Symmetric Tensor Decomposition

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Abstract

We present an algorithm for decomposing a symmetric tensor, of dimension n and order d, as a sum of rank-1 symmetric tensors, extending the algorithm of Sylvester devised in 1886 for binary forms.

We recall the correspondence between the decomposition of a homogeneous polynomial in n variables of total degree d as a sum of powers of linear forms (Waring's problem), incidence properties on secant varieties of the Veronese variety and the representation of linear forms as a linear combination of evaluations at distinct points. Then we reformulate Sylvester's approach from the dual point of view. Exploiting this duality, we propose necessary and sufficient conditions for the existence of such a decomposition of a given rank, using the properties of Hankel (and quasi-Hankel) matrices, derived from multivariate polynomials and normal form computations. This leads to the resolution of systems of polynomial equations of small degree in non-generic cases. We propose a new algorithm for symmetric tensor decomposition, based on this characterization and on linear algebra computations with Hankel matrices.

The impact of this contribution is two-fold. First it permits an efficient computation of the decomposition of any tensor of sub-generic rank, as opposed to widely used iterative algorithms with unproved global convergence (e.g. Alternate Least Squares or gradient descents). Second, it gives tools for understanding uniqueness conditions and for detecting the rank.

Key words: tensor decomposition

1. Introduction

Symmetric tensors appear in applications mainly as high-order derivatives of multivariate functions; for instance in Statistics, cumulant tensors are derivatives of the second characteristic function [42].

Tensors have been widely utilized in Electrical Engineering since the 1990s [52], and in particular in Antenna Array Processing [23, 10] and Telecommunications [55, 9, 49, 26, 20]. Even earlier, in the 1970s, tensors have been used in Chemometrics [5] and Psychometrics [33]. Also since the

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1970s tensor decompositions of the third order have been applied in the study of the arithmetic complexity of computing a set of bilinear forms [35, 4, 51, 46, 37].

Another important application field is Data Analysis, for instance, Independent Component Analysis, originally introduced for symmetric tensors whose rank did not exceed dimension [13] [7]. More recently, tensors whose rank exceeds dimension have raised a greater interest [24, 32]. In some applications, tensors may be symmetric only in some modes [15], or may neither be symmetric nor have equal dimensions [11, 50]. Numerous further applications of tensor decompositions may be found in [11, 50, 3].

Sometimes, tensors are encountered in the form of a collection of symmetric matrices [22, 27, 55, 48, 54], in which case they may enjoy symmetries in some modes but not in the others. Conversely, some algorithms treat symmetric tensors as a collection of symmetric matrix slices [56, 58, 21].

The problem of decomposition of a symmetric tensor, which we consider in this paper, extends the Singular Value Decomposition (SVD) for symmetric matrices which is an important tool in numerical linear algebra, routinely used in many applications [28]. As exhibited above, the extension to general symmetric tensors also appears in numerous application domains. However, many theoretical and algorithmic issues remain unsolved. Among the solved problems, let us mention the determination of the minimal number of terms in the decomposition of generic tensors [2], defined there in terms of dual interpolation problems. See [31, chap. 2] and section 4 for the link between these two points of view. Among open problems are the determination of the maximal rank of tensors of given degree and dimension, and the determination of the stratification of the set of symmetric tensors by the rank. See, however, [12] for a progress in the binary case. For a detailed presentation of the symmetric tensor decomposition problem, from a projective algebraic geometric point of view, we refer to [31]. The properties of so-called catalecticant matrices, related to the apolar duality induced by the symmetric tensor associated with homogeneous polynomials of a given degree, are extensively studied.

Independently tensor decompositions have been studied by numerical analysts inspired in a disconnected way, probably because of language barrier. Investigations of this problem in numerical analysis have been developed, inspired by the successful work on order two tensors, i.e. matrices. However, despite their obvious practical value, numerical algorithms presently used in most scientific communities are suboptimal, in the sense that they either do not fully exploit symmetries [1], minimize different successive criteria sequentially [56, 21], or are iterative and do not guarantee a global convergence [30, 47]. In addition, they often request the rank to be much smaller than generic. Among these popular methods, we refer to "PARAFAC" techniques [5], extensively applied to ill-posed problems... Indeed unlike the matrix case, the set of symmetric tensors of rank at most r is not closed, and its closure has singularities corresponding to tensors of rank greater than r. This explains why iterative numerical methods encounter difficulties in computing a tensor decomposition. For more details and open problems on symmetric tensors, see [16].

The goal of this paper is to describe a new algorithm that decomposes a symmetric tensor of arbitrary order and dimension into a sum of rank-one terms. The algorithm proposed in this paper is inspired by Sylvester's theorem [36] and extends its principle to larger dimensions. Using apolar duality on polynomials, we show that the symmetric tensor decomposition reduces to the decomposition of a linear form as a linear combination of evaluations at distinct points. We give a necessary and sufficient condition for the existence of a decomposition of rank r, based on rank conditions of Hankel operators or commutation properties. Instead of working, degree by degree, as in [31], we consider affine situations in order to treat simultaneously the various homogeneous components. In the binary case, the decomposition can be obtained directly by computing ranks of catalecticant matrices. In higher dimension, this is not so simple. An extension step is required to find the decomposition. This leads to the resolution of a polynomial system of small degree, from which we deduce the decomposition by solving a simple eigenvalue problem, by means of linear algebra manipulations.

The algorithm is not restricted to strictly sub-generic ranks as for the method proposed in [31][chap. 5]. In sub-generic cases, the decomposition is essentially unique (i.e. up to scale and permutation) when some rank conditions are satisfied. Our algorithm fully exploits this symmetry and provides a complete answer to the questions of uniqueness and computation, for any order [14].

In the following section, we recall the method deduced from Sylvester's theorem to decompose a binary form, and in section 3 we present the notation used throught the paper. In section 4, we give three equivalent formulations of the same problem, used and studied in different communities. In section 5, we develop the duality point of view, extending the notion of generalized additive decomposition, introduced in [31], to any dimension. Section 6 is devoted to the algebraic characterization of the extension property of linear forms, in terms of rank condition on multivariate Hankel operators, or on commutation properties. Finally in section 7, we describe the algorithm and give examples.

2. The binary case (Sylvester's algorithm)

The present contribution is a generalization of Sylvester's algorithm devised to decompose homogeneous polynomials in two variables into a sum of powers of linear forms [53] [12]. It is hence convenient to first recall the latter algorithm.

Theorem 2.1 (Sylvester, 1886). *A binary quantic* $p(x_1, x_2) = \sum_{i=0}^{d} \binom{d}{i} c_i \, x_1^i \, x_2^{d-i}$ *can be written as a sum of* d^{th} *powers of* r *distinct linear forms in* $\mathbb{C}$ *as:*

$$p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j \, (\alpha_j \, x_1 + \beta_j \, x_2)^d, \tag{1}$$

if and only if **(i)** *there exists a vector* $\mathbf{q} = (q_\ell)_{\ell=0}^{r}$*, such that*

$$\begin{bmatrix} c_0 & c_1 & \cdots & c_r \\ \vdots & & & \vdots \\ c_{d-r} & \cdots & c_{d-1} & c_d \end{bmatrix} \mathbf{q} = \mathbf{0}. \tag{2}$$

and **(ii)** *the polynomial* $q(x_1, x_2) = \sum_{\ell=0}^{r} q_\ell \, x_1^\ell \, x_2^{r-\ell}$ *admits* r *distinct roots, i.e. can be written as* $q(x_1, x_2) = \prod_{j=1}^{r} (\beta_j^* \, x_1 - \alpha_j^* \, x_2)$.

The proof of this theorem is constructive [53] [14] [16] and yields Algorithm 2.1. Note that step 5 is a specialization only if the dimension of the right kernel is strictly larger than one.

3. Notation and preliminaries

Let $\mathbb{K}$ be an algebraically closed field (e.g. $\mathbb{K} = \mathbb{C}$ the field of complex numbers). For a vector space E, its associated projective space is denoted $\mathbb{P}(E)$. For $\mathbf{v} \in E - \{0\}$ its class in $\mathbb{P}(E)$ is

Algorithm 2.1: BINARY FORM DECOMPOSITION

Input: Given a binary polynomial $p(x_1, x_2)$ of degree d with coefficients $a_i = \binom{d}{i} c_i$, $0 \le i \le d$, define the Hankel matrix $H[r]$ of dimensions $d - r + 1 \times r + 1$ with entries $H[r]_{ij} = c_{i+j-2}$.

Output: A decomposition of p as $p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j \, \mathbf{k}_j(\mathbf{x})^d$ with minimal r.

1. Initialize $r = 0$
2. Increment $r \leftarrow r + 1$
3. If the matrix H[r] has full column rank, then go to step 2
4. Else compute a basis $\{\mathbf{k}_1, \ldots, \mathbf{k}_l\}$ of the right kernel of $H[r]$.
5. Specialization:
   - Take a generic vector $\mathbf{q}$ in the kernel, e.g. $\mathbf{q} = \sum_i \mu_i \mathbf{k}_i$
   - Compute the roots of the associated polynomial $q(x_1, x_2) = \sum_{\ell=0}^{r} q_\ell \, x_1^\ell \, x_2^{d-\ell}$. Denote them $(\beta_j, -\alpha_j)$, where $|\alpha_j|^2 + |\beta_j|^2 = 1$.
   - If the roots are not distinct in $\mathbb{P}^2$, try another specialization. If distinct roots cannot be obtained, go to step 2.
   - Else if $q(x_1, x_2)$ admits r distinct roots then compute coefficients λ_j, $1 \le j \le r$, by solving the linear system below, where a_i denotes $\binom{d}{i} c_i$

$$\begin{bmatrix} \alpha_1^d & \ldots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \ldots & \alpha_r^{d-1}\beta_r \\ \alpha_1^{d-2}\beta_1^2 & \ldots & \alpha_r^{d-1}\beta_r^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \ldots & \beta_r^d \end{bmatrix} \lambda = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$$

6. The decomposition is $p(x_1, x_2) = \sum_{j=1}^{r} \lambda_j \, \mathbf{k}_j(\mathbf{x})^d$, where $\mathbf{k}_j(\mathbf{x}) = (\alpha_j \, x_1 + \beta_j \, x_2)$.

denoted $\overline{\mathbf{v}}$. Let $\mathbb{P}^n$ be the projective space of the field $\mathbb{K}^n$. A symmetric tensor is an element of the tensor algebra $T(E)$ which can be represented by an array $[t_{i_1,\ldots,i_d}]_{0\le i_j\le n}$ of coefficients in a basis of $T(E)$, with $t_{i_1,\ldots,i_d}\in\mathbb{K}$ and $t_{i_{\sigma(1)},\ldots,i_{\sigma(d)}} = t_{i_1,\ldots,i_d}$ for any permutation σ of $[1,\ldots,d]$. The set of all symmetric tensors forms an algebra $S(E)$, called the symmetric algebra of E.

