Antenna

Let k = K[t, K1, K2, Te, Kp, Kc] be the commutative polynomial ring in t, K1, K2, Te, Kp, Kc and let us define the Ore algebra A = k⟨∂,δ⟩, where ∂ is differential operator defined by $\partial f(t) = f'(t)$ and δ is time-delay operator defined by δf(t) = f(t-1).

```
A = OreAlgebra[Der[t], S[-1][t]];
```

We consider the matrix $R \in A^{6\times 9}$ given by

```
R0 = {{Der[t], -K1, 0, 0, 0, 0, 0, 0, 0},
   {0, K2/Te + Der[t], 0, 0, 0, 0, -(Kp S[-1][t])/Te, -(Kc S[-1][t])/Te, -(Kc S[-1][t])/Te},
   {0, 0, Der[t], -K1, 0, 0, 0, 0, 0},
   {0, 0, 0, K2/Te + Der[t], 0, 0, -(Kc S[-1][t])/Te, -(Kp S[-1][t])/Te, -(Kc S[-1][t])/Te},
   {0, 0, 0, 0, Der[t], -K1, 0, 0, 0},
   {0, 0, 0, 0, 0, K2/Te + Der[t], -(Kc S[-1][t])/Te, -(Kc S[-1][t])/Te, -(Kp S[-1][t])/Te}};
```

```
MatrixForm[R = ToOrePolynomial[R0, A]]
```

$$\begin{pmatrix} D_t & -K1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_t + \frac{K2}{Te} & 0 & 0 & 0 & 0 & -\frac{Kp}{Te} S[-1][t] & -\frac{Kc}{Te} S[-1][t] & -\frac{Kc}{Te} S[-1][t] \\ 0 & 0 & D_t & -K1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_t + \frac{K2}{Te} & 0 & 0 & -\frac{Kc}{Te} S[-1][t] & -\frac{Kp}{Te} S[-1][t] & -\frac{Kc}{Te} S[-1][t] \\ 0 & 0 & 0 & 0 & D_t & -K1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_t + \frac{K2}{Te} & -\frac{Kc}{Te} S[-1][t] & -\frac{Kc}{Te} S[-1][t] & -\frac{Kp}{Te} S[-1][t] \end{pmatrix}$$

and the left A-module, finitely presented by R, defined by $M = A^{1\times 9} / A^{1\times 6} R$.

Let us compute the adjoint of R:

```
MatrixForm[Radj = Involution[R, A]]
```

$$\begin{pmatrix} D_t & 0 & 0 & 0 & 0 & 0 \\ -K1 & D_t + \frac{K2}{Te} & 0 & 0 & 0 & 0 \\ 0 & 0 & D_t & 0 & 0 & 0 \\ 0 & 0 & -K1 & D_t + \frac{K2}{Te} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_t & 0 \\ 0 & 0 & 0 & 0 & -K1 & D_t + \frac{K2}{Te} \\ 0 & -\frac{Kp}{Te}\left(S_t^{-1}\right) & 0 & -\frac{Kc}{Te}\left(S_t^{-1}\right) & 0 & -\frac{Kc}{Te}\left(S_t^{-1}\right) \\ 0 & -\frac{Kc}{Te}\left(S_t^{-1}\right) & 0 & -\frac{Kp}{Te}\left(S_t^{-1}\right) & 0 & -\frac{Kc}{Te}\left(S_t^{-1}\right) \\ 0 & -\frac{Kc}{Te}\left(S_t^{-1}\right) & 0 & -\frac{Kc}{Te}\left(S_t^{-1}\right) & 0 & -\frac{Kp}{Te}\left(S_t^{-1}\right) \end{pmatrix}$$

