```
> restart:

> with(LinearAlgebra):
> with(OreModules):
```

Examples 1 & 2

```
> A := DefineOreAlgebra(diff=[d,t],polynom=[t]):

> M := Matrix([[4,6],[6,9]]);
```

$$M := \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \quad (1)$$

```
> l := Rank(M);
```

$$l := 1 \quad (2)$$

```
> D1 := DiagonalMatrix([1,1]);
```

$$D1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3)$$

```
> D2 := Matrix([[1,0],[0,2]]);
```

$$D2 := \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad (4)$$

```
> u := Matrix([[1]$2]); v := Matrix([[2,3],[2,3]]);
```

$$u := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v := \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \quad (5)$$

```
> simplify(Matrix([[D1.u,D2.u]]).v-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (6)$$

```
> L := SyzygyModule(M,A);
```

$$L := \begin{bmatrix} 3 & -2 \end{bmatrix} \quad (7)$$

```
> N := Matrix([[L.D1],[L.D2]]);
```

$$N := \begin{bmatrix} 3 & -2 \\ 3 & -4 \end{bmatrix} \quad (8)$$

```
> Determinant(N);
```

$$-6 \quad (9)$$

```
> NullSpace(N);
```

$$\{\,\} \quad (10)$$

Examples 3 & 4

```
> M := Matrix([[3,5],[4,7]]);
```

$$M := \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} \quad (11)$$

```
> l := Rank(M);
```

$$l := 2 \quad (12)$$

```
> X := M:
> L := SyzygyModule(M,A);
```

$$L := INJ(2) \quad (13)$$

```
> Z := DiagonalMatrix([1,1]);
```

$$Z := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (14)$$

```
> W1 := Involution(Factorize(Involution(D1.Z,A),Involution(X,A),A),A)
;
```

$$W1 := \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} \quad (15)$$

```
> W2 := Involution(Factorize(Involution(D2.Z,A),Involution(X,A),A),A)
;
```

$$W2 := \begin{bmatrix} 7 & -10 \\ -4 & 6 \end{bmatrix} \quad (16)$$

```
> Phi := Matrix([[psi[1]],[psi[2]]]);
```

$$\Phi := \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad (17)$$

```
> B := Matrix([[W1.Phi,W2.Phi]]);
```

$$B := \begin{bmatrix} 7\,\psi_1 - 5\,\psi_2 & 7\,\psi_1 - 10\,\psi_2 \\ -4\,\psi_1 + 3\,\psi_2 & -4\,\psi_1 + 6\,\psi_2 \end{bmatrix} \quad (18)$$

```
> Determinant(B);
```

$$\psi_1\,\psi_2 \quad (19)$$

```
> E := MatrixInverse(B);
```

$$E := \begin{bmatrix} -\frac{2\,(2\,\psi_1 - 3\,\psi_2)}{\psi_1\,\psi_2} & -\frac{7\,\psi_1 - 10\,\psi_2}{\psi_1\,\psi_2} \\ \frac{4\,\psi_1 - 3\,\psi_2}{\psi_1\,\psi_2} & \frac{7\,\psi_1 - 5\,\psi_2}{\psi_1\,\psi_2} \end{bmatrix} \quad (20)$$

```
> u := Z.Phi;
```

(21)

$$u := \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad (21)$$

```
> v := E;
```

$$v := \begin{bmatrix} -\frac{2\left(2\,\psi_1 - 3\,\psi_2\right)}{\psi_1\,\psi_2} & -\frac{7\,\psi_1 - 10\,\psi_2}{\psi_1\,\psi_2} \\ \frac{4\,\psi_1 - 3\,\psi_2}{\psi_1\,\psi_2} & \frac{7\,\psi_1 - 5\,\psi_2}{\psi_1\,\psi_2} \end{bmatrix} \quad (22)$$

```
> simplify(Matrix([[D1.u,D2.u]]).v-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (23)$$

Example 5

```
> M := Matrix([[1,0,0],[0,0,0],[0,0,1]]);
```

$$M := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (24)$$

```
> D1 := Matrix([[1,0,0],[0,0,0],[0,0,0]]);
```

$$D1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (25)$$

```
> D2 := Matrix([[0,0,0],[0,1,0],[0,0,0]]);
```

$$D2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (26)$$

```
> l := Rank(M);
```

$$l := 2 \quad (27)$$

```
> X := SubMatrix(M,1..3,[1,3]);
```

$$X := \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad (28)$$

```
> Y := Involution(Factorize(Involution(M,A),Involution(X,A),A),A);
```

$$Y := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (29)$$

```
> L := SyzygyModule(M,A);
```

$$L := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad (30)$$

```
> N := Matrix([[L.D1],[L.D2]]);
```

$$N := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (31)$$

```
> K := NullSpace(N);
```

$$K := \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (32)$$

```
> Z := Matrix([[K[1],K[2]]]);
```

$$Z := \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad (33)$$

```
> W1 := Involution(Factorize(Involution(D1.Z,A),Involution(X,A),A),A)
  ;
```

$$W1 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (34)$$

```
> W2 := Involution(Factorize(Involution(D2.Z,A),Involution(X,A),A),A)
  ;
```

$$W2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (35)$$

```
> Phi := Matrix([[psi[1]],[psi[2]]]);
```

$$\Phi := \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad (36)$$

```
> B := Matrix([[W1.Phi,W2.Phi]]);
```

$$B := \begin{bmatrix} \psi_2 & 0 \\ 0 & 0 \end{bmatrix} \quad (37)$$

```
> A := DefineOreAlgebra(diff=[psi[1],s1],diff=[psi[2],s2],polynom=
  [s1,s2]):
> SyzygyModule(B,A);
```