If $\mathbf{a} = (a_1,\ldots,a_n)$ is a vector in $\mathbb{N}^n$, then $|\mathbf{a}|$ is the sum of its elements, i.e. $|\mathbf{a}| = \sum_{i=a}^{n} a_i$. We also use the greek letters α and β for vectors in $\mathbb{N}^n$. We denote by $\mathbf{x}^\alpha$ the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. For a set $B = \{b_1,\ldots,b_m\}$, we denote by $\langle B\rangle$, respectively (B), the corresponding vector space, resp. ideal, generated by B.

Let R be the ring of polynomials $\mathbb{K}[x_1,\ldots,x_n]$, while R_d denotes the ring of polynomials of (total) degree at most d. The set $\{\mathbf{x}^\alpha\}_{|\alpha|\le d} = \{x_1^{\alpha_1}\cdots x_n^{\alpha_n}\}_{\alpha_1+\cdots+\alpha_n\le d}$ represents the elements of the monomial basis of the vector space R_d and contains $\binom{n+d}{d}$ elements. Hereafter, the superscript h denotes the homogenization of a polynomial. We denote by S_d the vector space of homogeneous polynomials in $n+1$ variables $x_0, x_1,\ldots,x_n$. This is also the symmetric d-th power $S^d(E)$ where $E = \langle x_0,\ldots,x_n\rangle$. The dehomogenization of a polynomial $f\in S_d$ with respect to the variable x_0 is denoted $f^a := f(1,x_1,\ldots,x_n)$.

Duality is an important ingredient of our approach. For a comprehensive treatment of duality of multivariate polynomials, we refer the reader to [44]. Recall that the dual $E^* = \mathrm{Hom}_{\mathbb{K}}(E,\mathbb{K})$ of a $\mathbb{K}$-vector space E is the set of $\mathbb{K}$-linear forms from E to $\mathbb{K}$. A basis of the dual space R_d^*, is the set of linear forms that compute the coefficients of a polynomial in the primal basis. It is denoted by $\{\mathbf{d}^\alpha\}_{|\alpha|\le d}$.

We identify R^* with the (vector) space of formal power series, i.e. $\mathbb{K}[[\mathbf{d}]] = \mathbb{K}[[d_1,\ldots,d_n]]$. Any element $\Lambda\in R^*$ can be decomposed as $\Lambda = \sum_{\mathbf{a}}\Lambda(\mathbf{x}^{\mathbf{a}})\mathbf{d}^{\mathbf{a}}$. Typical elements of R^* are the linear forms that correspond to the evaluation at a point $\zeta\in\mathbb{K}^n$:

$$\begin{aligned}\mathbb{1}_\zeta \ :\ & R\to\mathbb{K}\\ & p\mapsto p(\zeta)\end{aligned}$$

The decomposition of $\mathbb{1}_\zeta$ in the basis $\{\mathbf{d}^{\mathbf{a}}\}_{|\mathbf{a}|\le d}$ is $\mathbb{1}_\zeta = \sum_{\mathbf{a}}\zeta^{\mathbf{a}}\mathbf{d}^{\mathbf{a}}$. Such an evaluation form can be composed with differentiation. In fact, if $\theta(\partial_1,\ldots,\partial_n)$ is a differential polynomial, then

$$\begin{aligned}\mathbb{1}_\zeta\circ\theta(\partial_1,\ldots,\partial_n) \ :\ & R\to\mathbb{K}\\ & p\mapsto\theta(\partial_1,\ldots,\partial_n)(p)(\zeta).\end{aligned}$$

The dual space R^* has a natural structure of R-module [25] defined as follows: for all $p\in R$, and for all $\Lambda\in R^*$ consider the linear operator

$$\begin{aligned}p\star\Lambda \ :\ & R\to\mathbb{K}\\ & q\mapsto\Lambda(pq).\end{aligned}$$

In particular, we have $x_i\star\mathbf{d}^{\mathbf{a}} = \begin{cases} d_1^{a_1}\cdots d_{i-1}^{a_{i-1}}d_i^{a_i-1}d_{i+1}^{a_{i+1}}\cdots d_n^{a_n} & \text{if } a_i>0,\\ 0 & \text{otherwise.}\end{cases}$

4. Problem formulations

In this section, we present three different formulations of the same problem.

4.1. Polynomial decomposition

A symmetric tensor $[a_{j_0,\ldots,j_n}]$ of order d and dimension n can be associated with a homogeneous polynomial $f(\mathbf{x}) \in S_d$:

$$f(\mathbf{x}) = \sum_{j_0+j_1+\cdots+j_n=d} a_{j_0,j_1,\ldots,j_n} x_0^{j_0} x_1^{j_1} \cdots x_n^{j_n}. \tag{3}$$

Our goal is to compute a decomposition of f as a sum of d^{th} powers of linear forms, i.e.

$$f(\mathbf{x}) = \sum_{i=1}^{r} \lambda_i \left(k_{i,0}x_0 + k_{i,1}x_1 + \cdots + k_{i,n}x_n\right)^d = \lambda_1\, \mathbf{k}_1(\mathbf{x})^d + \lambda_2\, \mathbf{k}_2(\mathbf{x})^d + \cdots + \lambda_r\, \mathbf{k}_r(\mathbf{x})^d, \tag{4}$$

where $\lambda_i \neq 0$, $\mathbf{k}_i \neq 0$, and r is the smallest possible. This minimal r is called the *rank* of f.

Here is the *direct* approach to solving this decomposition problem. Consider the relation

$$f(\mathbf{x}) = \sum_{i=1}^{r} \left(k_{i,0}x_0 + k_{i,1}x_1 + \cdots + k_{i,n}x_n\right)^d,$$

where $\mathbf{k}_i \neq 0$. We assume that r, the rank, is known and the smallest possible. We consider the $r(n+1)$ coefficients $k_{i,j}$ of the linear forms as unknowns. We expand (symbolically) the right hand side of the equation. The two polynomials on the left and right hand sides are equal. Thus by equating the coefficients of the same monomials we get a polynomial system in the coefficients $k_{i,j}$. This is an over-constrained polynomial system of $\binom{n+d}{d}$ equations and $r(n+1)$ unknowns. The polynomials of the system are homogeneous of degree d and the magnitude of their coefficients is at most $r\binom{n+d}{d}$. This approach describes the problem of decomposition in a non-optimal way. It introduces $r!$ redundant solutions, since every permutation of the linear forms is also a solution. Another drawback of this approach is that the polynomials involved are of high degree, that is, d. The reader can compare this with the degree two polynomial system, described in Section 6, containing the polynomials that we have to solve in order to extend the matrix.

In the following sections, we are going to describe a new method, which is much more efficient for solving this decomposition problem.

4.2. Veronese and Secant Varieties

Let us recall the well-known correspondence between the symmetric outer product decomposition and secant varieties for symmetric tensors. The set of symmetric tensors or homogeneous polynomials of the form $\mathbf{k}(\mathbf{x})^d = (k_0x_0 + k_1x_1 + \cdots + k_nx_n)^d$ for $\mathbf{k} = (k_0, k_1, \ldots, k_n) \in \mathbb{K}^n$ is a closed algebraic set. Scaling the vector $\mathbf{k}$ by a non-zero scalar λ yields a homogeneous polynomial scaled by λ^d. Thus, we can also consider this construction as a map $\mathbf{k} \mapsto \mathbf{k}(\mathbf{x})^d$ from the projective space $\mathbb{P}^{n-1}$ to the projective space of symmetric tensors:

$$\begin{array}{cccc} \nu & : & \mathbb{P}(S_1) & \rightarrow & \mathbb{P}(S_d) \\ & & \mathbf{k}(\mathbf{x}) & \mapsto & \mathbf{k}(\mathbf{x})^d. \end{array}$$

The image of ν is called the Veronese variety $\mathcal{V}_{n,d}$ [57, 29]. Following this point of view, a tensor is of rank one if it corresponds to a point on the Veronese variety. A tensor is of rank at most r if it is a linear combination of r tensors of rank one. In other words, it is in the linear space spanned by r points of the Veronese variety. The closure of the $r-$dimensional linear space spanned by r points of the Veronese variety $\mathcal{V}_{n,d}$ is called the $(r-1)$-secant variety of $\mathcal{V}_{n,d}$ and denoted $\mathcal{S}^{r-1}(\mathcal{V}_{n,d})$. We refer the reader to [57, 29] for examples and general properties of these algebraic sets. In the non-symmetric case, the so-called Segre variety of the projective space of tensors is considered instead of the Veronese variety. It corresponds to the set of (possibly non-symmetric) tensors of rank one.

For any $f \in S_d - \{0\}$, the smallest r such that $\overline{f} \in \mathcal{S}^{r-1}(\mathcal{V}_{n,d})$ is called the *typical rank* or *border rank* of f [16, 54, 6].

4.3. Decomposition using duality

Let $f, g \in S_d$, where $f = \sum_{|\alpha|=d} f_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ and $g = \sum_{|\alpha|=d} g_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We define the apolar inner product on S_d as

$$\langle f, g \rangle = \sum_{|\alpha|=d} f_\alpha \, g_\alpha \binom{d}{\alpha_0, \ldots, \alpha_n}^{-1}.$$

Using this non-degenerate inner product, we can associate an element of S_d with an element S_d^*, through the following map:

$$\begin{aligned} \tau : S_d &\rightarrow S_d^* \\ f &\mapsto f^*, \end{aligned}$$

where the linear form f^* is defined as $f^* : g \mapsto \langle f, g \rangle$. A simple calculation shows that $\langle f, \mathbf{k}(\mathbf{x})^d \rangle = f(\mathbf{k})$ so that under this duality it holds that $\tau(\mathbf{k}(\mathbf{x})^d) = \mathbb{1}_{\mathbf{k}} \in S_d^*$. Moreover, under τ, the polynomial $f = \sum_{|\alpha|=d} c_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha \in S_d$ is mapped to $f^* = \sum_{|\alpha|=d} c_\alpha \, \mathbf{d}^\alpha \in S_d^*$. The decomposition of f can then be restated as follows:

> Given $f^* \in S_d^*$, find the minimal number of non-zero vectors $\mathbf{k_1}, \ldots, \mathbf{k_r} \in \mathbb{K}^{n+1}$ and non-zero scalars $\lambda_1, \ldots, \lambda_r \in \mathbb{K} - \{0\}$ such that
>
> $$f^* = \sum_{i=1}^{r} \lambda_i \, \mathbb{1}_{\mathbf{k}_i}.$$

By scaling $\mathbf{k_i}$ and multiplying λ_i by the inverse of the d^{th} power of this scaling factor, we may assume that the first non-zero coordinate of $\mathbf{k}_i$ is 1.