Let us check whether or not the left A-module M is torsion-free, i.e. whether or not the system admits some autonomous elements.

```
{L0, L1, L2} = Ext1[Radj, A];
```

```
MatrixForm@L0
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
MatrixForm@L1
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -\mathrm{Te}\, D_t - \mathrm{K2} & \mathrm{Kc}\, (S_t^{-1}) & \mathrm{Kc}\, (S_t^{-1}) & \mathrm{Kp}\, (S_t^{-1}) \\ 0 & 0 & 0 & 0 & D_t & -\mathrm{K1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{Te}\, D_t - \mathrm{K2} & 0 & 0 & \mathrm{Kc}\, (S_t^{-1}) & \mathrm{Kp}\, (S_t^{-1}) & \mathrm{Kc}\, (S_t^{-1}) \\ 0 & 0 & -D_t & \mathrm{K1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{Te}\, D_t + \mathrm{K2} & 0 & 0 & 0 & 0 & -\mathrm{Kp}\, (S_t^{-1}) & -\mathrm{Kc}\, (S_t^{-1}) & -\mathrm{Kc}\, (S_t^{-1}) \\ D_t & -\mathrm{K1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
L2
```

$$\{\{0,\ 0,\ (-2\,\mathrm{K1}\,\mathrm{Kc}^2 + \mathrm{K1}\,\mathrm{Kc}\,\mathrm{Kp} + \mathrm{K1}\,\mathrm{Kp}^2)\ (S_t^{-1})\},\ \{0,\ 0,\ (-2\,\mathrm{Kc}^2 + \mathrm{Kc}\,\mathrm{Kp} + \mathrm{Kp}^2)\ D_t\ (S_t^{-1})\},$$
$$\{0,\ (2\,\mathrm{K1}\,\mathrm{Kc}^2 - \mathrm{K1}\,\mathrm{Kc}\,\mathrm{Kp} - \mathrm{K1}\,\mathrm{Kp}^2)\ (S_t^{-1}),\ 0\},\ \{0,\ (2\,\mathrm{Kc}^2 - \mathrm{Kc}\,\mathrm{Kp} - \mathrm{Kp}^2)\ D_t\ (S_t^{-1}),\ 0\},$$
$$\{(-2\,\mathrm{K1}\,\mathrm{Kc}^2 + \mathrm{K1}\,\mathrm{Kc}\,\mathrm{Kp} + \mathrm{K1}\,\mathrm{Kp}^2)\ (S_t^{-1}),\ 0,\ 0\},\ \{(-2\,\mathrm{Kc}^2 + \mathrm{Kc}\,\mathrm{Kp} + \mathrm{Kp}^2)\ D_t\ (S_t^{-1}),\ 0,\ 0\},$$
$$\{-\mathrm{Kc}\,\mathrm{Te}\,D_t^2 - \mathrm{K2}\,\mathrm{Kc}\,D_t,\ \mathrm{Kc}\,\mathrm{Te}\,D_t^2 + \mathrm{K2}\,\mathrm{Kc}\,D_t,\ (\mathrm{Kc}\,\mathrm{Te} + \mathrm{Kp}\,\mathrm{Te})\ D_t^2 + (\mathrm{K2}\,\mathrm{Kc} + \mathrm{K2}\,\mathrm{Kp})\ D_t\},$$
$$\{-\mathrm{Kc}\,\mathrm{Te}\,D_t^2 - \mathrm{K2}\,\mathrm{Kc}\,D_t,\ (-\mathrm{Kc}\,\mathrm{Te} - \mathrm{Kp}\,\mathrm{Te})\ D_t^2 + (-\mathrm{K2}\,\mathrm{Kc} - \mathrm{K2}\,\mathrm{Kp})\ D_t,\ -\mathrm{Kc}\,\mathrm{Te}\,D_t^2 - \mathrm{K2}\,\mathrm{Kc}\,D_t\},$$
$$\{(\mathrm{Kc}\,\mathrm{Te} + \mathrm{Kp}\,\mathrm{Te})\ D_t^2 + (\mathrm{K2}\,\mathrm{Kc} + \mathrm{K2}\,\mathrm{Kp})\ D_t,\ \mathrm{Kc}\,\mathrm{Te}\,D_t^2 + \mathrm{K2}\,\mathrm{Kc}\,D_t,\ -\mathrm{Kc}\,\mathrm{Te}\,D_t^2 - \mathrm{K2}\,\mathrm{Kc}\,D_t\}\}$$