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \quad (38)$$

Examples 6 & 7 & 11

```
> M := Matrix(3, 2, {(1, 1) = 0, (1, 2) = 0, (2, 1) = -147360, (2, 2)
  = -96804, (3, 1) = 0, (3, 2) = 0});
```

$$M := \begin{bmatrix} 0 & 0 \\ -147360 & -96804 \\ 0 & 0 \end{bmatrix} \tag{39}$$

```
> D1 := Matrix(3, 3, {(1, 1) = 0, (1, 2) = 0, (1, 3) = 0, (2, 1) = 0,
  (2, 2) = 54, (2, 3) = -31, (3, 1) = 0, (3, 2) = 0, (3, 3) = 0});
```

$$D1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 54 & -31 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

```
> D2 := Matrix(3, 3, {(1, 1) = 0, (1, 2) = 0, (1, 3) = 0, (2, 1) = 0,
  (2, 2) = -58, (2, 3) = -77, (3, 1) = 0, (3, 2) = 0, (3, 3) = 0});
```

$$D2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -58 & -77 \\ 0 & 0 & 0 \end{bmatrix} \tag{41}$$

```
> D3 := Matrix(3, 3, {(1, 1) = 0, (1, 2) = 0, (1, 3) = 0, (2, 1) =
  79, (2, 2) = 0, (2, 3) = 0, (3, 1) = 0, (3, 2) = 0, (3, 3) = 0});
```

$$D3 := \begin{bmatrix} 0 & 0 & 0 \\ 79 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{42}$$

```
> l := Rank(M);
```

$$l := 1 \tag{43}$$

```
> X := SubMatrix(M,1..3,1);
```

$$X := \begin{bmatrix} 0 \\ -147360 \\ 0 \end{bmatrix} \tag{44}$$

```
> Y := Involution(Factorize(Involution(M,A),Involution(X,A),A),A);
```

$$Y := \begin{bmatrix} 1 & \frac{8067}{12280} \end{bmatrix} \tag{45}$$

```
> L := SyzygyModule(M,A);
```

$$L := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{46}$$

```
> N := Matrix([[L.D1],[L.D2]]);
```

$$N := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{47}$$

```
> K := NullSpace(N);
```

$$K := \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (48)$$

```
> Z := DiagonalMatrix([1,1,1]);
```

$$Z := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (49)$$

```
> W1 := Involution(Factorize(Involution(D1.Z,A),Involution(X,A),A),A)
;
```

$$W1 := \begin{bmatrix} 0 & -\frac{9}{24560} & \frac{31}{147360} \end{bmatrix} \quad (50)$$

```
> W2 := Involution(Factorize(Involution(D2.Z,A),Involution(X,A),A),A)
;
```

$$W2 := \begin{bmatrix} 0 & \frac{29}{73680} & \frac{77}{147360} \end{bmatrix} \quad (51)$$

```
> W3 := Involution(Factorize(Involution(D3.Z,A),Involution(X,A),A),A)
;
```

$$W3 := \begin{bmatrix} -\frac{79}{147360} & 0 & 0 \end{bmatrix} \quad (52)$$

```
> c :=1/147360: W1/c; W2/c; W3/c;
```

$$\begin{bmatrix} 0 & -54 & 31 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 58 & 77 \end{bmatrix}$$

$$\begin{bmatrix} -79 & 0 & 0 \end{bmatrix} \quad (53)$$

```
> Phi := Matrix([[psi[1]],[psi[2]],[psi[3]]]);
```

$$\Phi := \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \quad (54)$$

```
> B_psi := Matrix([[W1.Phi,W2.Phi,W3.Phi]]);
```

$$B\_psi := \begin{bmatrix} -\frac{9}{24560}\psi_2 + \frac{31}{147360}\psi_3 & \frac{29}{73680}\psi_2 + \frac{77}{147360}\psi_3 & -\frac{79}{147360}\psi_1 \end{bmatrix} \quad (55)$$

```
> B_psi/c;
```

$$\begin{bmatrix} -54\psi_2 + 31\psi_3 & 58\psi_2 + 77\psi_3 & -79\psi_1 \end{bmatrix} \quad (56)$$

```
> A := DefineOreAlgebra(diff=[x1,y1],diff=[x2,y2],diff=[x3,y3],
polynom=[y1,y2,y3]):
```

```
> X := Matrix([[x1],[x2],[x3]]);
```

$$X := \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} \quad (57)$$

```
> B := Matrix([[W1.X,W2.X,W3.X]]);
```

