+t]\, x_1[-1+t]\, x_1[t]}\, \mathrm{d}x_2[t]\right\}$$

```
BookForm[sp = SimplifyBasis[SpanK[difs, t]]]
```

SpanK[$\mathrm{d}x_2$[t]]

```
IntegrateOneForms[sp]
```

{x_2[t]}

Finally we obtain that x_2 defines a flat output of the nonlinear system.