

Computations with real algebraic numbers of degree up to 4

Elias P. Tsigaridas, Ioannis Z. Emiris,
Department of Informatics and Telecommunications
National and Kapodistrian University of Athens
Panepistimiopolis, 15771 Athens
Greece
{et, emiris}@di.uoa.gr

Abstract

Our motivation comes from computer-aided design and geometric modeling, where algorithms and predicates rely on computations of real algebraic numbers (usually of small degree). Predicates must be decided exactly in all cases, including degeneracies. In computational geometry there is a growing need for robust manipulation of curved objects (see [2] for a modern approach), since the critical points and the intersection points of such objects have coordinates which are real algebraic numbers. Efficiency is critical because solutions of polynomials and comparisons on algebraic numbers lie in the inner loop of most algorithms in non-linear computational geometry.

We focus on real algebraic numbers of degree up to 4 and polynomials in one variable of arbitrary degree, or in 2 variables of degree ≤ 2 , but we can extend our methods to arbitrary degree.

In order to decide the number of real roots and their multiplicity of a polynomial of degree up to 4, we consider the coefficients as parameters and we use the discrimination systems, first introduced by Yang [3]. In our approach, the computation of such systems relies on Sturm-Habicht sequences ([6]), the coefficients of which are factorized by the use of minors of the Bézoutian matrix of the polynomial and its derivative ([5]) and by the use of invariants. In [1] we presented the full discrimination system of a quartic, where we corrected a small typographical error from Yang ([3]).

We represent a real algebraic number by a polynomial and an isolating interval. We compute rational numbers that isolate the roots of every polynomial of degree up to 4, which are computed as functions in the coefficients of the polynomial.

In order to compare two algebraic numbers of degree up to 4, or to find the sign of a polynomial of arbitrary degree, evaluated over such a number, we use Sturm-based algorithms (see [7]), which rely on computations of the signs of certain quantities. We precompute these quantities, we factorize them by the use of invariants and/or by minors of the Bézoutian

matrix; for our implementation, this is done in an automated way. We developed programs that produce all possible sign combinations of the tested quantities, so as to test as few quantities as possible. Our algorithms, for algebraic numbers of degree up to 3, are optimal with respect to the algebraic degree of the tested quantities since the degree agrees with that of the resultant. It is an open problem whether for higher degree algebraic numbers, the degree of the resultant is tight lower bound for comparisons.

Additionally, we solve bivariate problems, such as the computation of the sign of a bivariate polynomial of degree 2 evaluated over two algebraic numbers and the solution of a system of two such polynomials. We tackle these problems by the use of Sturm-Habicht sequences ([6]), where again certain quantities are precomputed and factorized. To be more specific, we consider the resultants R_x, R_y , of the two bivariate polynomials thus obtaining degree-4 polynomials in x and y . We solve R_x, R_y and the isolating points of the computed algebraic numbers define a grid of boxes. We can decide if a box is empty or if it contains a simple or a multiple root of the system by computing the signs of f_1 and f_2 over these 2 algebraic numbers, that define the box. We do an optimization by noticing that the intersections in a column (row) cannot exceed 2 nor the multiplicity of the algebraic number and thus excluding various boxes. Our algorithm does not make any assumption, such as that the boxes cannot contain any critical points of the intersecting polynomials, hence there is no need for refinement. Moreover, we are trying another approach, by considering the rational univariate representation that can be obtained by the specialization of the subresultant chain.

Our Sturm-based algorithms rely on isolating points and precomputed quantities so we avoid iterative methods (which depend on separation bounds) and the explosion of the algebraic degree of the tested quantities.

Almost all the code of our algorithms was first implemented in `MAPLE` where we use packages `codegeneration` and `optimize` so as to produce efficient `C++` code.

We have implemented a package of algebraic numbers as part of the `SYNAPS 2.1` ([4]) library and show that it compares favorably with other software. We performed tests against `MAPLE 9`, the package of Rioobo in `AXIOM` ([9]), the iterative solver of `SYNAPS` which is based on Berstein's basis ([10]), and the package from [8]. Current results show the superiority of our approach, as well as its robustness, since we have similar running times for all kinds of tests and we handle degenerate cases faster than the general ones. This is not the case for any of the other approaches. However, more experimentation is needed and we are planning tests with `RS` ([11]) and `NiX` ([12]).

References:

- [1] I. Emiris and E. Tsigaridas; *Comparing real algebraic numbers of small degree*, In *ESA*, volume 3221 of *LNCS*, pages 652-663, 2004.
- [2] I. Emiris, A. Kakargias, M. Teillaud, E. Tsigaridas, and S. Pion; *Towards an open curved kernel*, In *Proc. Annual ACM Symp. on Computational Geometry*, p. 438-446, New York, 2004. ACM Press.

- [3] L. Yang; *Recent advances on determining the number of real roots of parametric polynomials*, J. Symbolic Computation, 28:225–242, 1999.
- [4] D. Dos Reis, B. Mourrain, R. Rouillier, and P. Trébuchet; *An environment for symbolic and numeric computation*, In Proc. of the International Conference on Mathematical Software 2002, World Scientific, p. 239–249, 2002.
- [5] P. Bikker and A. Y. Uteshev; *On the Bézout construction of the resultant*, J. Symbolic Computation, 28(1–2):45–88, July/Aug. 1999.
- [6] S. Basu, R. Pollack, and M-F.Roy; *Algorithms in Real Algebraic Geometry*, volume 10 of Algorithms and Computation in Mathematics, Springer-Verlag, 2003.
- [7] C. Yap; *Fundamental Problems of Algorithmic Algebra*, Oxford Univ. Press, 2000.
- [8] L. Guibas, M. Karavelas, and D. Russel. *A computational framework for handling motion*, In Proc. 6th Workshop (ALLENEX), Jan. 2004.
- [9] R. Rioboo; *Real algebraic closure of an ordered field: implementation in axiom*, In Proc. Annual ACM ISSAC, pages 206–215. ACM Press, 1992.
- [10] B. Mourrain, M. Vrahatis, and J.C. Yakoubsohn; *On the Complexity of Isolating Real Roots and Computing with Certainty the Topological Degree*, J. of Complexity, 18(2):612–640, 2002.
- [11] F. Rouillier and P. Zimmermann; *Efficient isolation of polynomial real roots*, J. of Computational and Applied Mathematics, 162(1):33-50, 2003.
- [12] <http://www.mpi-sb.mpg.de/projects/EXACUS/>

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