Definition 4.1. *We say that f^* has an affine decomposition if for every $\mathbf{k}_i$ in the decomposition, $\mathbf{k}_{i,0} \neq 0$.*

By a generic change of coordinates, any decomposition of f^* can be transformed into an affine decomposition. To any $f^* \in S_d^*$, we can associate an element in R_d^*, defined by $\tilde{\Lambda}_f : p \in R_d \mapsto f^*(p^h)$, where p^h is the homogenization in degree d of p. If f^* admits an affine decomposition with $\mathbf{k}_{i,0} = 1$ then $\tilde{\Lambda}_f$ coincides with the linear form

$$\tilde{\Lambda} = \sum_{i=1}^{r} \lambda_i \, \mathbb{1}_{\tilde{\mathbf{k}}_i}$$

up to degree d, where $\tilde{\mathbf{k}}_i$ is the vector made up of the last n coordinates of $\mathbf{k}_i$.

5. Hankel operators and quotient algebra

In this section, we recall the algebraic tools we need to describe and analyze our algorithm. For any $\Lambda \in R^*$, we define the bilinear form Q_Λ, such that

$$\begin{aligned} Q_\Lambda \ &: \ R \times R \to \mathbb{K} \\ &\quad (a, b) \mapsto \Lambda(ab). \end{aligned}$$

The matrix of Q_Λ, in the monomial basis of R, is $\mathbb{Q}_\Lambda = (\Lambda(\mathbf{x}^{\alpha+\beta}))_{\alpha,\beta}$, where $\alpha, \beta \in \mathbb{N}^n$. For any $\Lambda \in R^*$, we define the Hankel operator H_Λ from R to R^* as

$$\begin{aligned} H_\Lambda \ &: \ R \to R^* \\ &\quad p \mapsto p \star \Lambda. \end{aligned}$$

The matrix of the linear operator H_Λ in the monomial basis and in the dual basis, $\{\mathbf{d}^\alpha\}$, is $\mathbb{H}_\Lambda = (\Lambda(\mathbf{x}^{\alpha+\beta}))_{\alpha,\beta}$, where $\alpha, \beta \in \mathbb{N}^n$. The following relates the Hankel operators to the bilinear forms. For all $a, b \in R$, due to the R-module structure, it holds

$$Q_\Lambda(a, b) = \Lambda(ab) = (a \star \Lambda)(b) = (b \star \Lambda)(a) = H_\Lambda(a)(b) = H_\Lambda(b)(a).$$

In what follows we identify H_Λ and Q_Λ.

Definition 5.1. *Given $B = \{b_1, \ldots, b_r\}, B' = \{b'_1, \ldots, b'_{r'}\} \subset R$ we define*

$$H_\Lambda^{B,B'} : \langle B \rangle \to \langle B' \rangle^*,$$

as the restriction of H_Λ to the vector space $\langle B \rangle$ and inclusion of R^* in $\langle B' \rangle^*$. Let $\mathbb{H}_\Lambda^{B,B'} = (\Lambda(b_i\, b'_j))_{1 \le i \le r, 1 \le j \le r'}$. If $B' = B$, we also use the notation H_Λ^B and $\mathbb{H}_\Lambda^B$.

If B, B' are linearly independent, then $\mathbb{H}_\Lambda^{B,B'}$ is the matrix of $H_\Lambda^{B,B'}$ in this basis $\{b_1, \ldots, b_r\}$ of $\langle B \rangle$ and the dual basis of B' in $\langle B' \rangle^*$. The *catalecticant* matrices of [31] correspond to the case where B and B' are the set of monomials of degree at most k and $d - k$ ($k = 0, \ldots, d$), respectively.

From the definition of the Hankel operators, we can deduce that a polynomial $p \in R$ belongs to the kernel of $\mathbb{H}_\Lambda$ if and only if $p \star \Lambda = 0$, which in turn holds if and only if for all $q \in R$, $\Lambda(pq) = 0$.

Proposition 5.2. *Let I_Λ be the kernel of H_Λ. Then, I_Λ is an ideal of R.*

Proof. Let $p_1, p_2 \in I_\Lambda$. Then for all $q \in R$, $\Lambda((p_1+p_2)q) = \Lambda(p_1 q) + \Lambda(p_2 q) = 0$. Thus, $p_1 + p_2 \in I_\Lambda$. If $p \in I_\Lambda$ and $p' \in R$, then $\Lambda(pp'q) = 0$ holds for all $q \in R$. Thus $pp' \in I_\Lambda$ and I_Λ is an ideal. □

Let $\mathcal{A}_\Lambda = R/I_\Lambda$ be the quotient algebra of polynomials modulo the ideal I_Λ, which, as Proposition 5.2 states, is the kernel of H_Λ. The rank of H_Λ is the dimension of $\mathcal{A}_\Lambda$ as a $\mathbb{K}$-vector space.

A quotient algebra $\mathcal{A}$ is a Gorenstein algebra if there exists a non-degenerate bilinear form Q on $\mathcal{A}$, such that for all polynomials $f, g, h \in \mathcal{A}$ it holds that $Q(f, gh) = Q(fg, h)$, or equivalently, if there exists $\Lambda \in \mathcal{A}^*$ such that $(f, g) \in \mathcal{A} \times \mathcal{A} \mapsto \Lambda(f\, g)$ is non-degenerate. Equivalently, $\mathcal{A}$ is a Gorenstein algebra iff $\mathcal{A}^*$ is a free $\mathcal{A}$-module generated by one element $\Lambda \in \mathcal{A}^*$: $\mathcal{A}^* = \mathcal{A} \star \Lambda$. See e.g. [25] for more details. The set $R \star \Lambda$ is also called the inverse system generated by Λ [41].

Proposition 5.3. *The dual space $\mathcal{A}_\Lambda^*$ of $\mathcal{A}_\Lambda$, can be identified with the set $D = \{q \star \Lambda \mid q \in R\}$ and $\mathcal{A}_\Lambda$ is a Gorenstein algebra.*

Proof. Let $D = \{q \star \Lambda; q \in R\}$ be the inverse system generated by Λ. By definition,

$$D^\perp = \{p \in R \mid \forall q \in R, q \star \Lambda(p) = \Lambda(pq) = 0\}.$$

Thus $D^\perp = I_\Lambda$, which is the ideal of the kernel of H_Λ (Proposition 5.2). Since ${\mathcal{A}_\Lambda}^* = I_\Lambda^\perp$ is the set of linear forms in R^* which vanish on I_Λ, we deduce that ${\mathcal{A}_\Lambda}^* = I_\Lambda^\perp = D^{\perp\perp} = D$.

It holds that $p \in I_\Lambda$ or $p \equiv 0$ in $\mathcal{A}_\Lambda$, because $p \star \Lambda = 0$. Hence, ${\mathcal{A}_\Lambda}^*$ is a free rank 1 $\mathcal{A}_\Lambda$-module (generated by Λ). Thus $\mathcal{A}_\Lambda$ is a Gorenstein algebra. □

Definition 5.4. *For any $B \subset R$ let $B^+ = B \cup x_1 B \cup \cdots \cup x_n B$ and $\partial B = B^+ - B$.*

Proposition 5.5. *Assume that* $\mathrm{rank}(H_\Lambda) = r < \infty$ *and let $B = \{b_1, \ldots, b_r\} \subset R$ such that $\mathbb{H}_\Lambda^B$ is invertible. Then $b_1, \ldots, b_r$ is a basis of $\mathcal{A}_\Lambda$. If $1 \in \langle B\rangle$ then the ideal I_Λ is generated by* $\ker H_\Lambda^{B^+}$.

Proof. Let us first prove that $\{b_1, \ldots, b_r\} \cap I_\Lambda = \{0\}$. Let $p \in \langle b_1, \ldots, b_r\rangle \cap I_\Lambda$. Then $p = \sum_i p_i\, b_i$ with $p_i \in \mathbb{K}$ and $\Lambda(p\, b_j) = 0$. The second equation implies that $\mathbb{H}_\Lambda^B \cdot \mathbf{p} = \mathbf{0}$, where $\mathbf{p} = [p_1, \ldots, p_r]^t \in \mathbb{K}^r$. Since $\mathbb{H}_\Lambda^B$ is invertible, this implies that $\mathbf{p} = \mathbf{0}$ and $p = 0$.

As a consequence, we deduce that $b_1 \star \Lambda, \ldots, b_r \star \Lambda$ are linearly independent elements of R^*. This is so, because otherwise there exists $\mathbf{m} = [\mu_1, \ldots, \mu_r]^\top \neq \mathbf{0}$, such that $\mu_1(b_1 \star \Lambda) + \cdots + \mu_r(b_r \star \Lambda) = (\mu_1 b_1 + \cdots + \mu_r b_r) \star \Lambda = 0$. Since $\{b_1, \ldots, b_r\} \cap \mathsf{Kernel}(\mathbb{H}_\Lambda) = \{0\}$, we have a contradiction.

Consequently, $\{b_1 \star \Lambda, \ldots, b_r \star \Lambda\}$ span the image of H_Λ. For any $p \in R$, it holds that $p \star \Lambda = \sum_{i=1}^r \mu_i (b_i \star \Lambda)$ for some $\mu_1, \ldots, \mu_r \in \mathbb{K}$. We deduce that $p - \sum_{i=1}^r \mu_i b_i \in I_\Lambda$. This yields the decomposition $R = B \oplus I_\Lambda$, and shows that $b_1, \ldots, b_r$ is a basis of $\mathcal{A}_\Lambda$.

If $1 \in \langle B\rangle$, the ideal I_Λ is generated by the relations $x_j b_k - \sum_{i=1}^r \mu_i^{j,k} b_i \in I_\Lambda$. These are precisely in the kernel of $H_\Lambda^{B^+}$. □

Proposition 5.6. *If* $\mathrm{rank}(H_\Lambda) = r < \infty$, *then $\mathcal{A}_\Lambda$ is of dimension r over $\mathbb{K}$ and there exist $\zeta_1, \ldots, \zeta_d \in \mathbb{K}^n$ where $d \leq r$, and $p_i \in \mathbb{K}[\partial_1, \ldots, \partial_n]$, such that*

$$\Lambda = \sum_{i=1}^{d} \mathbb{1}_{\zeta_i} \circ p_i(\partial) \tag{5}$$

Moreover the multiplicity of ζ_i is the dimension of the vector space spanned by the inverse system generated by $\mathbb{1}_{\zeta_i} \circ p_i(\partial)$.