Since the first matrix is identity, M is torsion-free. Let us now compute the obstructions for M to be projective:

```
P = ObstructionToProjectiveness[R, A]
```

$$\{\{\{(S_t^{-1})\},\ \{\mathrm{Te}\,D_t^2 + \mathrm{K2}\,D_t\}\}\}$$

Let us deduce the obstructions in δ for M to be a projective left A-module.

```
OreIntersection[Flatten@P, {S[-1][t]}, A]
```

$$\{(S_t^{-1})\}$$

Hence we obtain that B ⊗ M is projective over the ring B = A⟨σ⟩ / ⟨$\delta\ \sigma$ -1, $\sigma\delta$ - 1⟩, where σ is defined by σf(t)=f(t+1) (i.e. the two sided inverse of δ).

```
AnT = AnnihilatorTorsionModule[R, A]
```

```
Module is not torsion
```

Let us compute the paramterization of the system

```
param = Parametrization[R, ξ, A]
```

```
{K1 (-2 Kc^2 + Kc Kp + Kp^2) ξ[3][-1 + t],
  (-2 Kc^2 + Kc Kp + Kp^2) ξ[3]'[-1 + t], K1 (2 Kc^2 - Kc Kp - Kp^2) ξ[2][-1 + t],
  (2 Kc^2 - Kc Kp - Kp^2) ξ[2]'[-1 + t], K1 (-2 Kc^2 + Kc Kp + Kp^2) ξ[1][-1 + t],
  (-2 Kc^2 + Kc Kp + Kp^2) ξ[1]'[-1 + t], -K2 Kc ξ[1]'[t] + K2 Kc ξ[2]'[t] +
   K2 (Kc + Kp) ξ[3]'[t] - Kc Te ξ[1]''[t] + Kc Te ξ[2]''[t] + (Kc + Kp) Te ξ[3]''[t],
  -K2 Kc ξ[1]'[t] - K2 (Kc + Kp) ξ[2]'[t] - K2 Kc ξ[3]'[t] - Kc Te ξ[1]''[t] -
   (Kc + Kp) Te ξ[2]''[t] - Kc Te ξ[3]''[t], K2 (Kc + Kp) ξ[1]'[t] + K2 Kc ξ[2]'[t] -
   K2 Kc ξ[3]'[t] + (Kc + Kp) Te ξ[1]''[t] + Kc Te ξ[2]''[t] - Kc Te ξ[3]''[t]}
```

We note that this parametrization is a minimal one:

```
L = MinimalParametrizations[R, A]
```

```
{{{0, 0, (-2 K1 Kc^2 + K1 Kc Kp + K1 Kp^2) (S_t^-1)}, {0, 0, (-2 Kc^2 + Kc Kp + Kp^2) D_t (S_t^-1)},
   {0, (2 K1 Kc^2 - K1 Kc Kp - K1 Kp^2) (S_t^-1), 0}, {0, (2 Kc^2 - Kc Kp - Kp^2) D_t (S_t^-1), 0},
   {(-2 K1 Kc^2 + K1 Kc Kp + K1 Kp^2) (S_t^-1), 0, 0}, {(-2 Kc^2 + Kc Kp + Kp^2) D_t (S_t^-1), 0, 0},
   {-Kc Te D_t^2 - K2 Kc D_t, Kc Te D_t^2 + K2 Kc D_t, (Kc Te + Kp Te) D_t^2 + (K2 Kc + K2 Kp) D_t},
   {-Kc Te D_t^2 - K2 Kc D_t, (-Kc Te - Kp Te) D_t^2 + (-K2 Kc - K2 Kp) D_t, -Kc Te D_t^2 - K2 Kc D_t},
   {(Kc Te + Kp Te) D_t^2 + (K2 Kc + K2 Kp) D_t, Kc Te D_t^2 + K2 Kc D_t, -Kc Te D_t^2 - K2 Kc D_t}}}
```

