Car model

Let us consider the following nonlinear ordinary differential system:

```
eqs = {x'[t] → v[t] Cos[ψ[t]],
   y'[t] → v[t] Sin[ψ[t]],
   z'[t] → -v[t] θ[t],
   ψ'[t] → ω[t],
   θ'[t] → -ϕ[t] ω[t],
   ϕ'[t] → θ[t] ω[t]};
vars = {x[t], y[t], z[t], ψ[t], θ[t], ϕ[t], v[t], ω[t]};
TableForm[eqs]
```

$$\begin{array}{l} x'[t] \to \mathrm{Cos}[\psi[t]]\, v[t] \\ y'[t] \to \mathrm{Sin}[\psi[t]]\, v[t] \\ z'[t] \to -v[t]\, \theta[t] \\ \psi'[t] \to \omega[t] \\ \theta'[t] \to -\phi[t]\, \omega[t] \\ \phi'[t] \to \theta[t]\, \omega[t] \end{array}$$

Let us introduce the Ore algebra A defined by

```
replA = ModelToReplacementRules[eqs, t];
A = OreAlgebraWithRelations[Der[t], replA]
```

$\mathbb{K}(t)[D_t; 1, D_t]$

and the matrix R of ordinary differential operators, which defines the generic linearization of the above nonlinear system

```
MatrixForm[R = ToOrePolynomialD[eqs, vars, A]]
```

$$\begin{pmatrix} D_t & 0 & 0 & \mathrm{Sin}[\psi[t]]\, v[t] & 0 & 0 & -\mathrm{Cos}[\psi[t]] & 0 \\ 0 & D_t & 0 & -\mathrm{Cos}[\psi[t]]\, v[t] & 0 & 0 & -\mathrm{Sin}[\psi[t]] & 0 \\ 0 & 0 & D_t & 0 & v[t] & 0 & \theta[t] & 0 \\ 0 & 0 & 0 & D_t & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & D_t & \omega[t] & 0 & \phi[t] \\ 0 & 0 & 0 & 0 & -\omega[t] & D_t & 0 & -\theta[t] \end{pmatrix}$$

and the left A-module, finitely presented by R, defined by $M = A^{1\times 8}/A^{1\times 6} R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} -D_t & 0 & 0 & 0 & 0 & 0 \\ 0 & -D_t & 0 & 0 & 0 & 0 \\ 0 & 0 & -D_t & 0 & 0 & 0 \\ \mathrm{Sin}[\psi[t]]\, v[t] & -\mathrm{Cos}[\psi[t]]\, v[t] & 0 & -D_t & 0 & 0 \\ 0 & 0 & v[t] & 0 & -D_t & -\omega[t] \\ 0 & 0 & 0 & 0 & \omega[t] & -D_t \\ -\mathrm{Cos}[\psi[t]] & -\mathrm{Sin}[\psi[t]] & \theta[t] & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \phi[t] & -\theta[t] \end{pmatrix}$$

Let us check whether or not M is torsion-free:

```
{Ann, Rp, Q} = Exti[Radj, A, 1];
Ann
```

$$\{\{D_t, 0, 0, 0, 0, 0\}, \{0, \omega[t]\, D_t^2 - \omega'[t]\, D_t + \omega[t]^3, 0, 0, 0, 0\}, \{0, 0, 0, -\omega[t]\, D_t + \omega'[t], 0, 0\}, \{0, 0, 0, 0, v[t]\, D_t - v'[t], 0\}, \{0, 0, 0, 0, 0, -\mathrm{Cos}[\psi[t]]^2\, v[t]^2\, \omega[t]\, D_t^2 + (-2\, \mathrm{Cos}[\psi[t]]\, \mathrm{Sin}[\psi[t]]\, v[t]^2\, \omega[t]^2 + 2\, \mathrm{Cos}[\psi[t]]^2\, v[t]\, \omega[t]\, v'[t] + \mathrm{Cos}[\psi[t]]^2\, v[t]^2\, \omega'[t])\, D_t + (-2\, \mathrm{Cos}[\psi[t]]^2\, v[t]^2\, \omega[t]^3 - 2\, \mathrm{Sin}[\psi[t]]^2\, v[t]^2\, \omega[t]^3 + 2\, \mathrm{Cos}[\psi[t]]\, \mathrm{Sin}[\psi[t]]\, v[t]\, \omega[t]^2\, v'[t] - 2\, \mathrm{Cos}[\psi[t]]^2\, \omega[t]\, v'[t]^2 - \mathrm{Cos}[\psi[t]]^2\, v[t]\, v'[t]\, \omega'[t] + \mathrm{Cos}[\psi[t]]^2\, v[t]\, \omega[t]\, v''[t])\}\}$$

```
Rp
```