Proof. Since $\mathrm{rank}(\mathbb{H}_\Lambda) = r$, the dimension of the vector space $\mathcal{A}_\Lambda$ is also r. Thus the number of zeros of the ideal I_Λ, denoted $\{\zeta_1, \ldots, \zeta_d\}$, is at most r, viz. $d \leq r$. We can apply the structure Theorem [25, Th. 7.34, p. 185] in order to get the decomposition (5) and the multplicity of the roots. □

If the field $\mathbb{K}$ is of characteristic 0, the inverse system of $\mathbb{1}_{\zeta_i} \circ p_i(\partial)$ is isomorphic to the vector space generated by p_i and its derivatives of any order with respect to the variables ∂_i. In general characteristic, we replace the derivatives by the product by the "inverse" of the variables [44], [25].

Definition 5.7. *For $f \in S^d$, we call generalized decomposition of f^* a decomposition such that $f^* = \sum_{i=1}^{d} \mathbb{1}_{\zeta_i} \circ p_i(\partial)$ where the sum for $i = 1, \ldots, d$ of the dimensions of the vector spaces spanned by the inverse system generated by $\mathbb{1}_{\zeta_i} \circ p_i(\partial)$ is minimal. This minimal sum of dimensions is called the length of f.*

This definition extends the definition introduced in [31] for binary forms. The length of f^* is the rank of the corresponding Hankel operator H_Λ.

Theorem 5.8. *Let $\Lambda \in R^*$. $\Lambda = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{\zeta_i}$ with $\lambda_i \neq 0$ and ζ_i distinct points of $\mathbb{K}^n$, iff $\operatorname{rank} H_\Lambda = r$ and I_Λ is a radical ideal.*

Proof. If $\Lambda = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{\zeta_i}$, with $\lambda_i \neq 0$ and ζ_i distinct points of $\mathbb{K}^n$. Let $\{e_1, \ldots, e_r\}$ be a family of interpolation polynomials at these points: $e_i(\zeta_j) = 1$ if $i = j$ and 0 otherwise. Let I_ζ be the ideal of polynomials which vanish at $\zeta_1, \ldots, \zeta_r$. It is a radical ideal. Clearly we have $I_\zeta \subset I_\Lambda$. For any $p \in I_\Lambda$, and $i = 1, \ldots, r$, we have $p \star \Lambda(e_i) = \Lambda(p\, e_i) = p(\zeta_i) = 0$, which proves that $I_\Lambda = I_\zeta$ is a radical ideal. H_Λ is of rank r because the quotient $\mathcal{A}_\Lambda$ is generated by the interpolation polynomials $e_1, \ldots, e_r$.

Conversely if $\operatorname{rank} H_\Lambda = r$ and I_Λ is radical, then by Proposition 5.6, $\Lambda = \sum_{i=1}^{r} \mathbb{1}_{\zeta_i} \circ p_i(\partial)$ with polynomials p_i of degree 0 since the multiplicity of ζ_i is 1. This concludes the proof of the equivalence. □

In order to compute the zeros of an ideal I_Λ when we know a basis of $\mathcal{A}_\Lambda$, we exploit the properties of the operators of multiplication in $\mathcal{A}_\Lambda$: $M_a : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, such that $\forall b \in \mathcal{A}_\Lambda, M_a(b) = a\, b$ and its transposed operator $M_a^t : \mathcal{A}_\Lambda^* \to \mathcal{A}_\Lambda^*$, such that for $\forall \gamma \in \mathcal{A}_\Lambda^*, M_a^\top(\gamma) = a \star \gamma$.

The following proposition connects the multiplication tables with Hankel matrices, based on the properties of duality.

Proposition 5.9. *For any linear form $\Lambda \in R^*$ such that $\operatorname{rank} H_\Lambda < \infty$ and any $a \in \mathcal{A}_\Lambda$, we have*

$$H_{a\star\Lambda} = M_a^t \circ H_\Lambda \tag{6}$$

Proof. By definition, $\forall p \in R, H_{a\star\Lambda}(p) = a\, p \star \Lambda = a \star (p \star \Lambda) = M_a^\top \circ H_\Lambda(p)$. □

We have the following well-known theorem:

Theorem 5.10. *Assume that $\mathcal{A}_\Lambda$ is a finite dimensional vector space. Then $\Lambda = \sum_{i=1}^{d} \mathbb{1}_{\zeta_i} \circ p_i(\partial)$ for $\zeta_i \in \mathbb{K}^n$ and $p_i(\partial) \in \mathbb{K}[\partial_1, \ldots, \partial_n]$ and*

- *the eigenvalues of the operators M_a and M_a^t, are given by $\{a(\zeta_1), \ldots, a(\zeta_r)\}$.*
- *the common eigenvectors of the operators $(M_{x_i}^t)_{1 \leq i \leq n}$ are (up to scalar) $\mathbb{1}_{\zeta_i}$.*

Proof. [18, 17, 25] □

Using the previous proposition and theorem 5.10, we can recover the points $\zeta_i \in \mathbb{K}^n$ by eigenvector computation as follows. Assume that $B \subset R$ with $|B| = \operatorname{rank}(H_\Lambda)$ and H_Λ^B invertible, then equation (6) and its transposition yield

$$\mathbb{H}_{a\star\Lambda}^B = \mathbb{M}_a^t \mathbb{H}_\Lambda^B = \mathbb{H}_\Lambda^B\, \mathbb{M}_a,$$

where $\mathbb{M}_a$ is the matrix of multiplication by a in the basis B of $\mathcal{A}_\Lambda$. By virtue of theorem 5.10, the solutions v of the generalized eigenvalue problem

$$(\mathbb{H}_{a\star\Lambda} - \lambda\, \mathbb{H}_\Lambda)\mathbf{v} = \mathbb{O} \tag{7}$$

for all $a \in R$, yield the common eigenvectors $\mathbb{H}_\Lambda^B \mathbf{v}$ of $\mathbb{M}_a^t$, that is, the evaluation $\mathbb{1}_{\zeta_i}$ at the roots. Therefore, these eigenvectors $\mathbb{H}_\Lambda^B \mathbf{v}$ are up to a scalar, the vectors $[b_1(\zeta_i), \ldots, b_r(\zeta_i)]$ $(i = 1, \ldots, r)$. Notice that it is sufficient to compute the common eigenvectors of $(\mathbb{H}_{x_i \star \Lambda}, \mathbb{H}_\Lambda)$ for $i = 1, \ldots, n$.

If $\Lambda = \sum_{i=1}^d \lambda_i \mathbb{1}_{\zeta_i}$ $(\lambda_i \neq 0)$, then the roots are simple, and the computation of one eigenvector is enough: for any $a \in R$, $\mathbb{M}_a$ is diagonalizable and the generalized eigenvectors $\mathbb{H}_\Lambda^B \mathbf{v}$ are, up to a scalar factor, the evaluations $\mathbb{1}_{\zeta_i}$ at the roots.

6. Truncated Hankel operators

Coming back to our problem of symmetric tensor decomposition, $f = \sum_{|\alpha| \leq d} c_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha \in R_d$ admits an affine decomposition of rank r, iff $\Lambda(\mathbf{x}^\alpha) = c_\alpha$ for all $|\alpha| \leq d$ where

$$\Lambda = \sum_{i=1}^{r} \lambda_i \, \mathbb{1}_{\zeta_i},$$

for some distinct $\zeta_1, \ldots, \zeta_r \in \mathbb{K}^n$ and some $\lambda_i \in \mathbb{K} - \{0\}$. Then, by virtue of theorem 5.8, H_Λ is of rank r and I_Λ is radical. Conversely, let H_Λ of rank r and a radical ideal I_Λ, which coincides up to degree d with Λ_d. Then by Proposition 5.6, $\Lambda = \sum_{i=1}^r \lambda_i \, \mathbb{1}_{\zeta_i}$ and f can be decomposed as a sum of r d^{th}-powers of linear forms. The problem of decomposition of f can thus be reformulated as follows:

> *Given $f^* \in {R_d}^*$ find the smallest r such that there exists $\Lambda \in R^*$ which extends f^* with H_Λ of rank r and I_Λ a radical ideal.*

In this section, we are going to characterize under which conditions f^* can be extended to $\Lambda \in R^*$ with H_Λ of rank r.

We need the following technical property on the bases of $\mathcal{A}_\Lambda$:

Definition 6.1. *Let B be a subset of monomials in R. We say that B is connected to 1 if $\forall m \in B$ either $m = 1$ or there exists $i \in [1, n]$ and $m' \in B$ such that $m = x_i \, m'$.*

Let $B \subset R_d$ be a set of monomials of degree at most d, connected to 1. We consider the Hankel matrix

$$\mathcal{H}_\Lambda^B = (h_{\alpha+\beta})_{\alpha,\beta \in B},$$

with $h_\alpha = f^*(\mathbf{x}^\alpha) = c_\alpha$ if $|\alpha|$ is at most d and otherwise h_α is a variable. The set of all these new variables is denoted $\mathbf{h}$.

Suppose that $\mathcal{H}_\Lambda^B$ is invertible in $\mathbb{K}(\mathbf{h})$, then we define the multiplication operators

$$\mathcal{M}_i^B(\mathbf{h}) := (\mathcal{H}_\Lambda^B)^{-1} \mathcal{H}_{x_i \star \Lambda}^B.$$

The following result characterizes the cases where $\mathbb{K}[\mathbf{x}] = \langle B \rangle \oplus I_\Lambda$:

Theorem 6.2. *Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ be a set of monomials of degree at most d, connected to 1 and let Λ be a linear form in $\langle B \cdot B^+ \rangle_d^*$. Let $\Lambda(\mathbf{h})$ be the linear form of $\langle B \cdot B^+ \rangle^*$ defined by $\Lambda(\mathbf{h})(\mathbf{x}^\alpha) = \Lambda(\mathbf{x}^\alpha)$ if $|\alpha|$ is at most d and $h_\alpha \in \mathbb{K}$ otherwise. Then $\Lambda(\mathbf{h})$ admits an extension $\tilde{\Lambda} \in R^*$ such that $H_{\tilde{\Lambda}}$ is of rank r with B a basis of $A_{\tilde{\Lambda}}$ iff*

$$\mathcal{M}_i^B(\mathbf{h}) \circ \mathcal{M}_j^B(\mathbf{h}) - \mathcal{M}_j^B(\mathbf{h}) \circ \mathcal{M}_i^B(\mathbf{h}) = 0 \quad (1 \leq i < j \leq n) \tag{8}$$

and $\det(\mathcal{H}_\Lambda^B)(\mathbf{h}) \neq 0$. *Moreover, such a $\tilde{\Lambda}$ is unique.*

Proof. If there exists $\tilde{\Lambda} \in R^*$ which extends $\Lambda(\mathbf{h})$, with $H_{\tilde{\Lambda}}$ of rank r then the tables of multiplications by the variables x_i are $M_i = (\mathcal{H}_{\Lambda}^{B})^{-1}\mathcal{H}_{x_i \star \Lambda}^{B}$ (proposition 5.9) and they commute.