$$B := \begin{bmatrix} -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{29}{73680}\,x2 + \frac{77}{147360}\,x3 & -\frac{79}{147360}\,x1 \end{bmatrix} \quad (58)$$

```
> SyzygyModule(B,A);
```

$$INJ(1) \quad (59)$$

```
> ann_N := PiPolynomial(Transpose(B),A);
```

$$ann\_N := [x3, x2, x1] \quad (60)$$

```
> E_x1 := Transpose(LocalLeftInverse(Transpose(B),[x1],A));
```

$$E\_x1 := \begin{bmatrix} 0 \\ 0 \\ -\frac{147360}{79\,x1} \end{bmatrix} \quad (61)$$

```
> simplify(B.E_x1);
```

$$\begin{bmatrix} 1 \end{bmatrix} \quad (62)$$

```
> c.E_x1;
```

$$\begin{bmatrix} 0 \\ 0 \\ -\frac{1}{79\,x1} \end{bmatrix} \quad (63)$$

```
> E_x2 := Transpose(LocalLeftInverse(Transpose(B),[x2],A));
```

$$E\_x2 := \begin{bmatrix} -\frac{2836680}{1489\,x2} \\ \frac{1142040}{1489\,x2} \\ 0 \end{bmatrix} \quad (64)$$

```
> simplify(B.E_x2);
```

$$\begin{bmatrix} 1 \end{bmatrix} \quad (65)$$

```
> 5956*c*E_x2;
```

$$\begin{bmatrix} -\frac{77}{x2} \\ \frac{31}{x2} \\ 0 \end{bmatrix} \quad (66)$$

```
> E_x3 := Transpose(LocalLeftInverse(Transpose(B),[x3],A));
```

$$E\_x3 := \begin{bmatrix} \frac{2136720}{1489\,x3} \\ \frac{1989360}{1489\,x3} \\ 0 \end{bmatrix} \tag{67}$$

```
> simplify(B.E_x3);
```

$$\begin{bmatrix} 1 \end{bmatrix} \tag{68}$$

```
> 2978*c*E_x3;
```

$$\begin{bmatrix} \frac{29}{x3} \\ \frac{27}{x3} \\ 0 \end{bmatrix} \tag{69}$$

```
> with(OreModules):
> C := Involution(SyzygyModule(Involution(B,A),A),A);
```

$$C := \begin{bmatrix} -58\,x2 - 77\,x3 & -79\,x1 & 0 \\ -54\,x2 + 31\,x3 & 0 & -79\,x1 \\ 0 & 54\,x2 - 31\,x3 & -58\,x2 - 77\,x3 \end{bmatrix} \tag{70}$$

```
> u := Z.Phi;
```

$$u := \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} \tag{71}$$

```
> Y_p := Matrix(3,2,symbol=y);
```

$$Y\_p := \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \\ y_{3,1} & y_{3,2} \end{bmatrix} \tag{72}$$

```
> v_x1 := subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_x1.Y+C.Y_p);
```

$$v\_x1 := \Big[\Big[\big[(-58\,\psi_2 - 77\,\psi_3)\,y_{1,1} - 79\,\psi_1\,y_{2,1},\ (-58\,\psi_2 - 77\,\psi_3)\,y_{1,2} - 79\,\psi_1\,y_{2,2}\big], \tag{73}$$

$$\big[(-54\,\psi_2 + 31\,\psi_3)\,y_{1,1} - 79\,\psi_1\,y_{3,1},\ (-54\,\psi_2 + 31\,\psi_3)\,y_{1,2} - 79\,\psi_1\,y_{3,2}\big],$$

$$\Big[-\frac{147360}{79\,\psi_1} + (54\,\psi_2 - 31\,\psi_3)\,y_{2,1} + (-58\,\psi_2 - 77\,\psi_3)\,y_{3,1},\ -\frac{96804}{79\,\psi_1} + (54\,\psi_2$$

$- 31\,\psi_3\big)\,y_{2,\,2} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{3,\,2}\big]\big]$

```
> simplify(Matrix([[D1.u,D2.u,D3.u]]).v_x1-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (74)$$

```
> v_x2 := subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_x2.Y+C.Y_p);
```

$$v\_x2 := \Big[\Big[-\frac{2836680}{1489\,\psi_2} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{1,\,1} - 79\,\psi_1\,y_{2,\,1},\ -\frac{1863477}{1489\,\psi_2} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{1,\,2} - 79\,\psi_1\,y_{2,\,2}\Big],$$
$$\Big[\frac{1142040}{1489\,\psi_2} + \big(-54\,\psi_2 + 31\,\psi_3\big)\,y_{1,\,1} - 79\,\psi_1\,y_{3,\,1},\ \frac{750231}{1489\,\psi_2} + \big(-54\,\psi_2 + 31\,\psi_3\big)\,y_{1,\,2} - 79\,\psi_1\,y_{3,\,2}\Big],$$
$$\Big[\big(54\,\psi_2 - 31\,\psi_3\big)\,y_{2,\,1} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{3,\,1},\ \big(54\,\psi_2 - 31\,\psi_3\big)\,y_{2,\,2} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{3,\,2}\Big]\Big] \qquad (75)$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u]]).v_x2-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (76)$$