Let us check whether or not L admits a left inverse:

```
LeftInverse[L[[1]], A]
```

```
{}
```

We obtain that L is not an injective parametrization over A. Let now define the following Ore algebra

```
B = OreAlgebra[Der[t]]
```

```
K(t)[D_t; 1, D_t]
```

and if we substitute S[-1][t] by δ, i.e.

```
L1 = L[[1]] /. S[-1][t] → δ
```

```
{{0, 0, (-2 K1 Kc^2 + K1 Kc Kp + K1 Kp^2) δ}, {0, 0, (-2 Kc^2 + Kc Kp + Kp^2) D_t δ},
  {0, (2 K1 Kc^2 - K1 Kc Kp - K1 Kp^2) δ, 0}, {0, (2 Kc^2 - Kc Kp - Kp^2) D_t δ, 0},
  {(-2 K1 Kc^2 + K1 Kc Kp + K1 Kp^2) δ, 0, 0}, {(-2 Kc^2 + Kc Kp + Kp^2) D_t δ, 0, 0},
  {-Kc Te D_t^2 - K2 Kc D_t, Kc Te D_t^2 + K2 Kc D_t, (Kc Te + Kp Te) D_t^2 + (K2 Kc + K2 Kp) D_t},
  {-Kc Te D_t^2 - K2 Kc D_t, (-Kc Te - Kp Te) D_t^2 + (-K2 Kc - K2 Kp) D_t, -Kc Te D_t^2 - K2 Kc D_t},
  {(Kc Te + Kp Te) D_t^2 + (K2 Kc + K2 Kp) D_t, Kc Te D_t^2 + K2 Kc D_t, -Kc Te D_t^2 - K2 Kc D_t}}
```

```
L2 = ChangeOreAlgebra[L1, B]
```

```
{{0, 0, -2 K1 Kc^2 δ + K1 Kc Kp δ + K1 Kp^2 δ}, {0, 0, (-2 Kc^2 δ + Kc Kp δ + Kp^2 δ) D_t},
  {0, 2 K1 Kc^2 δ - K1 Kc Kp δ - K1 Kp^2 δ, 0}, {0, (2 Kc^2 δ - Kc Kp δ - Kp^2 δ) D_t, 0},
  {-2 K1 Kc^2 δ + K1 Kc Kp δ + K1 Kp^2 δ, 0, 0}, {(-2 Kc^2 δ + Kc Kp δ + Kp^2 δ) D_t, 0, 0},
  {-Kc Te D_t^2 - K2 Kc D_t, Kc Te D_t^2 + K2 Kc D_t, (Kc Te + Kp Te) D_t^2 + (K2 Kc + K2 Kp) D_t},
  {-Kc Te D_t^2 - K2 Kc D_t, (-Kc Te - Kp Te) D_t^2 + (-K2 Kc - K2 Kp) D_t, -Kc Te D_t^2 - K2 Kc D_t},
  {(Kc Te + Kp Te) D_t^2 + (K2 Kc + K2 Kp) D_t, Kc Te D_t^2 + K2 Kc D_t, -Kc Te D_t^2 - K2 Kc D_t}}
```

then we obtain that the matrix L2 admits the following left inverse:

```
MatrixForm[T = Factor[iOrePolynomialToNormal[LeftInverse[L2, B]]]]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{1}{K1\,(Kc-Kp)\,(2\,Kc+Kp)\,\delta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{K1\,(Kc-Kp)\,(2\,Kc+Kp)\,\delta} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{K1\,(Kc-Kp)\,(2\,Kc+Kp)\,\delta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore if we use the advance operator $\sigma = \delta^{-1}$ then flat outputs of the system are defined by $\xi = T(x_1, \ldots, x_6, u_1, u_2, u_3)$.