Conversely suppose that these matrices commute. Then by [43], we have $\mathbb{K}[\mathbf{x}] = \langle B\rangle \oplus (K)$, where K is the set of border relations $x_i m - \mathcal{M}_i(m)$ for $m \in B$ and $i = 1, \ldots, n$. Let π_B be the projection of R on $\langle B\rangle$ along (K).

We define $\tilde{\Lambda} \in R^*$ as follows: $\forall p \in R, \tilde{\Lambda}(p) = \Lambda(p(\mathcal{M})(1))$ where $p(\mathcal{M})$ is the operator obtained by substitution of the variables x_i by the commuting operators $\mathcal{M}_i$. Notice that $p(\mathcal{M})$ is also the operator of multiplication by p modulo (K).

By construction, $(K) \subset \ker H_{\tilde{\Lambda}}$ and B is a generating set of $\mathcal{A}_{\tilde{\Lambda}}$.

Let us prove by induction on the degree of $b \in B$ that for all $b' \in B$, we have $\Lambda(b\,b') = \Lambda(b(M)\,(b'))$. The property is true for $b = 1$. If $b \neq 1$, then $b = x_i\, b''$ for some variable x_i and some element $b'' \in B$ of degree smaller than b, because B is connected to 1,

By construction of the operators M_i, we have $\Lambda(x_i b''\, b') = \Lambda(b'' M_i(b'))$. By induction hypothesis, we deduce that $\Lambda(b\,b') = \Lambda(b''(M) \circ M_i(b')) = \Lambda(b(M)(b'))$. It holds

$$\Lambda(b\,b') = \Lambda(b''(M) \circ M_i(b')) = \Lambda(b(M) \circ b'(M)(1)) = \Lambda((b\,b')(M)(1)) = \tilde{\Lambda}(b\,b'),$$

because $b' = b'(M)(1)$ for all $b' \in B$ (the multiplication of 1 by b is represented by $b \in \langle B\rangle$ modulo (K)). This shows that $\Lambda = \tilde{\Lambda}$ on $B \cdot B$. As $\det(\mathcal{H}_{\Lambda}^{B}) \neq 0$, we deduce that B is a basis of $\mathcal{A}_{\tilde{\Lambda}}$ and that $H_{\tilde{\Lambda}}$ is of rank r.

Suppose there exists another $\Lambda' \in R^*$ which extends $\Lambda(\mathbf{h}) \in \langle B \cdot B^+\rangle^*$ with $\operatorname{rank} H_{\Lambda'} = r$. By proposition 5.5, $\ker H_{\Lambda'}$ is generated by $\ker H_{\Lambda'}^{B \cdot B^+}$ and thus coincides with $\ker H_{\tilde{\Lambda}}$. The two elements of R^* must be equal, because Λ' coincides with $\tilde{\Lambda}$ on B. This completes the proof of the theorem. □

The degree of these commutation relations is at most 2 in the coefficients of the multiplications matrices $\mathcal{M}_i$. Direct computation yields the following results, for $m \in B$:

- If $x_i, m \in B, x_j\, m \in B$ then $(\mathcal{M}_i^B \circ \mathcal{M}_j^B - \mathcal{M}_j^B \circ \mathcal{M}_i^B)(m) \equiv 0$ in $\mathbb{K}(\mathbf{h})$.
- If $x_i m \in B$, $x_j\, m \notin B$ then $(\mathcal{M}_i^B \circ \mathcal{M}_j^B - \mathcal{M}_j^B \circ \mathcal{M}_i^B)(m)$ is of degree 1 in the coefficients of $\mathcal{M}_i, \mathcal{M}_j$.
- If $x_i m \notin B$, $x_j\, m \notin B$ then $(\mathcal{M}_i^B \circ \mathcal{M}_j^B - \mathcal{M}_j^B \circ \mathcal{M}_i^B)(m)$ is of degree 2 in the coefficients of $\mathcal{M}_i, \mathcal{M}_j$.

We are going to give an equivalent characterization of the extension property, based on rank bounds.

Theorem 6.3. *Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ be a set of monomials of degree at most d, connected to 1. Then, the linear form $f^* \in S_d^*$ admits an extension $\Lambda \in R^*$ such that H_Λ is of rank r with B a basis of $\mathcal{A}_\Lambda$ iff there exists an $\mathbf{h}$ such that all $(r+1) \times (r+1)$ minors of $\mathcal{H}_{\Lambda}^{B^+}(\mathbf{h})$ vanish and $\det(\mathcal{H}_{\Lambda}^{B})(\mathbf{h}) \neq 0$.*

Proof. Clearly, if there exists $\Lambda \in R^*$ which extends $f^* \in S_d^*$ with H_Λ of rank r, then all $(r+1) \times (r+1)$ minors of $\mathcal{H}_{\Lambda}^{B^+}(\mathbf{h})$ vanish.

Conversely, if $\mathcal{H}_{\Lambda}^{B^+}(\mathbf{h})$ and $\mathcal{H}_{\Lambda}^{B}(\mathbf{h})$ are of rank r, then by [40, Theorem 1.4] there exists a unique $\tilde{\Lambda} \in R^*$ such that H_Λ is of rank r, and which coincides with Λ on $\langle B^+ \cdot B^+\rangle$. □

Proposition 6.4. *Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ be a set of monomials of degree at most d, connected to 1. Then, the linear form $f^* \in S_d^*$ admits an extension $\Lambda \in R^*$ such that H_Λ is of rank r with B a basis of $\mathcal{A}_\Lambda$ iff*

$$\mathbb{H}_\Lambda^{B^+} = \begin{pmatrix} \mathbb{H} & \mathbb{G} \\ \mathbb{G}^t & \mathbb{J} \end{pmatrix}, \tag{9}$$

with $\mathbb{H} = \mathbb{H}_\Lambda^B$ and

$$\mathbb{G} = \mathbb{H}\,\mathbb{W}, \mathbb{J} = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}. \tag{10}$$

for some matrix $\mathbb{W} \in \mathbb{K}^{B \times \partial B}$.

Proof. In the virtue of theorem 6.3, $f^* \in S_d^*$ admits a (unique) extension $\Lambda \in R^*$ such that H_Λ is of rank r with B a basis of $\mathcal{A}_\Lambda$, iff H^{B^+} is of rank r. Let us decompose its matrix $\mathbb{H}^{B^+}$ as (9) with $\mathbb{H} = \mathbb{H}_\Lambda^B$.

If we have $\mathbb{G} = \mathbb{H}\,\mathbb{W}, \mathbb{J} = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}$, then

$$\begin{pmatrix} \mathbb{H} & \mathbb{H}\,\mathbb{W} \\ \mathbb{W}^t\mathbb{H} & \mathbb{W}^t\,\mathbb{H}\,\mathbb{W} \end{pmatrix}$$

is clearly of rank at most rank $\mathbb{H}$.

Conversely, suppose that $\mathbb{H}_\Lambda^{B^+} = \operatorname{rank} \mathbb{H}$. This implies that the image of $\mathbb{G}$ is in the image of $\mathbb{H}$. Thus, there exists $\mathbb{W} \in \mathbb{K}^{B \times \partial B}$ such that $\mathbb{G} = \mathbb{H}\,\mathbb{W}$. Without loss of generality, we can assume that the r first columns of $\mathbb{H}$ ($r = \operatorname{rank} \mathbb{H}$) are linearly independent. Assume that we choose $\mathbb{W}$ such that the i^{th} column of $\mathbb{G}$ is the linear combination of the r first columns with coefficients corresponding to the i column $\mathbb{W}_i$ of $\mathbb{W}$. The same relation holds for the whole column of this matrix, because $\operatorname{rank} \mathbb{H}_\Lambda^{B^+} = r$. Thus we have $\mathbb{J} = \mathbb{G}^t\,\mathbb{W} = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}$. □

Notice that if $\mathbb{H}$ is invertible, $\mathbb{W}$ is uniquely determined. In this case, we easily check that $\ker \mathbb{H}_\Lambda^{B^+} = \begin{pmatrix} \mathbb{W} \\ -\mathbb{I} \end{pmatrix}$.

This leads to the following system in the variables $\mathbf{h}$ and the coefficients $\mathbf{w}$ of matrix $\mathbb{W}$. It characterizes the linear forms $f^* \in S_d^*$ that admit an extension $\Lambda \in R^*$ such that H_Λ is of rank r with B a basis of $\mathcal{A}_\Lambda$.

$$\mathcal{H}_\Lambda^{B,\partial B}(\mathbf{h}) - \mathcal{H}_\Lambda^B(\mathbf{h})\,\mathbb{W}(\mathbf{w}) = 0, \quad \mathcal{H}_\Lambda^{\partial B,\partial B}(\mathbf{h}) - \mathbb{W}^t(\mathbf{w})\,\mathcal{H}_\Lambda^B(\mathbf{h})\,\mathbb{W}(\mathbf{w}) = 0 \tag{11}$$

with $\det(\mathcal{H}_\Lambda^B(\mathbf{h})) \neq 0$.

The matrix $\mathcal{H}_\Lambda^{B^+}$ is a quasi-Hankel matrix [44], whose structure is imposed by equality (linear) constraints on its entries. If $\mathbb{H}$ is known (ie. $B \times B \subset R_d$), the number of independent parameters in $\mathcal{H}_\Lambda^{B,B^+}(\mathbf{h})$ or in $\mathbb{W}$ is the number of monomials in $B \times \partial B - R_d$. By Proposition 6.4, the rank condition is equivalent to the quadratic relations $\mathbb{J} - \mathbb{W}^t\mathbb{H}^t\,\mathbb{W} = 0$ in these unknowns.

If $\mathbb{H}$ is not completely known, the number of parameters in $\mathbb{H}$ is the number of monomials in $B \times B - R_d$. The number of independent parameters in $\mathcal{H}_\Lambda^{B,\partial B}(\mathbf{h})$ or in $\mathbb{W}$ is then $B \times \partial B - R_d$.

The system (11) is composed of linear equations deduced from quasi-Hankel structure, quadratic relations for the entries in $B \times \partial B$ and cubic relations for the entries in $B \times \partial B$ in the unknown parameters $\mathbf{h}$ and $\mathbf{w}$.