$$\{\{0, 0, 0, 0, -\theta[t], -\phi[t], 0, 0\}, \{0, 0, 0, -\theta[t], 0, 1, 0, 0\}, \{-\mathrm{Cos}[\psi[t]]\, \theta[t] - \mathrm{Sin}[\psi[t]]\, \phi[t], -\mathrm{Sin}[\psi[t]]\, \theta[t] + \mathrm{Cos}[\psi[t]]\, \phi[t], -1, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, -\theta[t]\, D_t - \phi[t]\, \omega[t], 0, \theta[t]^2\}, \{0, 0, -\theta[t]\, D_t, 0, 0, v[t]\, \phi[t], -\theta[t]^2, 0\}, \{0, \theta[t]\, D_t, 0, 0, 0, -\mathrm{Cos}[\psi[t]]\, v[t], -\mathrm{Sin}[\psi[t]]\, \theta[t], 0\}\}$$

```
MatrixForm[Q]
```

$$\begin{pmatrix} \mathrm{Cos}[\psi[t]]\, v[t] & -\mathrm{Sin}[\psi[t]]\, v[t]^2 \\ \mathrm{Sin}[\psi[t]]\, v[t] & \mathrm{Cos}[\psi[t]]\, v[t]^2 \\ -v[t]\, \theta[t] & v[t]^2\, \phi[t] \\ \omega[t] & v[t]\, D_t + 2\, v'[t] \\ -\phi[t]\, \omega[t] & -v[t]\, \phi[t]\, D_t - 2\, \phi[t]\, v'[t] \\ \theta[t]\, \omega[t] & v[t]\, \theta[t]\, D_t + 2\, \theta[t]\, v'[t] \\ v[t]\, D_t + v'[t] & -v[t]^2\, \omega[t] \\ \omega[t]\, D_t + \omega'[t] & v[t]\, D_t^2 + 3\, v'[t]\, D_t + 2\, v''[t] \end{pmatrix}$$

Since the first matrix is not identity, we deduce that M admits nontrivial torsion elements and thus the corresponding system admits autonomous elements τ[1], ..., τ[6], defined by:

```
AutonomousElements[R,
 {dx[t], dy[t], dz[t], dψ[t], dθ[t], dϕ[t], dv[t], dω[t]}, τ, A, Relations → True]
```

```
{{τ[1][t] → -dθ[t] θ[t] - dϕ[t] ϕ[t], τ[2][t] → dϕ[t] - dψ[t] θ[t],
  τ[3][t] → -dz[t] + dy[t] (-Sin[ψ[t]] θ[t] + Cos[ψ[t]] ϕ[t]) -
    dx[t] (Cos[ψ[t]] θ[t] + Sin[ψ[t]] ϕ[t]),
  τ[4][t] → dω[t] θ[t]^2 - dϕ[t] ϕ[t] ω[t] - θ[t] dϕ'[t],
  τ[5][t] → -dv[t] θ[t]^2 + dϕ[t] v[t] ϕ[t] - θ[t] dz'[t],
  τ[6][t] → -Cos[ψ[t]] dϕ[t] v[t] + θ[t] (-dv[t] Sin[ψ[t]] + dy'[t])},
 {τ[1]'[t] == 0, ω[t]^3 τ[2][t] - ω'[t] τ[2]'[t] + ω[t] τ[2]''[t] == 0,
  τ[4][t] ω'[t] - ω[t] τ[4]'[t] == 0, -τ[5][t] v'[t] + v[t] τ[5]'[t] == 0,
  Cos[ψ[t]] v[t] (2 Cos[ψ[t]] ω[t] v'[t] + v[t] (-2 Sin[ψ[t]] ω[t]^2 + Cos[ψ[t]] ω'[t]))
     τ[6]'[t] - τ[6][t] (2 v[t]^2 ω[t]^3 + 2 Cos[ψ[t]]^2 ω[t] v'[t]^2 - Cos[ψ[t]] v[t]
        (2 Sin[ψ[t]] ω[t]^2 v'[t] - Cos[ψ[t]] v'[t] ω'[t] + Cos[ψ[t]] ω[t] v''[t])) -
    Cos[ψ[t]]^2 v[t]^2 ω[t] τ[6]''[t] == 0},
 {(-Sin[ψ[t]] v[t] (Cos[ψ[t]] θ[t] + Sin[ψ[t]] ϕ[t]) τ[2][t] + τ[5][t] -
     Sin[ψ[t]] θ[t] τ[6][t] + Cos[ψ[t]] ϕ[t] τ[6][t] - θ[t] τ[3]'[t]) /
   (θ[t] (Cos[ψ[t]] θ[t] + Sin[ψ[t]] ϕ[t])),
  1/θ[t] (Cos[ψ[t]] v[t] τ[2][t] + τ[6][t]), -(v[t] τ[1][t] + τ[5][t])/θ[t],
  -1/θ[t]^2 (ϕ[t] ω[t] τ[2][t] + τ[4][t] + θ[t] τ[2]'[t]),
  1/θ[t]^2 (ϕ[t] (-ω[t] τ[1][t] + τ[4][t]) - θ[t] τ[1]'[t]),
  (ω[t] τ[1][t] - τ[4][t])/θ[t]}}
```