```
> v_x3 := subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_x3.Y+C.Y_p);
```

$$v\_x3 := \Big[\Big[\frac{2136720}{1489\,\psi_3} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{1,\,1} - 79\,\psi_1\,y_{2,\,1},\ \frac{1403658}{1489\,\psi_3} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{1,\,2} - 79\,\psi_1\,y_{2,\,2}\Big],$$
$$\Big[\frac{1989360}{1489\,\psi_3} + \big(-54\,\psi_2 + 31\,\psi_3\big)\,y_{1,\,1} - 79\,\psi_1\,y_{3,\,1},\ \frac{1306854}{1489\,\psi_3} + \big(-54\,\psi_2 + 31\,\psi_3\big)\,y_{1,\,2} - 79\,\psi_1\,y_{3,\,2}\Big],$$
$$\Big[\big(54\,\psi_2 - 31\,\psi_3\big)\,y_{2,\,1} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{3,\,1},\ \big(54\,\psi_2 - 31\,\psi_3\big)\,y_{2,\,2} + \big(-58\,\psi_2 - 77\,\psi_3\big)\,y_{3,\,2}\Big]\Big] \qquad (77)$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u]]).v_x3-M);
```

(78)

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (78)$$

```
> with(QuillenSuslin):
```

```
> C_x1 := SubMatrix(C,1..3,2..3);
```

$$C\_x1 := \begin{bmatrix} -79\,x1 & 0 \\ 0 & -79\,x1 \\ 54\,x2 - 31\,x3 & -58\,x2 - 77\,x3 \end{bmatrix} \quad (79)$$

```
> simplify(B.C_x1);
```

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \quad (80)$$

```
> A_x1 := DefineOreAlgebra(diff=[y1,x1],diff=[x2,y2],diff=[x3,y3],
polynom=[x1,y2,y3]):
```

```
> V_x1 := Transpose(FactorizeRat(Transpose(C),Transpose(C_x1),A_x1));
```

$$V\_x1 := \begin{bmatrix} \frac{58}{79}\,\frac{x2}{x1} + \frac{77}{79}\,\frac{x3}{x1} & 1 & 0 \\ \frac{54}{79}\,\frac{x2}{x1} - \frac{31}{79}\,\frac{x3}{x1} & 0 & 1 \end{bmatrix} \quad (81)$$

```
> simplify(C-C_x1.V_x1);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (82)$$

```
> W_x1 := Transpose(FactorizeRat(Transpose(C_x1),Transpose(C),A_x1));
```

$$W\_x1 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (83)$$

```
> simplify(C_x1-C.W_x1);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (84)$$

```
> Y_pp := Matrix(2,2,symbol=y);
```

$$Y\_pp := \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \quad (85)$$

```
> v_x1_2 := subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_x1.Y+C_x1.Y_pp);
```

$$v\_x1\_2 := \left[\left[-79\,\psi_1\,y_{1,1},\ -79\,\psi_1\,y_{1,2}\right],\left[-79\,\psi_1\,y_{2,1},\ -79\,\psi_1\,y_{2,2}\right],\left[-\frac{147360}{79\,\psi_1} + (54\,\psi_2 - 31\,\psi_3)\,y_{1,1} + (-58\,\psi_2 - 77\,\psi_3)\,y_{2,1},\ -\frac{96804}{79\,\psi_1} + (54\,\psi_2 - 31\,\psi_3)\,y_{1,2} + (-58\,\psi_2 - 77\,\psi_3)\,y_{2,2}\right]\right] \tag{86}$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u]]).v_x1_2-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{87}$$

```
> Determinant(Matrix([[E_x1,C_x1]]));
```

$$-11641440\,x1 \tag{88}$$

```
> Y2 := Matrix([[Y],[Y_pp]]);
```

$$Y2 := \begin{bmatrix} 1 & \frac{8067}{12280} \\ y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \tag{89}$$

```
> U_x2 := ParkAlgorithm(B,[x1,x3]);
```

$$U\_x2 := \begin{bmatrix} -\frac{2836680}{1489\,x2} & -\frac{77}{147360}\,x3 - \frac{29}{73680}\,x2 & -\frac{6083}{5956}\,\frac{x1}{x2} \\ \frac{1142040}{1489\,x2} & -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{2449}{5956}\,\frac{x1}{x2} \\ 0 & 0 & 1 \end{bmatrix} \tag{90}$$

```
> Determinant(U_x2);
```

$$1 \tag{91}$$

```
> V_x2 := MatrixInverse(U_x2);
```

$$V\_x2 := \begin{bmatrix} -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{29}{73680}\,x2 + \frac{77}{147360}\,x3 & -\frac{79}{147360}\,x1 \\ -\frac{1142040}{1489\,x2} & -\frac{2836680}{1489\,x2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{92}$$

```
> C_x2 := SubMatrix(U_x2,1..3,2..3);
```

$$C\_x2 := \begin{bmatrix} -\frac{77}{147360}\,x3 - \frac{29}{73680}\,x2 & -\frac{6083}{5956}\,\frac{x1}{x2} \\ -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{2449}{5956}\,\frac{x1}{x2} \\ 0 & 1 \end{bmatrix} \tag{93}$$

```
> simplify(B.C_x2);
```