We are going to use these characterizations explicitly in our new algorithm for minimal tensor decomposition.

7. Symmetric tensor decomposition algorithm

Our algorithm for decomposing a symmetric tensor as sum of rank one symmetric tensors generalizes the algorithm of Sylvester [53], devised for dimension two tensors. See also [12].

Consider the homogeneous polynomial $f(\mathbf{x})$ in (3) that we want to decompose. We may assume without loss of generality, that for at least one variable, say x_0, all its coefficients in the decomposition are nonzeros, i.e. $k_{i,0} \neq 0$, for $1 \leq i \leq r$. We dehomogenize f with respect to this variable and we denote this polynomial by $f^a := f(1, x_1, \ldots, x_n)$. We want to decompose the polynomial $f^a(\mathbf{x}) \in R_d$ as a sum of powers of linear forms, i.e.

$$f(\mathbf{x}) = \sum_{i=1}^{r} \lambda_i \,(1 + k_{i,1}x_1 + \cdots + k_{i,n}x_n)^d = \sum_{i=1}^{r} \lambda_i \, \mathbf{k}_i(\mathbf{x})^d$$

Equivalently, we want to decompose its corresponding dual element $f^* \in {R_d}^*$ as a linear combination of evaluations over the distinct points $\mathbf{k}_i := (k_{i,1}, \cdots, k_{i,n})$:

$$f^* = \sum_{i=1}^{r} \lambda_i \, \mathbb{1}_{\mathbf{k}_i}$$

(we refer the reader to the end of Section 4.3).

Assume that we know the value of r. As we have seen previously, knowing the value of Λ on polynomials of degree high enough allows us to compute the table of multiplications modulo the kernel of $\mathbb{H}_\Lambda$. By Theorem 5.10, solving the generalized eigenvector problem $(\mathbb{H}_{x_1 \star \Lambda} - \lambda \, \mathbb{H}_\Lambda)\mathbf{v} = \mathbb{O}$, we recover the points of evaluation $\mathbf{k}_i$. By solving a linear system, we then deduce the value of $\lambda_1, \ldots, \lambda_r$. Thus, the goal of the following algorithm is to extend f^* to a large enough set of polynomials, in order to be able to run this eigenvalue computation.

The critical part in this algorithm is the completion of step two. Instead of the commutation relations, one can use the result of Proposition 6.4.

The two examples that follow demonstrate how the algorithm works.

7.1. First Example

1. Convert the symmetric tensor to the corresponding homogeneous polynomial.

> Assume that a tensor of dimension three and order 5, or equivalently a three-way array of dimension 5, corresponds to the following homogeneous polynomial
>
> $f = -1549440\, x_0 x_1 {x_2}^3 + 2417040\, x_0 {x_1}^2 {x_2}^2 + 166320\, {x_0}^2 x_1 {x_2}^2 - 829440\, x_0 {x_1}^3 x_2 - 5760\, {x_0}^3 x_1 x_2 - 222480\, {x_0}^2 {x_1}^2 x_2 + 38\, {x_0}^5 - 497664\, {x_1}^5 - 1107804\, {x_2}^5 - 120\, {x_0}^4 x_1 + 180\, {x_0}^4 x_2 + 12720\, {x_0}^3 {x_1}^2 + 8220\, {x_0}^3 {x_2}^2 - 34560\, {x_0}^2 {x_1}^3 - 59160\, {x_0}^2 {x_2}^3 + 831840\, x_0 {x_1}^4 + 442590\, x_0 {x_2}^4 - 5591520\, {x_1}^4 x_2 + 7983360\, {x_1}^3 {x_2}^2 - 9653040\, {x_1}^2 {x_2}^3 + 5116680\, x_1 {x_2}^4$.
>
> The minimum decomposition of the polynomial as a sum of powers of linear forms is
>
> $$(x_0 + 2\,x_1 + 3\,x_2)^5 + (x_0 - 2\,x_1 + 3\,x_2)^5 + \frac{1}{3}(x_0 - 12\,x_1 - 3\,x_2)^5 + \frac{1}{5}(x_0 + 12\,x_1 - 13\,x_2)^5,$$
>
> that is, the corresponding tensor is of rank 4.

Algorithm 7.1: Symmetric tensor decomposition

Input: A homogeneous polynomial $f(x_0, x_1, \ldots, x_n)$ of degree d.
Output: A decomposition of f as $f = \sum_{i=1}^{r} \lambda_i \, \mathbf{k}_i(\mathbf{x})^d$ with r minimal.

- Compute the coefficients of f^*: $c_\alpha = a_\alpha \binom{d}{\alpha}^{-1}$, for $|\alpha| \leq d$;
- $r := 1$;
- **Repeat**
  1. Compute a set B of monomials of degree at most d connected to one with $|B| = r$;
  2. Find parameters $\mathbf{h}$ s.t. $\det(\mathbb{H}_\Lambda^B) \neq 0$ and the operators $\mathbb{M}_i = \mathbb{H}_{x_i\Lambda}^B (\mathbb{H}_\Lambda^B)^{-1}$ commute.
  3. If there is no solution, restart the loop with $r := r + 1$.
  4. Else compute the $n \times r$ eigenvalues $\zeta_{i,j}$ and the eigenvectors $\mathbf{v}_j$ s.t. $\mathbb{M}_i \mathbf{v}_j = \zeta_{i,j} \mathbf{v}_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$.

  until the eigenvalues are simple.
- Solve the linear system in $(\ell_j)_{j=1,\ldots,k}$: $\Lambda = \sum_{j=1}^{r} \ell_j \mathbf{1}_{\zeta_j}$ where $\zeta_j \in \mathbb{K}^n$ are the eigenvectors found in step 4.

2. Compute the actual number of variables needed.
For algorithms computing the so-called number of *essential* variables, the reader may refer to the work of Oldenburger [45] or Carlini [8].

In our example the number of essential variable is three, so we have nothing to do.

3. Compute the matrix of the quotient algebra.
We form a $\binom{n+d-1}{d} \times \binom{n+d-1}{d}$ matrix, the rows and the columns of which correspond to the coefficients of the polynomial in the dual base. The map for this is

$$a_{j_0\, j_1 \ldots j_n} \mapsto c_{j_0\, j_1 \ldots j_n} := a_{j_0\, j_1 \ldots j_n} \binom{d}{j_0, \ldots, j_n}^{-1},$$

where $a_{d_0\, d_1 \ldots d_n}$ is the coefficient of the monomial $x_0^{j_0} \cdots x_n^{j_n}$ in f. Recall that, since the polynomial is homogeneous, $\sum_{i=1}^{n} j_i = d$.
This matrix is called quasi-Hankel [44] or Catalecticant [31].

Part of the corresponding matrix follows. The whole matrix is 21×21. We show only the 10×10 principal minor.

	1	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3
1	38	−24	36	1272	−288	822	−3456	−7416	5544	−5916
x_1	−24	1272	−288	−3456	−7416	5544	166368	−41472	80568	−77472
x_2	36	−288	822	−7416	5544	−5916	−41472	80568	−77472	88518
x_1^2	1272	−3456	−7416	166368	−41472	80568	−497664	−1118304	798336	−965304
x_1x_2	−288	−7416	5544	−41472	80568	−77472	−1118304	798336	−965304	1023336
x_2^2	822	5544	−5916	80568	−77472	88518	798336	−965304	1023336	−1107804
x_1^3	−3456	166368	−41472	−497664	−1118304	798336	$h_{6,0,0}$	$h_{5,1,0}$	$h_{4,2,0}$	$h_{3,3,0}$
$x_1^2x_2$	−7416	−41472	80568	−1118304	798336	−965304	$h_{5,1,0}$	$h_{4,2,0}$	$h_{3,3,0}$	$h_{2,4,0}$
$x_1x_2^2$	5544	80568	−77472	798336	−965304	1023336	$h_{4,2,0}$	$h_{3,3,0}$	$h_{2,4,0}$	$h_{1,5,0}$
x_2^3	−5916	−77472	88518	−965304	1023336	−1107804	$h_{3,3,0}$	$h_{2,4,0}$	$h_{1,5,0}$	$h_{0,6,0}$

Notice that we do not know the elements in some positions of the matrix. In general we do not know the elements that correspond to monomials with (total) degree higher than 5.

4. Extract a principal minor of full rank.
   We re-arrange the rows and the columns of the matrix so that there is a principal minor of full rank, R. We call this minor $\mathbb{H}_\Lambda$. In order to do that we try to put the matrix in row echelon form, using elementary row and column operations.

In our example the 4×4 principal minor is of full rank, so there is no need for re-arranging the matrix. The matrix $\mathbb{H}_\Lambda$ is

$$\mathbb{H}_\Lambda = \begin{bmatrix} 38 & -24 & 36 & 1272 \\ -24 & 1272 & -288 & -3456 \\ 36 & -288 & 822 & -7416 \\ 1272 & -3456 & -7416 & 166368 \end{bmatrix}$$

Notice that the columns of the matrix correspond to the monomials $\{1, x_1, x_2, x_1^2\}$.

5. We compute the "shifted" matrix $\mathbb{H}_{x_1\Lambda} = x_1\mathbb{H}_\Lambda$. If $\{\mathbf{x}^{\boldsymbol{\alpha}}\}$ is the set of monomials indexing the columns of $\mathbb{H}_\Lambda$, then the columns of $\mathbb{H}_{x_1\Lambda}$ correspond to the set of monomials $\{x_1\,\mathbf{x}^{\boldsymbol{\alpha}}\}$.

The shifted matrix $\mathbb{H}_{x_1\Lambda}$ is

$$\mathbb{H}_{x_1\Lambda} = \begin{bmatrix} -24 & 1272 & -288 & -3456 \\ 1272 & -3456 & -7416 & 166368 \\ -288 & -7416 & 5544 & -41472 \\ -3456 & 166368 & -41472 & -497664 \end{bmatrix}$$

Notice that the columns correspond to the monomials $\{x_1, x_1^2, x_1x_2, x_1^3\}$, which are just the corresponding monomials of the columns of $\mathbb{H}_\Lambda$, i.e. $\{1, x_1, x_2, x_1^2\}$, multiplied by x_1.

We assume for the moment that all the elements of the matrices $\mathbb{H}_\Lambda$ and $\mathbb{H}_{x_1\Lambda}$ are known. If this is not the case, then we can compute the unknown entries of the matrix, using either necessary and sufficient conditions of the quotient algebra, e.g. it holds that $\mathbb{M}_{x_i}\mathbb{M}_{x_j} -$

$\mathbb{M}_{x_j}\mathbb{M}_{x_i} = \mathbb{O}$ [43] for any $i, j \in \{1, \ldots, n\}$. There are other algorithms to extend a moment matrix, e.g. [39, 38, 19].