The first list gives the definition of the autonomous elements τ[1], ..., τ[6], the second list gives the equations they satisfy and the last list gives the relations between τ[1], ..., τ[6].

Using an implementation of Frobenius theorem in NLControl (see NLControl website http://www.nlcontrol.ioc.ee) we can prove that the one-forms defining the autonomous elements are closed.

```
aut = AutonomousElementsD[R, vars, τ, A, Relations → True];
```

```
BookForm[sp = SpanK[Last /@ aut[[1]], t]]
```

```
SpanK[-θ[t] dθ[t] - ϕ[t] dϕ[t], dϕ[t] - θ[t] dψ[t],
 (-Cos[ψ[t]] θ[t] - Sin[ψ[t]] ϕ[t]) dx[t] + (-Sin[ψ[t]] θ[t] + Cos[ψ[t]] ϕ[t])
  dy[t] - dz[t], -ϕ[t] ω[t] dϕ[t] + θ[t]^2 dω[t] - θ[t] dϕ'[t],
 -θ[t]^2 dv[t] + v[t] ϕ[t] dϕ[t] - θ[t] dz'[t],
 -Sin[ψ[t]] θ[t] dv[t] - Cos[ψ[t]] v[t] dϕ[t] + θ[t] dy'[t]]
```

```
Integrability[sp]
```

```
True
```

Let us compute a flat output of the controllable part of the generic linearization, defined by:

```
Thread[ApplyMatrix[Rp, {x[t], y[t], z[t], ψ[t], θ[t], ϕ[t], v[t], ω[t]}] == 0]
```

```
{-θ[t]^2 - ϕ[t]^2 == 0, ϕ[t] - θ[t] ψ[t] == 0,
 -z[t] + y[t] (-Sin[ψ[t]] θ[t] + Cos[ψ[t]] ϕ[t]) -
   x[t] (Cos[ψ[t]] θ[t] + Sin[ψ[t]] ϕ[t]) == 0,
 θ[t]^2 ω[t] - ϕ[t]^2 ω[t] - θ[t] ϕ'[t] == 0, v[t] (-θ[t]^2 + ϕ[t]^2) - θ[t] z'[t] == 0,
 -v[t] (Sin[ψ[t]] θ[t] + Cos[ψ[t]] ϕ[t]) + θ[t] y'[t] == 0}
```

It is given by

```
ApplyMatrix[LeftInverse[Q, A], {x[t], y[t], z[t], ψ[t], θ[t], ϕ[t], v[t], ω[t]}]
```

```
{(-Cos[ψ[t]] z[t] + y[t] ϕ[t]) / (Cos[ψ[t]] v[t] θ[t] + Sin[ψ[t]] v[t] ϕ[t]),
 (Sin[ψ[t]] z[t] + y[t] θ[t]) / (v[t]^2 (Cos[ψ[t]] θ[t] + Sin[ψ[t]] ϕ[t]))}
```