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \tag{94}$$

```
> A_x2 := DefineOreAlgebra(diff=[x1,y1],diff=[y2,x2],diff=[x3,y3],
polynom=[y1,x2,y3]):
```

```
> V_x2 := Transpose(FactorizeRat(Transpose(C),Transpose(C_x2),A_x2));
```

$$V\_x2 := \begin{bmatrix} 147360 & \frac{90221160}{1489}\,\frac{x1}{x2} & \frac{224097720}{1489}\,\frac{x1}{x2} \\ 0 & 54\,x2 - 31\,x3 & -58\,x2 - 77\,x3 \end{bmatrix} \tag{95}$$

```
> simplify(C-C_x2.V_x2);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{96}$$

```
> W_x2 := Transpose(FactorizeRat(Transpose(C_x2),Transpose(C),A_x2));
```

$$W\_x2 := \begin{bmatrix} \frac{1}{147360} & 0 \\ 0 & \frac{77}{5956\,x2} \\ 0 & -\frac{31}{5956\,x2} \end{bmatrix} \tag{97}$$

```
> simplify(C_x2-C.W_x2);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{98}$$

```
> Y_pp := Matrix(2,2,symbol=y);
```

$$Y\_pp := \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \tag{99}$$

```
> v_x2_2 := subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_x2.Y+C_x2.Y_pp);
```

$$v\_x2\_2 := \left[\left[-\frac{2836680}{1489\,\psi_2} + \left(-\frac{77}{147360}\,\psi_3 - \frac{29}{73680}\,\psi_2\right) y_{1,1} - \frac{6083}{5956}\,\frac{\psi_1\,y_{2,1}}{\psi_2}, \right.\right. \tag{100}$$

$$-\frac{1863477}{1489\,\psi_2} + \left(-\frac{77}{147360}\,\psi_3 - \frac{29}{73680}\,\psi_2\right) y_{1,2} - \frac{6083}{5956}\,\frac{\psi_1\, y_{2,2}}{\psi_2}\Bigg],$$

$$\Bigg[\frac{1142040}{1489\,\psi_2} + \left(-\frac{9}{24560}\,\psi_2 + \frac{31}{147360}\,\psi_3\right) y_{1,1} + \frac{2449}{5956}\,\frac{\psi_1\, y_{2,1}}{\psi_2}, \frac{750231}{1489\,\psi_2} + \Bigg($$

$$-\frac{9}{24560}\,\psi_2 + \frac{31}{147360}\,\psi_3\Bigg) y_{1,2} + \frac{2449}{5956}\,\frac{\psi_1\, y_{2,2}}{\psi_2}\Bigg],$$

$$\left[y_{2,1}, y_{2,2}\right]\Big]$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u]]).v_x2_2-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (101)$$

```
> Determinant(Matrix([[E_x2,C_x2]]));
```

$$1 \qquad (102)$$

```
> U_x3 := ParkAlgorithm(B,[x1,x2]);
```

$$U\_x3 := \begin{bmatrix} \frac{2136720}{1489\,x3} & -\frac{77}{147360}\,x3 - \frac{29}{73680}\,x2 & \frac{2291}{2978}\,\frac{x1}{x3} \\ \frac{1989360}{1489\,x3} & -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{2133}{2978}\,\frac{x1}{x3} \\ 0 & 0 & 1 \end{bmatrix} \qquad (103)$$

```
> Determinant(U_x3);
```

$$1 \qquad (104)$$

```
> V_x3 := MatrixInverse(U_x3);
```

$$V\_x3 := \begin{bmatrix} -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{29}{73680}\,x2 + \frac{77}{147360}\,x3 & -\frac{79}{147360}\,x1 \\ -\frac{1989360}{1489\,x3} & \frac{2136720}{1489\,x3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (105)$$

```
> C_x3 := SubMatrix(U_x3,1..3,2..3);
```

$$C\_x3 := \begin{bmatrix} -\frac{77}{147360}\,x3 - \frac{29}{73680}\,x2 & \frac{2291}{2978}\,\frac{x1}{x3} \\ -\frac{9}{24560}\,x2 + \frac{31}{147360}\,x3 & \frac{2133}{2978}\,\frac{x1}{x3} \\ 0 & 1 \end{bmatrix} \qquad (106)$$

```
> simplify(B.C_x3);
```

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \quad (107)$$

```
> A_x3 := DefineOreAlgebra(diff=[x1,y1],diff=[x2,y2],diff=[y3,x3],
polynom=[y1,y2,x3]):
```

```
> V_x3 := Transpose(FactorizeRat(Transpose(C),Transpose(C_x3),A_x3));
```

$$V\_x3 := \begin{bmatrix} 147360 & \frac{157159440}{1489}\frac{x1}{x3} & -\frac{168800880}{1489}\frac{x1}{x3} \\ 0 & 54\,x2 - 31\,x3 & -58\,x2 - 77\,x3 \end{bmatrix} \quad (108)$$

```
> simplify(C-C_x3.V_x3);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (109)$$