6. We solve the equation $(\mathbb{H}_{x_1\Lambda} - \lambda\mathbb{H}_{\Lambda})X = 0$.
   We solve the generalized eigenvalue/eigenvector problem using one of the well-known techniques [28]. We normalize the elements of the eigenvectors so that the first element is one, and we read the solutions from the coordinates of the (normalized) eigenvectors.

The normalized eigenvectors of the generalized eigenvalue problem are

$$\begin{bmatrix} 1 \\ -12 \\ -3 \\ 144 \end{bmatrix}, \begin{bmatrix} 1 \\ 12 \\ -13 \\ 144 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

The coordinates of the eigenvectors correspond to the elements $\{1, x_1, x_2, x_1^2\}$. Thus, we can recover the coefficients of x_1 and x_2 in the decomposition from coordinates of the eigenvectors. Recall that the coefficients of x_0 are considered to be one. Thus, the polynomial admits a decomposition

$$f = \ell_1(x_0 + 2x_1 + 3x_2)^5 + \ell_2(x_0 - 2x_1 + 3x_2)^5 + \ell_3(x_0 - 12x_1 - 3x_2)^5 + \ell_4(x_0 + 12x_1 - 13x_2)^5$$

It remains to compute ℓ_i's. We can do this easily by solving an over-determined linear system, which we know has always a solution, since the decomposition exists. Doing that, we deduce $\ell_1 = 1$, $\ell_2 = 1$, $\ell_3 = 1/3$ and $\ell_4 = 1/5$.

7.2. Second Example

One of the assumptions that the previous example fulfills is that all the entries of the matrices needed for the computations are known. However, this is not always the case as the following example shows.

1. Convert the symmetric tensor to the corresponding homogeneous polynomial.

Consider a tensor of dimension three and order 4, that corresponds to the following homogeneous polynomial

$$f = 79\, x_0x_1^3 + 56\, x_0^2x_2^2 + 49\, x_1^2x_2^2 + 4\, x_0x_1x_2^2 + 57\, x_0^3x_1,$$

the rank of which is 6.

2. Compute the actual number of variables needed.

In our example the number of essential variables is three, so we have nothing to do.

3. Compute the matrix of the quotient algebra.

The matrix is 15×15.

	1	x_1	x_2	x_1^2	x_1x_2	x_2^2	x_1^3	$x_1^2x_2$	$x_1x_2^2$	x_2^3	x_1^4	$x_1^3x_2$	$x_1^2x_2^2$	$x_1x_2^3$	x_2^4
1	0	$\frac{57}{4}$	0	0	0	$\frac{28}{3}$	$\frac{79}{4}$	0	$\frac{1}{3}$	0	0	0	$\frac{49}{6}$	0	0
x_1	$\frac{57}{4}$	0	0	$\frac{79}{4}$	0	$\frac{1}{3}$	0	0	$\frac{49}{6}$	0	h_{500}	h_{410}	h_{320}	h_{230}	h_{140}
x_2	0	0	$\frac{28}{3}$	0	$\frac{1}{3}$	0	0	$\frac{49}{6}$	0	0	h_{410}	h_{320}	h_{230}	h_{140}	h_{050}
x_1^2	0	$\frac{79}{4}$	0	0	0	$\frac{49}{6}$	h_{500}	h_{410}	h_{320}	h_{230}	h_{600}	h_{510}	h_{420}	h_{330}	h_{240}
x_1x_2	0	0	$\frac{1}{3}$	0	$\frac{49}{6}$	0	h_{410}	h_{320}	h_{230}	h_{140}	h_{510}	h_{420}	h_{330}	h_{240}	h_{150}
x_2^2	$\frac{28}{3}$	$\frac{1}{3}$	0	$\frac{49}{6}$	0	0	h_{320}	h_{230}	h_{140}	h_{050}	h_{420}	h_{330}	h_{240}	h_{150}	h_{060}
x_1^3	$\frac{79}{4}$	0	0	h_{500}	h_{410}	h_{320}	h_{600}	h_{510}	h_{420}	h_{330}	h_{700}	h_{610}	h_{520}	h_{430}	h_{340}
$x_1^2x_2$	0	0	$\frac{49}{6}$	h_{410}	h_{320}	h_{230}	h_{510}	h_{420}	h_{330}	h_{240}	h_{610}	h_{520}	h_{430}	h_{340}	h_{250}
$x_1x_2^2$	$\frac{1}{3}$	$\frac{49}{6}$	0	h_{320}	h_{230}	h_{140}	h_{420}	h_{330}	h_{240}	h_{150}	h_{520}	h_{430}	h_{340}	h_{250}	h_{160}
x_2^3	0	0	0	h_{230}	h_{140}	h_{050}	h_{330}	h_{240}	h_{150}	h_{060}	h_{430}	h_{340}	h_{250}	h_{160}	h_{070}
x_1^4	0	h_{500}	h_{410}	h_{600}	h_{510}	h_{420}	h_{700}	h_{610}	h_{520}	h_{430}	h_{800}	h_{710}	h_{620}	h_{530}	h_{440}
$x_1^3x_2$	0	h_{410}	h_{320}	h_{510}	h_{420}	h_{330}	h_{610}	h_{520}	h_{430}	h_{340}	h_{710}	h_{620}	h_{530}	h_{440}	h_{350}
$x_1^2x_2^2$	$\frac{49}{6}$	h_{320}	h_{230}	h_{420}	h_{330}	h_{240}	h_{520}	h_{430}	h_{340}	h_{250}	h_{620}	h_{530}	h_{440}	h_{350}	h_{260}
$x_1x_2^3$	0	h_{230}	h_{140}	h_{330}	h_{240}	h_{150}	h_{430}	h_{340}	h_{250}	h_{160}	h_{530}	h_{440}	h_{350}	h_{260}	h_{170}
x_2^4	0	h_{140}	h_{050}	h_{240}	h_{150}	h_{060}	h_{340}	h_{250}	h_{160}	h_{070}	h_{440}	h_{350}	h_{260}	h_{170}	h_{080}

4. Extract a principal minor of full rank.

In our example the 6×6 principal minor is of full rank. The matrix $\mathbb{H}_\Lambda$ is

$$\mathbb{H}_\Lambda = \begin{bmatrix} 0 & \frac{57}{4} & 0 & 0 & 0 & \frac{28}{3} \\ \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \end{bmatrix}$$

The columns (and the rows) of the matrix correspond to the monomials $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$.

5. We compute the "shifted" matrix $\mathbb{H}_{x_1\Lambda} = x_1\mathbb{H}_\Lambda$.

The shifted matrix $\mathbb{H}_{x_1\Lambda}$ is

$$\mathbb{H}_{x_1\Lambda} = \begin{bmatrix} \frac{57}{4} & 0 & 0 & \frac{79}{4} & 0 & \frac{1}{3} \\ 0 & \frac{79}{4} & 0 & 0 & 0 & \frac{49}{6} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{79}{4} & 0 & 0 & h_{500} & h_{410} & h_{320} \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \end{bmatrix}$$

The columns of the matrix correspond to the monomials $\{x_1, x_1^2, x_1x_2, x_1^3, x_1^2x_2, x_1x_2^2\}$ which are the monomials that correspond to the columns of $\mathbb{H}_\Lambda$, i.e. $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, multiplied by x_1.
Since not all the entries of $\mathbb{H}_{x_1\Lambda}$ are known, we need to compute them in order to proceed further.

Consider the following method to extend the matrix of a quotient algebra. In the quotient algebra it holds that $\mathbb{M}_{x_i}\mathbb{M}_{x_j} - \mathbb{M}_{x_j}\mathbb{M}_{x_i} = \mathbb{O}$ [43] for any $i, j \in \{1, \ldots, n\}$, i.e. the matrices of multiplications commute (cf. Section 6).

From Proposition 5.9 we know that $\Delta_i = \mathbb{M}_{x_i}^t \mathbb{H}_\Lambda$, and hence $\mathbb{M}_{x_i}^t = \Delta_i \mathbb{H}_\Lambda^{-1}$, for $1 \leq i \leq n$. We form all the possible matrix equations, $\mathbb{M}_{x_i}\mathbb{M}_{x_j} - \mathbb{M}_{x_j}\mathbb{M}_{x_i} = \mathbb{O}$, there are $\binom{n}{2}$, and we equate their elements to zero. Since the dimension of the matrices is $r \times r$, this leads to at most $\binom{n}{2}r^2$, or $\mathcal{O}(n^2 r^2)$ equations. Note that the equations are, at most of total degree two.

In our example the matrix Δ_2 is

$$\Delta_2 = \begin{bmatrix} 0 & 0 & \frac{28}{3} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{49}{6} & 0 \\ \frac{28}{3} & \frac{1}{3} & 0 & \frac{49}{6} & 0 & 0 \\ 0 & 0 & \frac{49}{6} & h_{410} & h_{320} & h_{230} \\ \frac{1}{3} & \frac{49}{6} & 0 & h_{320} & h_{230} & h_{140} \\ 0 & 0 & 0 & h_{230} & h_{140} & h_{050} \end{bmatrix}$$

Since we have only two variables, there is only one matrix equation,

$$\mathbb{M}_{x_i}\mathbb{M}_{x_j} - \mathbb{M}_{x_j}\mathbb{M}_{x_i} = \mathbb{H}_{x_1\Lambda}\mathbb{H}_\Lambda^{-1}\Delta_2\mathbb{H}_\Lambda^{-1} - \Delta_2\mathbb{H}_\Lambda^{-1}\mathbb{H}_{x_1\Lambda}\mathbb{H}_\Lambda^{-1} = \mathbb{O}.$$

Many of the resuling equations are trivial. After discarding them, we have 6 unknonws $\{h_{500}, h_{410}, h_{320}, h_{230}, h_{140}, h_{050}\}$ and 15 equations.
A solution of the system is the following

$$\{h_{500} = 1, h_{410} = 2, h_{320} = 3, h_{230} = 1.5060, h_{140} = 4.960, h_{050} = 0.056\}.$$

We subsitute these values to $\mathbb{H}_{x_1\Lambda}$ and we continue the algorithm as in the previous example.

Other algorithms to extend a moment matrix with rank constraints, e.g. [39, 38, 19], so called *flat extensions*, are applicable when the $\mathbb{H}_\Lambda$ is positive definite.