```
> W_x3 := Transpose(FactorizeRat(Transpose(C_x3),Transpose(C),A_x3));
```

$$W\_x3 := \begin{bmatrix} \frac{1}{147360} & 0 \\ 0 & -\frac{29}{2978\,x3} \\ 0 & -\frac{27}{2978\,x3} \end{bmatrix} \quad (110)$$

```
> simplify(C_x3-C.W_x3);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (111)$$

```
> Y_pp := Matrix(2,2,symbol=y);
```

$$Y\_pp := \begin{bmatrix} y_{1,1} & y_{1,2} \\ y_{2,1} & y_{2,2} \end{bmatrix} \quad (112)$$

```
> v_x3_2 := subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_x3.Y+C_x3.Y_pp);
```

$$v\_x3\_2 := \left[\left[\frac{2136720}{1489\,\psi_3} + \left(-\frac{77}{147360}\psi_3 - \frac{29}{73680}\psi_2\right)y_{1,1} + \frac{2291}{2978}\frac{\psi_1\,y_{2,1}}{\psi_3}, \frac{1403658}{1489\,\psi_3} + \left(-\frac{77}{147360}\psi_3 - \frac{29}{73680}\psi_2\right)y_{1,2} + \frac{2291}{2978}\frac{\psi_1\,y_{2,2}}{\psi_3}\right],\right. \quad (113)$$

$$\left[\frac{1989360}{1489\,\psi_3} + \left(-\frac{9}{24560}\psi_2 + \frac{31}{147360}\psi_3\right)y_{1,1} + \frac{2133}{2978}\frac{\psi_1\,y_{2,1}}{\psi_3}, \frac{1306854}{1489\,\psi_3} + \left(\right.\right.$$

$$-\frac{9}{24560}\psi_2+\frac{31}{147360}\psi_3\Big)y_{1,2}+\frac{2133}{2978}\frac{\psi_1 y_{2,2}}{\psi_3}\Big],$$

$$\Big[y_{2,1}, y_{2,2}\Big]\Big]$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u]]).v_x3_2-M);
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{114}$$

```
> Determinant(Matrix([[E_x3,C_x3]]));
```

$$1 \tag{115}$$

Example 9

```
> with(OreModules):
> A := DefineOreAlgebra(diff=[x1,s1],polynom=[s1]):
```

```
> D1 := Matrix([[1,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,-1]]);
```

$$D1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{116}$$

```
> D2 := Matrix([[0,0,0,0],[0,1,0,0],[0,0,-1,0],[0,0,0,0]]);
```

$$D2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{117}$$

```
> D3 := Matrix([[0,0,0,1],[0,0,0,0],[0,0,0,0],[-1,0,0,0]]);
```

$$D3 := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{118}$$

```
> D4 := Matrix([[0,0,0,0],[0,0,1,0],[0,-1,0,0],[0,0,0,0]]);
```

$$D4 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{119}$$

```
> M := Matrix([[1,0,0,1],[0,0,0,0],[0,0,0,0],[1,0,0,1]]);
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad (120)$$

```
> l := Rank(M);
```

$$l := 1 \qquad (121)$$

```
> X := SubMatrix(M,1..4,1);
```

$$X := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (122)$$

```
> Y := Involution(Factorize(Involution(M,A),Involution(X,A),A),A);
```

$$Y := \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \qquad (123)$$

```
> L := SyzygyModule(M,A);
```

$$L := \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (124)$$

```
> N := Matrix([[L.D1],[L.D2],[L.D3],[L.D4]]);
```

$$N := \begin{bmatrix} \textit{12 x 4 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \qquad (125)$$

```
> K := NullSpace(N);
```

$$K := \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad (126)$$

```
> Z := Matrix([K[1]]);
```

$$Z := \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (127)$$

```
> W1 := Involution(Factorize(Involution(D1.Z,A),Involution(X,A),A),A)
  ;
```

$$W1 := \begin{bmatrix} -1 \end{bmatrix} \qquad (128)$$

```
> W2 := Involution(Factorize(Involution(D2.Z,A),Involution(X,A),A),A)
;
```

$$W2 := \left[\begin{array}{c} 0 \end{array}\right] \quad (129)$$

```
> W3 := Involution(Factorize(Involution(D3.Z,A),Involution(X,A),A),A)
;
```

$$W3 := \left[\begin{array}{c} 1 \end{array}\right] \quad (130)$$

```
> W4 := Involution(Factorize(Involution(D4.Z,A),Involution(X,A),A),A)
;
```

$$W4 := \left[\begin{array}{c} 0 \end{array}\right] \quad (131)$$

```
> B := Matrix([[W1*x1,W2*x1,W3*x1,W4*x1]]);
```

$$B := \left[\begin{array}{cccc} -x1 & 0 & x1 & 0 \end{array}\right] \quad (132)$$

```
> ann_N := PiPolynomial(Transpose(B),A);
```

$$ann\_N := [x1] \quad (133)$$

```
> E_x1 := Transpose(LocalLeftInverse(Transpose(B),ann_N,A));
```

$$E\_x1 := \left[\begin{array}{c} 0 \\ 0 \\ \frac{1}{x1} \\ 0 \end{array}\right] \quad (134)$$