6. We solve the equation $(\mathbb{H}_{x_1\Lambda} - \lambda\mathbb{H}_\Lambda)X = 0$.

The normalized eigenvectors of the generalized eigenvalue problem are

$$\begin{bmatrix} 1 \\ -0.830+1.593\,i \\ -0.326-0.0501\,i \\ -1.849-2.645\,i \\ 0.350-0.478\,i \\ 0.103+0.0327\,i \end{bmatrix}, \begin{bmatrix} 1 \\ -0.830-1.593\,i \\ -0.326+0.050\,i \\ -1.849+2.645\,i \\ 0.350+0.478\,i \\ 0.103-0.032\,i \end{bmatrix}, \begin{bmatrix} 1.0 \\ 1.142 \\ 0.836 \\ 1.305 \\ 0.955 \\ 0.699 \end{bmatrix},$$

$$\begin{bmatrix} 10.956 \\ -0.713 \\ 0.914 \\ -0.682 \\ 0.509 \end{bmatrix}, \begin{bmatrix} 1 \\ -0.838+0.130\,i \\ 0.060+0.736\,i \\ 0.686-0.219\,i \\ -0.147-0.610\,i \\ -0.539+0.089\,i \end{bmatrix}, \begin{bmatrix} 1 \\ -0.838-0.130\,i \\ 0.060-0.736\,i \\ 0.686+0.219\,i \\ -0.147+0.610\,i \\ -0.539-0.089\,i \end{bmatrix}.$$

The coordinates of the eigenvectors correspond to the elements $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$ and we can recover the coefficients of x_1 and x_2 in the decomposition.
After, solving the over-constrained linear system for the coefficients of the linear forms we deduce the decomposition

$$\begin{aligned} &(0.517+0.044\,i)\,(x_0-(0.830-1.593\,i)x_1-(0.326+0.050\,i)x_2)^4 \\ +&(0.517-0.044\,i)\,(x_0-(0.830+1.593\,i)x_1-(0.326-0.050\,i)x_2)^4 \\ &+2.958\,(x_0+(1.142)x_1+0.836x_2)^4 \\ &+4.583\,(x_0+(0.956)x_1-0.713x_2)^4 \\ -&(4.288+1.119\,i)\,(x_0-(0.838-0.130\,i)x_1+(0.060+0.736\,i)x_2)^4 \\ -&(4.288-1.119\,i)\,(x_0-(0.838+0.130\,i)x_1+(0.060-0.736\,i)x_2)^4 \end{aligned}$$

8. Conclusions and future work

We proposed an algorithm that computes symmetric tensor decompositions, extending Sylvester's algorithm. The main ingredients were i) reformulation of the problem in a dual space, ii) exploitation of the properties of multivariate Hankel operators and Gorenstein algebra, iii) an effective method for solving the truncated Hankel problem, iv) deduction of the decomposition by solving a generalized eigenvalue problem.

There are still several questions that remain open, on which we are currently working. First, the (arithmetic and Boolean) complexity of the algorithm has not been evaluated. Second, one may ask oneself whether the decomposition can still be computed if some entries of the tensor are not known (case of missing data). Last, it is suitable to extend the procedure we have proposed to nonsymmetric tensors.

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A. Ternary cubics

As an application, we present the decomposition of all the types of ternary cubics. The decomposition allows us to classify, up to projective transformations of the variables, homogeneous polynomials of degree three in three variables, for instance with the help of the algorithm described in [14]. For another algorithm for decomposing ternary cubics, based on the method of moving frames and on triangular decompositions of algebraic varieties, we refer the reader to [34]. Two polynomials are equivalent in this classicifation if there exists a variable invertible transformation which maps one polynomial to the other.

The classification algorithm goes as follows. Given a ternary cubic, we compute its decomposition as a sum of powers of linear forms. We have the following cases:

- If the rank is one then the polynomial is a 3$^{\text{rd}}$ power of a linear form, that is, it is equivalent to x_0^3.
- If the rank is two, then the polynomial is equivalent to $x_0^3 + x_1^3$ and is in the orbit of $x_0x_1(x_0 + x_1)$. In fact, the decomposition of the latter polynomial is

$$(-0.34817 - 0.41842I)((-0.68827 - 0.16213I)x_0 + (-0.48454 + 0.51499I)x_1)^3$$
$$+(-0.34817 + 0.41842I)((-0.68827 + 0.16213I)x_0 + (-0.48454 - 0.51499I)x_1)^3.$$

- If the rank is three, then the polynomial is either in the orbit of $x_0^2x_1$ or in the orbit of $x_0^3+x_1^3+x_2^3$. To identify which the orbit, it suffices to check if the polynomial is square-free or not (that is, check whether the gcd between the polynomial and one of its derivatives is one). If it is not square-free then it is in the orbit of $x_0^2x_1$. Otherwise it is in the orbit of $x_0^3+x_1^3+x_2^3$.

  The approximate decomposition of $x_0^2x_1$ is

$$
\begin{aligned}
&(-0.16962-0.59162I)((-0.95338-0.09061I)x_0+(-0.00226+0.28783I)x_1)^3\\
&+(-0.16962+0.59162I)((-0.95338+0.09061I)x_0+(-0.00226-0.28783I)x_1)^3\\
&-0.04374(-0.50014x_0-0.86594x_1)^3.
\end{aligned}
$$

- If the rank is 4, then our polynomial is generic. As an example, consider the polynomial $150\,{x_0}^2x_2+{x_1}^2x_2+{x_2}^3-12\,{x_0}^3$; an approximate decomposition of which is

$$
\begin{aligned}
&0.53630(0.34496x_0+0.71403x_1+0.60923x_2)^3\\
&-201.24433(-0.99226x_0+0.00329x_1-0.12411x_2)^3\\
&+213.24332(-0.99270x_0+0.00311x_1+0.12054x_2)^3\\
&+0.53532(-0.34562x_0-0.71446x_1+0.60835x_2)^3.
\end{aligned}
$$

- If the rank is 5, then the polynomial is of maximal rank and it is in the orbit of $x_0^2x_1+x_0x_2^2$, an approximate decomposition of which is

$$
\begin{aligned}
&7.76492(0.37118x_0+0.65728x_1+0.65591x_2)^3\\
&-189.49893(0.31401x_0+0.74558x_1+0.58780x_2)^3\\
&+(-0.35985-0.39864I)((-0.76164-0.24098I)x_0+(0.05666+0.34433I)x_1+(0.46901-0.14176I)x_2)^3\\
&+(-0.35985+0.39864I)((-0.76164+0.24098I)x_0+(0.05666-0.34433I)x_1+(0.46901+0.14176I)x_2)^3\\
&-182.10065(-0.31276x_0-0.74846x_1-0.58481x_2)^3.
\end{aligned}
$$

B. An example of extreme rank

In this section we present in detail the decomposition of a ternary cubic of maximal rank, that is 5. Consider the polynomial

$$x_0^2x_1+x_0x_2^2.$$

The matrix of the quotient algebra is

$$
\begin{bmatrix}
0 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0\\
\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} & h_{2,2,0} & h_{1,3,0}\\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} & h_{1,3,0} & h_{0,4,0}\\
0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} & h_{2,2,0} & h_{5,0,0} & h_{4,1,0} & h_{3,2,0} & h_{2,3,0}\\
0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} & h_{1,3,0} & h_{4,1,0} & h_{3,2,0} & h_{2,3,0} & h_{1,4,0}\\
\frac{1}{3} & 0 & 0 & h_{2,2,0} & h_{1,3,0} & h_{0,4,0} & h_{3,2,0} & h_{2,3,0} & h_{1,4,0} & h_{0,5,0}\\
0 & h_{4,0,0} & h_{3,1,0} & h_{5,0,0} & h_{4,1,0} & h_{3,2,0} & h_{6,0,0} & h_{5,1,0} & h_{4,2,0} & h_{3,3,0}\\
0 & h_{3,1,0} & h_{2,2,0} & h_{4,1,0} & h_{3,2,0} & h_{2,3,0} & h_{5,1,0} & h_{4,2,0} & h_{3,3,0} & h_{2,4,0}\\
0 & h_{2,2,0} & h_{1,3,0} & h_{3,2,0} & h_{2,3,0} & h_{1,4,0} & h_{4,2,0} & h_{3,3,0} & h_{2,4,0} & h_{1,5,0}\\
0 & h_{1,3,0} & h_{0,4,0} & h_{2,3,0} & h_{1,4,0} & h_{0,5,0} & h_{3,3,0} & h_{2,4,0} & h_{1,5,0} & h_{0,6,0}
\end{bmatrix},
$$

and the matrices Δ_0, Δ_1 and Δ_2 are

$$\begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} \end{bmatrix}, \begin{bmatrix} 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{4,0,0} & h_{3,1,0} \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} \\ 0 & h_{4,0,0} & h_{3,1,0} & h_{5,0,0} & h_{4,1,0} \\ 0 & h_{3,1,0} & h_{2,2,0} & h_{4,1,0} & h_{3,2,0} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & h_{3,1,0} & h_{2,2,0} \\ 1/3 & 0 & 0 & h_{2,2,0} & h_{1,3,0} \\ 0 & h_{3,1,0} & h_{2,2,0} & h_{4,1,0} & h_{3,2,0} \\ 0 & h_{2,2,0} & h_{1,3,0} & h_{3,2,0} & h_{2,3,0} \end{bmatrix}.$$

If we form the matrix equation

$$\mathbb{M}_{x_i}\mathbb{M}_{x_j} - \mathbb{M}_{x_j}\mathbb{M}_{x_i} = \Delta_1\Delta_0^{-1}\Delta_2\Delta_0^{-1} - \Delta_2\Delta_0^{-1}\Delta_1\Delta_0^{-1} = \mathbb{O},$$

then we have a system of 8 equations in 8 unknowns. The unknowns are

$$\{h_{5,0,0}, h_{4,1,0}, h_{4,0,0}, h_{3,1,0}, h_{2,2,0}, h_{1,3,0}, h_{3,2,0}, h_{2,3,0}\}.$$

It turns out that the system is not zero dimensional, and that we can choose (randomly) the values of five of them, i.e. $\{h_{1,3,0} = 3, h_{3,1,0} = 1, h_{2,2,0} = 2, h_{4,1,0} = 4, h_{4,0,0} = 5\}$. Working as in the other examples we end up with the decomposition

$$\begin{aligned} &+0.000071(x_0 - 15.778x_1 + 0.510x_2)^3 \\ &+0.002916(x_0 + 3.517x_1 + 5.909x_2)^3 \\ &+0.178137(x_0 + 0.767x_1 - 0.513x_2)^3 \\ &(-0.09056 - 0.0879\, i)(x_0 + (-1.341 + 0.316\, i)x_1 + (-1.168 + 0.781\, i)x_2)^3 \\ &(-0.09056 + 0.0879\, i)(x_0 + (-1.341 + 0.316\, i)x_1 + (-1.168 - 0.781\, i)x_2)^3. \end{aligned}$$