```
> B.E_x1;
```

$$\left[\begin{array}{c} 1 \end{array}\right] \quad (135)$$

```
> C_x1 := Involution(SyzygyModule(Involution(B,A),A),A);
```

$$C\_x1 := \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \quad (136)$$

```
> B.C_x1;
```

$$\left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right] \quad (137)$$

```
> OreRank(B,A);
```

$$3 \quad (138)$$

```
> ColumnDimension(C_x1);
```

$$3 \quad (139)$$

```
> Y_p := Matrix(3,4,symbol=y);
```

$$Y\_p := \left[\begin{array}{cccc} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \end{array}\right] \quad (140)$$

```
> u := Z * psi[1];
```

$$u := \begin{bmatrix} -\psi_1 \\ 0 \\ 0 \\ \psi_1 \end{bmatrix} \quad (141)$$

```
> v := subs(x1=psi[1],E_x1.Y+C_x1.Y_p);
```

$$v := \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ \frac{1}{\psi_1} + y_{1,1} & y_{1,2} & y_{1,3} & \frac{1}{\psi_1} + y_{1,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \end{bmatrix} \quad (142)$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u,D4.u]]).v-M);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (143)$$

```
> Determinant(Matrix([[E_x1,C_x1]]));
```

$$\frac{1}{x1} \quad (144)$$

```
> Y2 := Matrix([[Y],[Y_p]]);
```

$$Y2 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \end{bmatrix} \quad (145)$$

```
> Determinant(Y2);
```

$$-y_{1,1} y_{2,2} y_{3,3} + y_{1,1} y_{2,3} y_{3,2} + y_{1,2} y_{2,1} y_{3,3} - y_{1,2} y_{2,3} y_{3,1} + y_{1,2} y_{2,3} y_{3,4} - y_{1,2} y_{2,4} y_{3,3} - y_{1,3} y_{2,1} y_{3,2} + y_{1,3} y_{2,2} y_{3,1} - y_{1,3} y_{2,2} y_{3,4} + y_{1,3} y_{2,4} y_{3,2} + y_{1,4} y_{2,2} y_{3,3} - y_{1,4} y_{2,3} y_{3,2} \quad (146)$$

```
> S,U,V := SmithForm(B,x1,output=['S','U','V']);
```

$$S, U, V := \begin{bmatrix} x1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (147)$$

```
> simplify(U.B.V-S);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \quad (148)$$

```
> MatrixInverse(V);
```

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (149)$$

```
> E_x1_2 := simplify(SubMatrix(V,1..4,1..1).S[1,1]^(-1).U);
```

$$E\_x1\_2 := \begin{bmatrix} -\frac{1}{x1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (150)$$

```
> B.E_x1_2;
```

$$\begin{bmatrix} 1 \end{bmatrix} \quad (151)$$

```
> C_x1_2 := SubMatrix(V,1..4,2..4);
```

$$C\_x1\_2 := \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (152)$$

```
> B.C_x1_2;
```

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad (153)$$

```
> v_2 := subs(x1=psi[1],E_x1_2.Y+C_x1_2.Y_p);
```

$$v\_2 := \begin{bmatrix} -\frac{1}{\psi_1} + y_{2,1} & y_{2,2} & y_{2,3} & -\frac{1}{\psi_1} + y_{2,4} \\ y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ y_{2,1} & y_{2,2} & y_{2,3} & y_{2,4} \\ y_{3,1} & y_{3,2} & y_{3,3} & y_{3,4} \end{bmatrix} \quad (154)$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u,D4.u]]).v-M);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (155)$$

```
> Determinant(Matrix([[E_x1_2,C_x1_2]]));
```

(156)

$$-\frac{1}{x1} \qquad (156)$$

Example 10

```
> D1 := Matrix([[1,0,0,0],[0,0,0,0],[0,0,0,0],[0,0,0,-1]]);
```

$$D1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (157)$$

```
> D2 := Matrix([[0,0,0,0],[0,1,0,0],[0,0,-1,0],[0,0,0,0]]);
```

$$D2 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (158)$$

```
> D3 := Matrix([[0,0,0,1],[0,0,0,0],[0,0,0,0],[-1,0,0,0]]);
```

$$D3 := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad (159)$$

```
> D4 := Matrix([[0,0,0,0],[0,0,1,0],[0,-1,0,0],[0,0,0,0]]);
```

$$D4 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (160)$$

```
> M := Matrix([[1,0,0,1],[0,1,-1,0],[0,1,1,0],[1,0,0,1]]);
```

$$M := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad (161)$$

```
> l := Rank(M);
```

$$l := 3 \qquad (162)$$

```
> X := SubMatrix(M,1..4,1..l);
```

(163)

$$X := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad (163)$$

```
> Rank(X);
```

$$3 \quad (164)$$

```
> Y := Involution(Factorize(Involution(M,A),Involution(X,A),A),A);
```

$$Y := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (165)$$

```
> L := SyzygyModule(M,A);
```

$$L := \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix} \quad (166)$$

```
> N := Matrix([[L.D1],[L.D2],[L.D3],[L.D4]]);
```

$$N := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (167)$$

```
> K := NullSpace(N);
```

$$K := \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad (168)$$

```
> Z := Matrix([K[1],K[2],K[3]]);
```

$$Z := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad (169)$$

```
> W1 := Involution(Factorize(Involution(D1.Z,A),Involution(X,A),A),A)
  ;
```

$$W1 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (170)$$

```
> W2 := Involution(Factorize(Involution(D2.Z,A),Involution(X,A),A),A)
  ;
```

$$W2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad (171)$$

```
> W3 := Involution(Factorize(Involution(D3.Z,A),Involution(X,A),A),A)
  ;
```

$$W3 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (172)$$

```
> W4 := Involution(Factorize(Involution(D4.Z,A),Involution(X,A),A),A)
  ;
```

$$W4 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad (173)$$

```
> X := Matrix([[x1],[x2],[x3]]);
```

$$X := \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} \quad (174)$$

```
> B := Matrix([[W1.X,W2.X,W3.X,W4.X]]);
```

$$B := \begin{bmatrix} -x1 & 0 & x1 & 0 \\ 0 & -\frac{1}{2}\,x2 + \frac{1}{2}\,x3 & 0 & \frac{1}{2}\,x2 - \frac{1}{2}\,x3 \\ 0 & -\frac{1}{2}\,x2 - \frac{1}{2}\,x3 & 0 & -\frac{1}{2}\,x2 - \frac{1}{2}\,x3 \end{bmatrix} \quad (175)$$

```
> A := DefineOreAlgebra(diff=[x1,s1],diff=[x2,s2],diff=[x3,
  s3],polynom=[s1,s2,s3]):
```

```
> ann_N := PiPolynomial(Transpose(B),A);
```

$$ann\_N := \left[x1\,x2^2 - x1\,x3^2\right] \quad (176)$$

```
> g1 := factor(ann_N[1]);
```

$$g1 := x1\,(x2 - x3)\,(x2 + x3) \quad (177)$$

```
> E_g1 := Transpose(LocalLeftInverse(Transpose(B),ann_N,A));
```

$$E\_g1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{-x1\,x2 - x1\,x3}{x1\,x2^2 - x1\,x3^2} & \frac{-x1\,x2 + x1\,x3}{x1\,x2^2 - x1\,x3^2} \\ \frac{x2^2 - x3^2}{x1\,x2^2 - x1\,x3^2} & 0 & 0 \\ 0 & \frac{x1\,x2 + x1\,x3}{x1\,x2^2 - x1\,x3^2} & \frac{-x1\,x2 + x1\,x3}{x1\,x2^2 - x1\,x3^2} \end{bmatrix} \quad \textbf{(178)}$$

```
> simplify(B.E_g1);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(179)}$$

```
> simplify(g1*E_g1);
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -x1\,(x2 + x3) & -x1\,(x2 - x3) \\ x2^2 - x3^2 & 0 & 0 \\ 0 & x1\,(x2 + x3) & -x1\,(x2 - x3) \end{bmatrix} \quad \textbf{(180)}$$

```
> C_g1 := Involution(SyzygyModule(Involution(B,A),A),A);
```

$$C\_g1 := \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \textbf{(181)}$$

```
> B.C_g1;
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \textbf{(182)}$$

```
> Y_p := Matrix(1,4,symbol=y);
```

$$Y\_p := \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \end{bmatrix} \quad \textbf{(183)}$$

```
> u := Z . Matrix([[psi[1]],[psi[2]],[psi[3]]]);
```

$$u := \begin{bmatrix} -\psi_1 \\ \psi_3 \\ \psi_2 \\ \psi_1 \end{bmatrix} \quad \textbf{(184)}$$

```
> v := simplify(subs([x1=psi[1],x2=psi[2],x3=psi[3]],E_g1.Y+C_g1.Y_p)
```

```
);
```

$$v := \begin{bmatrix} y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \\ 0 & -\frac{1}{\psi_2 - \psi_3} & -\frac{1}{\psi_2 + \psi_3} & 0 \\ \frac{\psi_1 y_{1,1} + 1}{\psi_1} & y_{1,2} & y_{1,3} & \frac{\psi_1 y_{1,4} + 1}{\psi_1} \\ 0 & \frac{1}{\psi_2 - \psi_3} & -\frac{1}{\psi_2 + \psi_3} & 0 \end{bmatrix} \tag{185}$$

```
> simplify(Matrix([[D1.u,D2.u,D3.u,D4.u]]).v-M);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{186}$$

```
> Determinant(Matrix([[E_g1,C_g1]]));
```

$$\frac{2}{x1\, x2^2 - x1\, x3^2} \tag{187}$$

```
> Y2 := Matrix([[Y],[Y_p]]);
```

$$Y2 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ y_{1,1} & y_{1,2} & y_{1,3} & y_{1,4} \end{bmatrix} \tag{188}$$

```
> Determinant(Y2);
```

$$y_{1,4} - y_{1,1} \tag{189}$$